Solutions de la série n°1

Solution 1 *Les solutions aux questions (1 + 3 + 5 + 6 + 7 + 8) sont données dans le tableau suivant,*

z	$\mathcal{R}e(z)$	$\mathrm{Im}(z)$	$\overline{z}$	$\lvert z\rvert$	$Arg(z)$	$\arg(z)$	*forme trigo.*	*f. exp.*
0	0	0	0	0	*ind*	*ind*	*ind*	*ind*
-1	-1	0	-1	1	π	$\pi+2k\pi,\ k\in\mathbb{Z}$	$(\cos\pi+i\sin\pi)$	$e^{\pi i}$
$+1$	$+1$	0	$+1$	1	0	$2k\pi,\ k\in\mathbb{Z}$	$(\cos 0+i\sin 0)$	e^{0i}
i	0	1	$-i$	1	$\frac{\pi}{2}$	$\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z}$	$(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$	$e^{\frac{\pi}{2}i}$
$-i$	0	-1	$+i$	1	$\frac{3\pi}{2}$	$\frac{3\pi}{2}+2k\pi,\ k\in\mathbb{Z}$	$(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2})$	$e^{\frac{3\pi}{2}i}$
$1+i$	1	1	$1-i$	$\sqrt{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$	$\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$	$\sqrt{2}e^{\frac{\pi}{4}i}$
$-1+i$	-1	1	$-1-i$	$\sqrt{2}$	$\frac{3\pi}{4}$	$\frac{3\pi}{4}+2k\pi,\ k\in\mathbb{Z}$	$\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$	$\sqrt{2}e^{\frac{3\pi}{4}i}$
$-1-i$	-1	-1	$-1+i$	$\sqrt{2}$	$\frac{5\pi}{4}$	$\frac{5\pi}{4}+2k\pi,\ k\in\mathbb{Z}$	$\sqrt{2}(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4})$	$\sqrt{2}e^{\frac{5\pi}{4}i}$
$1-i$	1	-1	$1+i$	$\sqrt{2}$	$\frac{7\pi}{4}$	$\frac{7\pi}{4}+2k\pi,\ k\in\mathbb{Z}$	$\sqrt{2}(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4})$	$\sqrt{2}e^{\frac{7\pi}{4}i}$

4- Forme algébrique,

$$z_2+z_9=-1+(1-i)=-i$$

$$z_4+z_6=i+(1+i)=1+2i$$

$$z_2-z_8=-1-(-1-i)=i$$

$$z_5-z_7=-i-(-1+i)=1-2i$$

$$z_4z_6=i(1+i)=-1+i$$

$$z_7z_8=(-1+i)(-1-i)=2$$

$$\frac{z_3}{z_7}=\frac{1}{-1+i}=\frac{-1-i}{2}=-\frac{1}{2}-\frac{1}{2}i$$

$$\frac{z_5}{z_9}=\frac{-i}{1-i}=\frac{-i(1+i)}{2}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i$$

$$\frac{z_7}{z_8}=\frac{-1+i}{-1-i}=\frac{(-1+i)(-1+i)}{2}=\frac{-2i}{2}=-i.$$

9- On a,

$$(z_6)^{25} = (1+i)^{25} = (2^{1/2}e^{\frac{\pi}{2}i})^{25} = 2^{25/2}e^{\frac{25\pi}{2}i} = 2^{12}\sqrt{2}e^{\frac{\pi}{2}i} = 2^{12}\sqrt{2}i.$$

$$\left(\frac{z_7}{z_8}\right)^{20} = \left(\frac{-1+i}{-1-i}\right)^{20} = (-i)^{20} = ((-i)^2)^{10} = (-1)^{10} = 1.$$

Solution 2

$$1 = |1+a-a-b+b| \leq |1+a| + |-a-b+b| \leq |1+a| + |a+b| + |b|.$$

$$\mathcal{R}e(a^2) = \mathcal{R}e((\alpha+i\beta)^2) = \alpha^2 - \beta^2 = -\alpha^2 - \beta^2 + 2\alpha^2 = 2(\mathcal{R}e(a))^2 - |a|^2.$$

$$|1+a|^2 + |1-a|^2 = (1+a)\overline{(1+a)} + (1-a)\overline{(1-a)} = (1+a)(1+\overline{a}) + (1-a)(1-\overline{a}) = 2 + 2a\overline{a} = 2 + 2|a|^2 = 4.$$

$$\overline{\left(i\left(\frac{a+1}{a-1}\right)\right)} = -i\left(\frac{\overline{a}+1}{\overline{a}-1}\right) = -i\left(\frac{\frac{1}{a}+1}{\frac{1}{a}-1}\right) = -i\left(\frac{a+1}{1-a}\right) = i\left(\frac{a+1}{a-1}\right) \Leftrightarrow i\left(\frac{a+1}{a-1}\right) \in \mathbb{R}$$

et

$$\begin{aligned}\mathcal{R}e(\frac{1}{1-a}) &= \frac{1}{2}\left[(\frac{1}{1-a}) + \overline{(\frac{1}{1-a})}\right] = \frac{1}{2}\left[(\frac{1}{1-a}) + (\frac{1}{1-\overline{a}})\right] = \frac{1}{2}\left[\frac{2-a-\overline{a}}{(1-a)(1-\overline{a})}\right] \\ &= \frac{1}{2}\left[\frac{2-a-\overline{a}}{(1-a-\overline{a}+a\overline{a}}\right] = \frac{1}{2}\left[\frac{2-a-\overline{a}}{2-a-\overline{a}}\right] = \frac{1}{2}.\end{aligned}$$

Solution 3

$$\left|\frac{\lambda-i}{\lambda+i}\right| = 1 \Leftrightarrow |\lambda-i| = |\lambda+i| \Leftrightarrow |x+i(y-1)| = |x+i(y+1)| \Leftrightarrow x^2+(y-1)^2 = x^2+(y+1)^2$$

$$\Leftrightarrow y = 0 \Leftrightarrow \lambda = x \in \mathbb{R}$$

$$\left|\frac{1+\lambda i}{1-\lambda i}\right| = 1 \Leftrightarrow \left|\frac{i(\lambda-i)}{-i(\lambda+i)}\right| = 1 \Leftrightarrow \left|\frac{\lambda-i}{\lambda+i}\right| = 1 \Leftrightarrow \lambda \in \mathbb{R}$$

Solution 4

$$\frac{1-z^n}{1-z} = \sum_{k=0}^{k=n-1} z^k$$

$$\left|\frac{1-z^{2019}}{1-z}\right| = \left|\sum_{k=0}^{k=2018} z^k\right| \leq \sum_{k=0}^{k=2018} \left|z^k\right| = \sum_{k=0}^{k=2018} |z|^k = \frac{1-|z|^{2019}}{1-|z|}.$$

Solution 5

$$(-1)^{1/5} = (e^{i(\pi+2k\pi)})^{1/5} = e^{(\frac{2k+1}{5})\pi i}, \qquad 0 \leq k < 5.$$

$$(+1)^{1/3} = (e^{2k\pi i})^{1/3} = e^{(\frac{2k}{3})\pi i}, \qquad 0 \leq k < 3.$$

$$(i)^{1/4} = (e^{i(\frac{\pi}{2}+2k\pi)})^{1/4} = e^{(\frac{4k+1}{8})\pi i}, \qquad 0 \leq k < 4.$$

$$(1+i)^{1/2} = (2^{1/2}e^{i(\frac{\pi}{4}+2k\pi)})^{1/2} = 2^{1/4}e^{(\frac{8k+1}{8})\pi i}, \qquad 0 \leq k < 2.$$

$$(-1+i)^{1/8} = (2^{1/2}e^{i(\frac{3\pi}{4}+2k\pi)})^{1/8} = 2^{1/16}e^{(\frac{8k+3}{32})\pi i}, \qquad 0 \leq k < 8.$$

Solution 6

$$(4+2i)x + (5-3i)y = 13+i$$

$$(4x+5y-13) + i(2x-3y-1) = 0$$

$$\begin{cases} 4x+5y-13 &= 0 \\ 2x-3y-1 &= 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x &= 2 \\ y &= 1 \end{cases}$$

$$\Leftrightarrow z = 2+i.$$

$4z^2 + 8\left|z\right|^2 - 3 = 0$

$$\Leftrightarrow 4(x+iy)^2 + 8(x^2+y^2) - 3 = 0$$

$$\Leftrightarrow 4(x^2 - y^2 + 2ixy) + 8(x^2+y^2) - 3 = 0$$

$$\Leftrightarrow 12x^2 + 4y^2 - 3 + i8xy = 0$$

$$\Leftrightarrow \left\{ \begin{array}{lcl} 12x^2 + 4y^2 - 3 & = & 0 \\ 8xy & = & 0 \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{lcl} x & = & 0 \\ y & = & \frac{\sqrt{3}}{2} \end{array} \right. ; \left\{ \begin{array}{lcl} x & = & 0 \\ y & = & -\frac{\sqrt{3}}{2} \end{array} \right. ; \left\{ \begin{array}{lcl} x & = & \frac{1}{2} \\ y & = & 0 \end{array} \right. ; \left\{ \begin{array}{lcl} x & = & -\frac{1}{2} \\ y & = & 0 \end{array} \right. ;$$

$$\Leftrightarrow z_1 = -i\frac{\sqrt{3}}{2}; z_2 = i\frac{\sqrt{3}}{2}; z_3 = -\frac{1}{2}; z_4 = \frac{1}{2}.$$

$$z^2 + (2i-3)z + 5 - i = 0$$

$$\Delta = (2i-3)^2 - 4(5-i) = -15 - 8i$$

$$\sqrt{\Delta} = \pm(1-4i)$$

$$z_1 = \frac{-(2i-3)-(1-4i)}{2} = 2 - 3i$$

$$z_2 = \frac{-(2i-3)+(1-4i)}{2} = 1 + i.$$

$$z^4 - (3+8i)z^2 - 16 + 12i = 0$$

$$X^2 - (3+8i)X - 16 + 12i = 0$$

$$\Delta = (3+8i)^2 - 4(-16+12i) = 9$$

$$X_1 = \frac{(3+8i)-3}{2} = 4i$$
$$X_2 = \frac{(3+8i)+3}{2} = 3-4i$$

$$\sqrt{X_1} = \pm(\sqrt{2} + i\sqrt{2})$$
$$\sqrt{X_2} = \pm(2+i)$$

$$z_1 = -\sqrt{2} - i\sqrt{2}\,;\; z_2 = \sqrt{2} + i\sqrt{2}\,;\; z_3 = -2 - i\,;\; z_4 = 2 + i.$$