Mathematical reminder (part 1)

This worksheet is a reminder of the main concepts and operations we will be using throughout the course. This worksheet is the first part of two series of reminders, the second of which is devoted to probability calculations. It can be found in the Sampling course.

First Part

Prerequisite : the theory of the set of numbers.

Reminder Mathematics & Statistics

1 Elementary calculation rules

Let x, y and z three real numbers ($x, y, z \in \mathbb{R}$). The following rules apply :

The commutative principle. Whatever the real numbers x and y,

$$x + y = y + x \quad \text{et} \quad xy = yx$$

Associativity. Whatever the real numbers x, y and z,

$$(x + y) + z = x + (y + z) \quad \text{et} \quad x(yz) = (xy)z$$

Distributivity. Whatever the real numbers x and y,

$$(x + y)z = xz + yz$$

The neutral elements. Whatever the real number x :

$$x + 0 = x \quad \text{et} \quad x \times 1 = x$$

2 Summation & Product

2.1. The summation sign

$\sum$ is read **Sigma**, is the capital letter of the Greek letter S.

Exemples

$$\sum_{i=1}^{10} x_i = 1+2+3+4+5+6+7+8+9+10 = 55$$

Remarque

- The summing expression $\sum_{i=k}^{n} x_i$ can also be written as : $\sum_{k \leq i \leq n} x_i$.

- The summation index i can have any integer value (positive, negative or zero).

- We say that the summation index is a **dummy variable**, i.e. it can be replaced by any other letter apart from the start and end letters (i and n in our case).

Some properties

Some properties apply to summation signs $\sum$.

Property 1 : Let k and l two integers such that $k \geq l$, and let x_i and y_i for $i = k, \ldots, l$ are any real numbers, then :

$$\sum_{i=k}^{l}(x_i + y_i) = \sum_{i=k}^{l} x_i + \sum_{i=k}^{l} y_i$$

For any real number β,

$$\sum_{i=k}^{l} \beta x_i = \beta \sum_{i=k}^{l} x_i$$

The factorisation of the previous property :

$$\sum_{i=k}^{l}(\beta x_i + \lambda y_i) = \beta\sum_{i=k}^{l} x_i + \lambda\sum_{i=k}^{l} y_i$$

Property of sums and products : Let n and m two integers and x_i, y_i for $i = 1,\dots,n$ and $j = 1,\dots,m$, are any real numbers, then :

$$\left(\sum_{i=0}^{n} x_i\right)\left(\sum_{j=0}^{m} y_j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m} x_i y_j$$

Telescopic sums

Let l and k be two natural numbers with $k \leq l$, and let x_i for $i = k,\dots,n$ be any real numbers, then :

$$\sum_{i=k}^{l}(x_{i+1} - x_i) = x_{l+1} - x_k.$$

The Sum of an Arithmetic Sequence

For any natural number n :

$$\sum_{x=1}^{n} x = 1+2+3+\cdots+(n-1)+n = \frac{n(n+1)}{2}.$$

Sums of Squares from 1 to n

For any natural number n :

$$\sum_{x=1}^{n} x^2 = 1^2+2^2+3^2+\cdots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Sums of Cubes from 1 to n

For any natural number n :

$$\sum_{x=1}^{n} x^3 = 1^3+2^3+3^3+\cdots+(n-1)^3+n^3 = \frac{n^2(n+1)^2}{4}.$$

Double Sums

Double sums were the subject of the chapter on bivariate analysis (cite the chapter and provide a reference).

In the case of two natural numbers i and j and $x_{i,j}$ as real numbers, with $i = 1,\dots,n$ and $j = 1,\dots,m$, the sum of $x_{i,j}$ is written as follows :

$$\sum_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} x_{i,j}.$$

Consequence of the notation : if $n = m$, we will have the following notation :

$$\sum_{1\leq i,j\leq n} x_{i,j}.$$

The following results derive from the use of double summation notation :

- For all real $x_{i,j}$ and $i,j = 1,\dots,n$, we have :

$$\sum_{1\leq i,j\leq n} x_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{n} x_{i,j}.$$

- The following notation holds for all i, j ordered in a broad sense :

$$\sum_{1\leq i\leq j\leq n} x_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{j} x_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j}.$$

- For all i, j ordered in a **strict** sense, the notation becomes :

$$\prod_{i=j}^{n} x_i = \sum_{j=2}^{n}\sum_{i=1}^{j-1} x_{i,j} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_{i,j}.$$

2.2. Product

Products are symbolized by the sign Π, which is the Greek letter P.

We will see its use in the following lines.

Let i and j be two natural numbers and let x_i for $i = j,\dots,n$ be any real numbers :

$$\prod_{i=j}^{n} x_i = \begin{cases} x_j \times x_{j+1} \times \cdots \times x_{n-1} \times x_n & \text{if } j \leq n,\\ 0 & \text{otherwise.}\end{cases}$$

The notation $\prod_{i=j}^{n} x_i$ is also written as $\prod_{j\leq i\leq n}$ and is pronounced as follows : *product of x_i for i ranging from j to n.*

Properties of Products

Let j and n be two natural numbers such that $j \leq n$,
the following properties apply to products :

— For any real number x,

$$\prod_{i=j}^{n} x = x \times x \times x \times \cdots \times x = x^{n-j+1}$$

— For all real numbers x_i, $i = j,\dots,n$ and for any integer s such that $j \leq s < n$, we have :

$$\prod_{i=j}^{n} x_i = \left(\prod_{i=j}^{s} x_i\right)\left(\prod_{i=s+1}^{n} x_i\right)$$

— For all real numbers x_i and y_i, $i = j, \dots, n$,

$$\prod_{i=j}^{n} x_i y_i = \left(\prod_{i=j}^{n} x_i\right)\left(\prod_{i=j}^{n} y_i\right)$$

— For all real numbers x_i and y_i, $i = j, \dots, n$, such that $y_i \neq 0$ for all i,

$$\prod_{i=j}^{n} x_i / y_i = \left(\prod_{i=j}^{n} x_i\right) / \left(\prod_{i=j}^{n} y_i\right)$$

— For all real numbers x_i, $i = j, \dots, n$ and for any $e \in \mathbb{N}$,

$$\prod_{i=j}^{n} x_i^e = \left(\prod_{i=j}^{n} x_i\right)^e$$

Factorial

Let n be a natural number, we denote $n!$ (read as factorial n) the natural number :

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ \prod_{i=1}^{n} i & \text{otherwise} \end{cases}$$

This means :

$$0! = 1 \text{ (by convention)}$$

$$n! = 1 \times 2 \times 3 \times 4 \times \cdots \times n$$

3 Real Number Intervals

An interval I of $\mathbb{R}$ satisfies, for all x and y in I and for any z in $\mathbb{R}$, if $x \leq z \leq y$, then z belongs to I.

Remarks

— The fact of considering a subset I of $\mathbb{R}$ is denoted $I \subset \mathbb{R}$ (read as I is included in $\mathbb{R}$) ;

— The fact of considering an element x of I is denoted $x \in I$ (read as x belongs to I).

Therefore, do not confuse the symbol $\subset$, which is used for subsets, and $\in$, which is used for elements. We could rewrite the previous definition as follows : An interval I of $\mathbb{R}$ is any subset $I \subset \mathbb{R}$ satisfying for all $x, y \in I$ and for any $z \in \mathbb{R}$, if $x \leq z \leq y$, then $z \in I$.

Closed and Bounded Interval (Segment)
Let x and y be two real numbers such that $x \leq y$. We call a closed and bounded interval (also called a *segment*) of $\mathbb{R}$ any set of the form :

$$[x, y] = \{z \in \mathbb{R}, \quad x \leq z \leq y\}.$$

Open Interval
Let x and y be two real numbers such that $x < y$. We call an open interval of $\mathbb{R}$ any set of the form :

$$]x, y[= \{z \in \mathbb{R}, \quad x < z < y\}.$$

Open and Bounded Interval
Let x and y be two real numbers such that $x \leq y$. We call an open and bounded interval of $\mathbb{R}$ any set of the form :

$$]x, y[= \{z \in \mathbb{R}, \quad x < z < y\}.$$

Semi-Open and Bounded Interval
Let x and y be two real numbers such that $x \leq y$. We call a semi-open and bounded interval of $\mathbb{R}$ any set of the form :

$$[x, y[= \{z \in \mathbb{R}, \quad x \leq z < y\}.$$

This configuration is also possible :

$$]x, y] = \{z \in \mathbb{R}, \quad x < z \leq y\}.$$

Remark :
It can also be sets of the form :

$$]x, +\infty[= \{z \in \mathbb{R}, \quad x < z\},$$

$$]-\infty, y[= \{z \in \mathbb{R}, \quad z < y\},$$

Closed and Unbounded Interval
Let x and y be two real numbers. By convention, we call a closed and unbounded interval of $\mathbb{R}$ any set of the form :

$$[x, +\infty[= \{z \in \mathbb{R}, \quad x \leq z\}.$$

This configuration is also possible :

$$]-\infty, y] = \{z \in \mathbb{R}, \quad z \leq y\}.$$

Remarks and Notations :

— We denote the following particular intervals :
$\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_+^* =]0, +\infty[$, $\mathbb{R}_- =]-\infty, 0]$, $\mathbb{R}_-^* =]-\infty, 0[$.

— The notation $\mathbb{R}^*$ denotes the set of $\mathbb{R}$ without 0.

— The interval that contains no real number is called the **empty set**, denoted as : $\emptyset$.

— The interval that contains only one number is called a **singleton**. It is denoted by braces : $\{x\}$.

The following table provides a classification of intervals in $\mathbb{R}$:

Intervals in $\mathbb{R}$	**Bounded**	**Unbounded**
Open	$]x, y[\;\; ; \;\; \emptyset$	$\mathbb{R} \;\; ; \;\;]-\infty, x[\;\; ; \;\;]x, +\infty[$
Closed	$[x, y] \;\; ; \;\; \{x\} \;\; ; \;\; \emptyset$	$\mathbb{R} \;\; ; \;\;]-\infty, x] \;\; ; \;\; [x, +\infty[$
Semi-open	$[x, y[\;\; ; \;\;]x, y]$	