EXO3 :

I - $V = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 2 ; 0 \leq y \leq 4 ; 0 \leq z \leq 3\}$

$$I = \iiint_V x y z^2 e^{x^2 y} \, dx\,dy\,dz$$

$$= \int_0^4 \left[\int_0^2 x y e^{x^2 y} dx \right] dy \times \int_0^3 z^2 dz$$

$$= \int_0^4 \left[\frac{1}{2} e^{x^2 y} \right]_0^2 dy \times \left[\frac{1}{3} z^3 \right]_0^3$$

$$= \frac{1}{2} \int_0^4 (e^{4y} - 1)\, dy \times 9$$

$$= \frac{9}{2} \left[\frac{1}{4} e^{4y} - y \right]_0^4$$

$$= \frac{9}{2} \left[\frac{1}{4} e^{16} - 4 - \frac{1}{4} \right]$$

$$= \frac{9}{2} \left[\frac{e^{16} - 17}{4} \right]$$

$$= \frac{9}{8} (e^{16} - 17).$$

II- $V_1 = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq 2 ; 0 \leq y \leq x ; 0 \leq z \leq x^2+y^2\}$

$$I_1 = \iiint_{V_1} (3x+2y)\,dx\,dy\,dz$$

$$= \int_0^2 \left[\int_0^x \left[\int_0^{x^2+y^2} (3x+2y)\,dz \right] dy \right] dx$$

$$= \int_0^2 \left[\int_0^x \left[(3x+2y)z \right]_0^{x^2+y^2} dy \right] dx$$

$$= \int_0^2 \left[\int_0^x (3x^3 + 2y^3 + 3xy^2 + 2yx^2)\,dy \right] dx$$

$$= \int_0^2 \left[3x^3y + \frac{1}{2}y^4 + xy^3 + x^2y^2 \right]_0^x dx$$

$$= \int_0^2 \frac{11}{2} x^4\,dx$$

$$= \left[\frac{11}{10} x^5 \right]_0^2$$

$$= \frac{176}{5}$$

(3) $V_2 = \{(x,y,z) \in \mathbb{R}^3 ; x \geq 0 ; y \geq 0, z \geq 0 ; z^2 \leq 1 - x^2 - y^2\}$

$$I_2 = \iiint_{V_2} z \sin(x^2+y^2)\, dx\, dy\, dz$$

On utilise les coordonnés cylindriques :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Big/ \quad \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq z \leq \sqrt{1-r^2} \end{array} \quad |J| = r$$

$$I_2 = \iiint_{\Delta} z \sin(r^2) \times r\, dr\, d\theta\, dz$$

$$= \int_0^1 \left[\int_0^{\sqrt{1-r^2}} z\, r \sin(r^2)\, dz \right] dr \times \int_0^{\frac{\pi}{2}} d\theta$$

$$= \int_0^1 \frac{1}{2}(1-r^2) \times r \sin(r^2)\, dr \times \left[\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4} \int_0^1 (1-r^2) \times r \sin(r^2)\, dr$$

$u(r) = 1 - r^2 \Rightarrow u'(r) = -2r$

$v'(r) = r\sin(r^2) \Rightarrow v(r) = -\frac{1}{2}\cos(r^2)$

$$I_2 = \frac{\pi}{4}\left(\left[(1-r^2) \times \frac{-1}{2} \cos(r^2) \right]_0^1 - \int_0^r r\cos(r^2)\, dr \right)$$

$$= \frac{\pi}{4}\left(\left[\frac{1}{2}\right] - \frac{1}{2}\left[\sin(r^2)\right]_0^1 \right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2} - \frac{1}{2}\sin(1) \right)$$

$$= \frac{\pi}{8}(1 - \sin(1))$$

$\boxed{3}$

VI. $V_3 = \{(x,y,z) \in \mathbb{R}^3 \mid z \geq 0;\ x^2+y^2+z^2 \leq 1\}$

$$I_3 = \iiint_{V_3} xyz \, dx\, dy\, dz$$

En utilisant les coordonnés sphériques :

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\sin\varphi \end{cases} \text{ tq } \quad \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{array}$$

$$|J| = r^2\cos\varphi$$

$$I_3 = \iiint_{\Delta} r\cos\theta\cos\varphi \; r\sin\theta\cos\varphi \; r\sin\varphi \; r^2\cos\varphi \, dr\,d\theta\,d\varphi$$

$$= \int_0^1 r^5 dr \times \int_0^{2\pi} \cos\theta\sin\theta \, d\theta \times \int_0^{\frac{\pi}{2}} \sin\varphi\cos^3\varphi \, d\varphi$$

$$= \left[\frac{1}{6} r^6\right]_0^1 \times \left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi} \times \left[-\frac{1}{4}\cos^4\varphi\right]_0^{\frac{\pi}{2}}$$

$$= 0$$