Exo 2 : (Intégrales doubles).

I - $D_1 = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq \sqrt{2} \ ; \ 0 \leq y \leq 2\}$.

$$I_1 = \iint_{D_1} y e^{xy} \, dx\,dy$$

$$= \int_0^2 \left[\int_1^{\sqrt{2}} y e^{xy} \, dx \right] dy$$

$$= \int_0^2 \left. e^{xy} \right]_1^{\sqrt{2}} dy.$$

$$= \int_0^2 \left(e^{\sqrt{2}y} - e^{y} \right) dy.$$

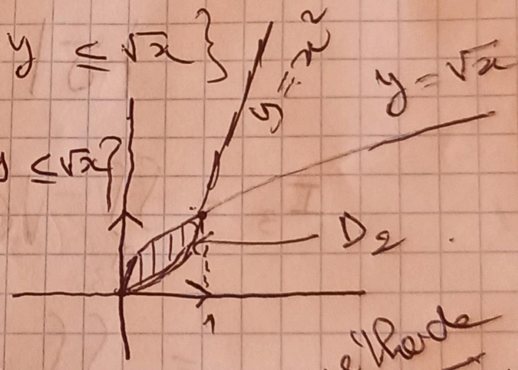
$$= \left[\frac{1}{\sqrt{2}} e^{\sqrt{2}y} - e^{y} \right]_0^2 =$$

$$= \frac{1}{\sqrt{2}} e^{2\sqrt{2}} - e^{2} - \frac{1}{\sqrt{2}} + 1.$$

$$= \frac{e^{2\sqrt{2}} - \sqrt{2}e^{2} - 1 + \sqrt{2}}{\sqrt{2}}$$

II : $D_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 \leq y \leq \sqrt{x}\}$

$D_2 = \{(x,y) \in \mathbb{R}^2 \ ; \ 0 \leq x \leq 1, \ x^2 \leq y \leq \sqrt{x}\}$



1ère méthode

$D_2 = \{ 0 \leq y \leq 1, \ y^2 \leq x \leq \sqrt{y} \}$

2ème méthode

$$I_2 = \iint_{D_2} y \cos x \, dx\,dy$$

$$= \int_0^1 \left[\int_{x^2}^{\sqrt{x}} y \cos x \, dy \right] dx$$

$$= \int_0^1 \left. \frac{1}{2} y^2 \cos x \right]_{x^2}^{\sqrt{x}} dx.$$

$$= \int_0^1 \left(\frac{1}{2} x \cos x - \frac{1}{2} x^4 \cos x \right) dx.$$

$$= \frac{1}{2} \int_0^1 (x - x^4) \cos x \, dx.$$

$$I_2 = \frac{1}{2}\left[(-x^4+12x^2+x-24)\sin x + (-4x^3+24x+1)\cos x\right]_0^1$$

$$= \frac{1}{2}\left[-12\sin 1 + 21\cos(1) - 1\right]$$

$$= -\frac{1}{2}\left[12\sin(1) - 21\cos(1) + 1\right]$$

III - $D_3 = \{(x,y) \in \mathbb{R}^2 / 0 \leq x-y \leq 1,\ 1 \leq x+2y \leq 2\}$

1- tracer le domaine D_3.

2- on pose : $\begin{cases} u = x-y \\ v = x+2y \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{2u+v}{3} \\ y = \dfrac{v-u}{3} \end{cases}$

$$J = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$|J| = \frac{1}{3}$$

$$I_3 = \iint_{D_3} \sqrt{(1-x^2+2xy-y^2)(x+2y)}\, dx\,dy$$

$$= \iint_{\Delta} \sqrt{(1-u^2)v}\ |J|\, du\, dv.$$

$$\Delta = \{(u,v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1 ; 1 \leq v \leq 2\}$$

$$I_3 = \frac{1}{3}\int_0^1 \sqrt{1-u^2}\, du \times \int_1^2 \sqrt{v}\, dv.$$

$$I_3 = \frac{1}{3}\int_0^1 \sqrt{1-u^2}\, du \times \left[\frac{2}{3} v^{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{9}(2\sqrt{2}-1) \int_0^1 \sqrt{1-u^2}\, du$$

$$J = \int_0^1 \sqrt{1-u^2}\, du = ?$$

on pose $u = \sin t$

$du = \cos t\, dt$

$u = 0 \Rightarrow t = 0$

$u = 1 \Rightarrow t = \frac{\pi}{2}$

$$J = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t} \times \cos t\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t}\, \cos t\, dt$$

$$= \int_0^{\frac{\pi}{2}} |\cos t| \times \cos t\, dt$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t\, dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [1+\cos(2t)]\, dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}$$

Donc

$$I_3 = \frac{2}{9}(2\sqrt{2}-1)\frac{\pi}{4}$$

$$= \frac{\pi}{18}(2\sqrt{2}-1)$$

VI)

$$D_4 = \{(x, y) \in \mathbb{R}^2 ; \ x \geq 0 ; \ y \geq 0 \quad 1 \leq x^2 + y^2 \leq 2\}.$$

$$I_4 = \iint_{D_4} \frac{1}{\sqrt{x^2+y^2+1}} \, dx\,dy.$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{matrix} 1 \leq r \leq \sqrt{2} \\ 0 \leq \theta \leq \frac{\pi}{2} \\ |J| = r \end{matrix} \quad (\Delta)$$

$$I_4 = \iint_{\Delta} \frac{1}{\sqrt{r^2+1}} \, r \, dr\, d\theta.$$

$$= \int_1^{\sqrt{2}} \frac{r}{\sqrt{r^2+1}} \, dr \times \int_0^{\frac{\pi}{2}} d\theta.$$

$$= \frac{1}{2} \int_1^{\sqrt{2}} \frac{2r}{\sqrt{r^2+1}} \, dr \times \left[\frac{\pi}{2}\right].$$

$$= \frac{\pi}{4} \left[2\sqrt{r^2+1} \right]_1^{\sqrt{2}}.$$

$$= \frac{\pi}{2} \left(\sqrt{3} - \sqrt{2} \right).$$

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$$D_5 = \{(x,y) \in \mathbb{R}^2 / \ x \leq 0, \quad 1 \leq x^2+y^2 \leq 4\}$$

$$I_5 = \iint_{D_5} (2x^2 - y^2)\, dx\, dy$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{tq} \quad \begin{matrix} 1 \leq r \leq 2 \\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \end{matrix} \quad \Delta$$

$$|J| = r$$

$$I_5 = \iint_{\Delta} (2r^2\cos^2\theta - r^2\sin^2\theta) \times r\, dr\, d\theta$$

$$= \iint_{\Delta} r^3(2\cos^2\theta - \sin^2\theta)\, dr\, d\theta$$

$$= \int_1^2 r^3\, dr \times \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2\cos^2\theta - \sin^2\theta)\, d\theta$$

$$= \left[\frac{1}{4}r^4\right]_1^2 \times \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (3\cos^2\theta - 1)\, d\theta$$

$$= \frac{15}{4} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{3}{2}(1 + \cos(2\theta)) - 1)\, d\theta$$

$$= \frac{15}{8} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 + 3\cos(2\theta))\, d\theta$$

$$= \frac{15}{8} \left[\theta + \frac{3}{2}\sin(2\theta)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{15}{8} \left[\frac{3\pi}{2} - \frac{\pi}{2}\right]$$

$$= \frac{15\pi}{8}$$