Corrigé de la Série 1 de TD de l'Analyse 2

Exercice 1. *Calculer les primitives suivantes:*

1) $\int (3x^2 - 8x + 2)\, dx$ 2) $\int (\cos 2x)\, dx$ 3) $\displaystyle\int \frac{-3}{x+2} dx.$

4) $\displaystyle\int \frac{1}{(x+1)^4} dx.$ 5) $\displaystyle\int \frac{1}{x \ln x} dx.$ 6) $\displaystyle\int (2x+1)^{10}\, dx.$

Corrigé:

1) $\int (3x^2 - 8x + 2)\, dx = x^3 - 4x^2 + 2x + c,\ \ c \in \mathbb{R}.$

2) $\int (\cos 2x)\, dx = \frac{1}{2} \sin 2x + c,\ \ c \in \mathbb{R}.$

3) $\displaystyle\int \frac{-3}{x+2} dx = -3 \ln |x+2| + c,\ \ c \in \mathbb{R}.$

4) $\displaystyle\int \frac{1}{(x+1)^4} dx = \frac{-1}{3} \frac{1}{(x+1)^3} + c,\ \ c \in \mathbb{R}.$

5) $\displaystyle\int \frac{1}{x \ln x} dx = \int \frac{\left(\frac{1}{x}\right)}{\ln x} dx = \ln |\ln x| + c,\ \ c \in \mathbb{R}.$

6) $\displaystyle\int (2x+1)^{10}\, dx = \frac{1}{22} \int (11)(2)(2x+1)^{10}\, dx = \frac{1}{22} (2x+1)^{11} + c,\ \ c \in \mathbb{R}.$

Exercice 2. *Calculer les primitives suivantes:*

1) $\int xe^x dx,$ $\int_0^1 xe^x dx.$

2) $\displaystyle\int \frac{1}{(x-1)^5} dx,$ $\displaystyle\int_2^3 \frac{1}{(x-1)^5} dx.$

3) $\int \sin^3 x \cos x dx,$ $\int_0^\pi \sin^3 x \cos x dx.$

4) $\displaystyle\int \frac{dx}{3 - 2e^x},$ $\displaystyle\int_0^1 \frac{dx}{3 - 2e^x}.$

Corrigé:

1) Calculons $\int xe^x dx.$

Posons : $\begin{cases} U(x) = x, \\ V'(x) = e^x, \end{cases} \Rightarrow \begin{cases} U'(x) = 1, \\ V(x) = e^x. \end{cases}$

Donc

$$\int xe^x dx = xe^x - \int 1 e^x dx = (x-1)\, e^x + c, \quad c \in \mathbb{R}.$$

Alors,

$$\int_0^1 xe^x dx = \left[(x-1)\, e^x\right]_0^1 = (1-1)\, e^1 - (0-1)\, e^0 = -1.$$

2) Calculons $\int \frac{1}{(x-1)^5}\ dx$.

On pose : $t = x-1$, *donc* $dt = dx$. *Alors*

$$\int \frac{1}{(x-1)^5}\ dx = \int \frac{1}{t^5} dt = \int t^{-5}\ dt = \frac{-1}{4} t^{-4} + c,\ c\in\mathbb{R}$$
$$= \frac{-1}{4(x-1)^4} + c,\ c\in\mathbb{R}.$$

Alors,

$$\int_2^3 \frac{1}{(x-1)^5} dx = \left[\frac{-1}{4(x-1)^4}\right]_2^3 = \left(\frac{-1}{4(3-1)^4}\right) - \left(\frac{-1}{4(2-1)^4}\right) = \frac{15}{64}$$

3) Calculons $\int \sin^3 x \cos x\ dx$.

On pose : $t = \cos x$, *donc* $dt = -(\sin x)\,dx$, *d'où* $dx = \frac{-1}{\sin x} dt$.

Alors

$$\int \sin^3 x \cos x dx = \int \left(\sin^3 x\right) t\ \frac{-1}{\sin x} dt = -\int \left(\sin^2 x\right) t\ dt$$
$$= -\int (1-\cos^2 x)\, t\ dt = -\int (1-t^2)\, t\ dt$$
$$= -\frac{1}{2} t^2 + \frac{1}{4} t^4 + c,\ c\in\mathbb{R}$$
$$= -\frac{1}{2}\cos^2 x + \frac{1}{4}\cos^4 x + c,\ c\in\mathbb{R}.$$

Alors,

$$\int_0^\pi \sin^3 x \cos x dx = \left[-\frac{1}{2}\cos^2 x + \frac{1}{4}\cos^4 x\right]_0^\pi = \left(-\frac{1}{2}\cos^2\pi + \frac{1}{4}\cos^4\pi\right) - \left(-\frac{1}{2}\cos^2 0 + \frac{1}{4}\cos^4 0\right)$$
$$= \left(-\frac{1}{2}(-1)^2 + \frac{1}{4}(-1)^4\right) - \left(-\frac{1}{2}(1)^2 + \frac{1}{4}(1)^4\right) = 0.$$

4) Calculer $\int \frac{dx}{3-2e^x}$.

On pose : $t = e^x$, *donc* $dt = e^x dx$, *d'où* $dx = \frac{1}{e^x} dt = \frac{1}{t} dt$.

On a :

$$\int \frac{dx}{3-2e^x} = \int \frac{dt}{t(3-2t)} = \frac{1}{3}\int \frac{dt}{t} - \frac{1}{3}\int \frac{(-2)\,dt}{3-2t}$$
$$= \frac{1}{3}\ln|t| - \frac{1}{3}\ln|3-2t| + c,\ c\in\mathbb{R}$$
$$= \frac{1}{3}x - \frac{1}{3}\ln|3-2e^x| + c,\ c\in\mathbb{R}.$$

Alors,

$$\int_0^1 \frac{dx}{3-2e^x} = \left[\frac{1}{3}x - \frac{1}{3}\ln|3-2e^x|\right]_0^1 = \left(\frac{1}{3} - \frac{1}{3}\ln|3-2e|\right) - \left(-\frac{1}{3}\ln\left|3-2e^0\right|\right)$$
$$= \frac{1}{3} - \frac{1}{3}\ln(2e-3).$$

Exercice 3.

$$1)\ \int \frac{1}{x^2-1}dx.\quad 2)\ \int \frac{x+4}{x^2+2x+5}dx.$$

Corrigé:

1) Calculons $\int \frac{1}{x^2-1}dx$.

On a :

$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

Par suite on a :

$$\begin{aligned}\int \frac{1}{x^2-1}dx &= \int \frac{1}{2(x-1)}dx - \int \frac{1}{2(x+1)}dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c,\ c\in\mathbb{R}.\end{aligned}$$

2) Calculer $\int \frac{x+4}{x^2+2x+5}dx$.

On a : $x^2+2x+5 = (x+1)^2+4$.

En posant : $x+1 = 2t$ *(et donc* $dx = 2dt$*), on obtient :*

$$\begin{aligned}\int \frac{x+4}{x^2+2x+5}dx &= \int \frac{2t+3}{4(t^2+1)}2dt = \int \frac{4t}{4(t^2+1)}dt + \frac{3}{2}\int \frac{1}{t^2+1}dt \\ &= \frac{1}{2}\ln(t^2+1) + \frac{3}{2}\arctan t + c \\ &= \frac{1}{2}\ln\left(\frac{x^2+2x+5}{4}\right) + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + c,\ c\in\mathbb{R}.\end{aligned}$$

Exercice 4. *Calculer*

$$1)\ \int \cos^5 x \sin^2 x dx. \qquad 2)\ \int \cos^6 x \sin^3 x dx.$$

Corrigé:

1) Calculons $\int \cos^5 x \sin^2 x dx$.

On a :

$$\begin{aligned}\int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx = \int \left(1-\sin^2 x\right)^2 \sin^2 x \cos x dx \\ &= \int (1-t^2)^2 t^2 dt = \int (t^6 - 2t^4 + t^2)\, dt \\ &= \frac{1}{7}t^7 - \frac{2}{5}t^5 + \frac{1}{3}t^3 + c,\ c\in\mathbb{R}. \\ &= \frac{1}{7}\sin^7 x - \frac{2}{5}\sin^5 x + \frac{1}{3}\sin^3 x + c,\ c\in\mathbb{R}.\end{aligned}$$

2) Calculer $\int \cos^6 x \sin^3 x dx$.

On a :

$$\begin{aligned}\int \cos^6 x \sin^3 x dx &= \int \cos^6 x \left(1-\cos^2 x\right) \sin x dx = -\int t^6 \left(1-t^2\right) dt \\ &= \int \left(t^8 - t^6\right) dt = \frac{1}{9} t^9 - \frac{1}{7} t^7 + c, \ \ c \in \mathbb{R}. \\ &= \frac{1}{9} \cos^9 x - \frac{1}{7} \cos^7 x + c, \ \ c \in \mathbb{R}.\end{aligned}$$

Exercice 5. *Calculer*

1) $\int (\cos 5x)(\cos x)\, dx$.
2) $\int (\sin 4x)(\cos 6x)\, dx$.
3) $\int (\sin 3x)(\sin 2x)\, dx$.
4) $\int (\cos^2 x)\, dx$.

1) Calculer $\int (\cos 5x)(\cos x)\, dx$.

On a :

$$\begin{aligned}\int (\cos 5x)(\cos x)\, dx &= \frac{1}{2}\int (\cos 6x)\, dx + \frac{1}{2}\int (\cos 4x)\, dx \\ &= \frac{1}{2}\left(\tfrac{1}{6}\sin 6x\right) + \frac{1}{2}\left(\frac{1}{4}\sin 4x\right) + c, c \in \mathbb{R}. \\ &= \frac{1}{12}\sin 6x + \frac{1}{8}\sin 4x + c, c \in \mathbb{R}.\end{aligned}$$

2) Calculer $\int (\sin 4x)(\cos 6x)\, dx$.

On a :

$$\begin{aligned}\int (\sin 4x)(\cos 6x)\, dx &= \frac{1}{2}\int \sin(10x)\, dx + \frac{1}{2}\int \sin(-2x)\, dx \\ &= \frac{1}{2}\left(\frac{-1}{10}\cos 10x\right) + \frac{1}{2}\left(\frac{-1}{-2}\cos(-2x)\right) + c, \ \ c \in \mathbb{R}. \\ &= \frac{-1}{20}\cos 10x + \frac{1}{4}\cos 2x + c, \ \ c \in \mathbb{R}.\end{aligned}$$

3) Calculer $\int (\sin 3x)(\sin 2x)\, dx$.

On a :

$$\begin{aligned}\int (\sin 3x)(\sin 2x)\, dx &= \frac{1}{2}\int \left[-\cos 5x + \cos x\right] dx \\ &= \frac{1}{2}\left[\frac{-1}{5}\sin 5x + \sin x\right] + c, c \in \mathbb{R}. \\ &= \frac{-1}{10}\sin 5x + \frac{1}{2}\sin x + c, c \in \mathbb{R}.\end{aligned}$$

4) Calculons $\int (\cos^2 x)\, dx$.

$$\int \left(\cos^2 x\right) dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c, c \in \mathbb{R}.$$

Rappel

$$(\cos\alpha x)(\cos\beta x) = \frac{1}{2}\left[\cos(\alpha+\beta)x + \cos(\alpha-\beta)x\right], \alpha\ et\ \beta \in \mathbb{R}^*.$$

$$(\sin\alpha x)(\cos\beta x) = \frac{1}{2}\left[\sin(\alpha+\beta)x + \sin(\alpha-\beta)x\right], \alpha\ et\ \beta \in \mathbb{R}^*.$$

$$(\sin\alpha x)(\sin\beta x) = \frac{1}{2}\left[-\cos(\alpha+\beta)x + \cos(\alpha-\beta)x\right], \alpha\ et\ \beta \in \mathbb{R}^*.$$