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NUMERICAL ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS: APPLICATION IN FINANCIAL MATHEMATICS

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Contents

1. Introduction

This course, offered at the University of Bouira and the University of Béjaïa, is aimed at Master's students in Operations Research, with a particular focus on Financial Mathematics. The main objective is to study certain linear partial differential equations (PDEs) and their applications in approximating option prices in financial markets, emphasizing fundamental concepts while maintaining academic rigor.

The course covers several linear PDE of first and second order, with analytical solutions provided in the first part. The second part focuses on solving the heat equation using both analytical methods and finite difference methods, drawing on the work of specialists in the field. Although this approach does not allow for as in-depth analysis as traditional numerical methods, it is sufficient to fully address the problems of option pricing, particularly those modeled by the Black, Scholes and Merton PDE.

The modeling of a real-world problem relies on the application of physical laws (such as mechanics, thermodynamics, electromagnetism, acoustics, etc.), which are generally expressed in the form of balances translated mathematically by ordinary differential equations (ODE) or partial differential equations (PDE). These equations also appear in many other fields, such as chemistry for modeling reactions, economics for studying market behavior, and finance for analyzing derivative products (options and bonds), among others.

Partial differential equations are a highly active area of research in mathematics and have led to the creation of many fundamental concepts, such as the Fourier transform. In many cases, it is extremely difficult, if not impossible, to obtain an explicit solution for a partial differential equation. However, in some cases, it is possible to show that the problem is well-posed (i.e., it admits a unique solution), and, occasionally, to calculate numerical approximations of the solutions.

A partial differential equation is an equation that involves partial derivatives, such as the following example:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where u is a function of x and t ($x \in \mathbb{R}^n$, $t \in \mathbb{R}_+^*$).
Unlike ordinary differential equations, which are based on a key theorem – the fundamental theorem of existence and uniqueness – there is no unified theory for partial differential equations. Instead, each type of partial differential equation has its own specific properties, often reflecting the physical phenomena the equation was originally used to model.

In this course, we will focus on a key example of partial differential equations: the heat equation. We will study it in two main parts. First, we will explore analytical methods, which will allow us to solve this equation exactly using traditional mathematical tools. Then, in the second part, we will introduce numerical methods, particularly finite difference approaches, which provide an effective solution for solving complex equations that are difficult to handle analytically.

The study of the heat equation is central to this work, as it serves as a fundamental model that we will apply in practical applications, especially in the field of finance. Indeed, partial differential equations, such as the heat equation, play a central role in modeling many physical and financial phenomena. A solid understanding of these equations is essential for modeling complex systems, especially in quantitative finance, where they are used to model processes such as price diffusion, interest rates, or option pricing.

Linear systems and Partiels differentiel Equations

2. Systems of differential equations

Introduction

Many fundamental theories in physics and engineering are expressed through systems of partial differential equations. Some of these are familiar to those who have studied physics, such as the Euler or Navier-Stokes equations in fluid mechanics, Maxwell's equations in electromagnetism, Einstein's field equations in general relativity, or the Black-Scholes-Merton formula in finance for option pricing. Therefore, it is crucial to develop methods and techniques for solving a wide variety of partial differential equations.

2.1 Differential equations

Differential equations are a fundamental tool in mathematics and physics. They allow us to describe the evolution of a quantity with respect to time (or another independent variable). In general, a differential equation relates an unknown function $x(t)$ and its derivatives with respect to an independent variable t.

Definition 2.1.1 A differential equation is a mathematical equation that relates a function with its derivatives such that the solution satisfies both the function and the derivatives.

Differential equations can be split into two classes:

Definition 2.1.2 (**Ordinary differential equation (ODE)**)
An ordinary differential equation (ODE) is an equation (or equation system) involving an unknown function of one independent variable and certain of its derivatives.

Definition 2.1.3 (**Partial differential equation (PDE)**)
A partial differential equation (PDE) is an equation (or equation system) involving an unknown function of two or more variables and certain of its partial derivatives.

2.1.1 General form of a differential equation

The most common form of a differential equation is:

$$\frac{dx}{dt} = F(t,x);$$

where $F(t,x)$ is a given function of t (the independent variable) and $x(t)$ (the unknown function).

This means that the derivative of $x(t)$ with respect to t is equal to a function of t and $x(t)$, and the goal is to find the function $x(t)$ that satisfies this relationship.

2.1.2 Origins and history

Differential equations played a crucial role in the development of modern mathematics. The advent of infinitesimal calculus in the 17th century through the work of Isaac Newton and Gottfried Wilhelm Leibnitz was a turning point. They developed the foundations of analysis and differentiation, which are essential for understanding the dynamic behavior of systems.

Newton, in particular, used differential equations to model the motion of the planets under the influence of universal gravitation—one of the first notable applications that demonstrated their power. The notation of Leibnitz, $\frac{dx}{dt}$, which we still use today, helped formalize and solve complex problems in physics, geometry, engineering and in finance.

2.1.3 Applications and importance

Differential equations are ubiquitous in many fields:

- **Physics**: To model phenomena such as the laws of motion, wave propagation, or electrical circuits;
- **Biology**: To study population growth, the spread of diseases, or chemical reactions;
- **Economics**: To analyze the evolution of economic systems, such as growth models or market dynamics;
- **Engineering**: To model mechanical, electrical, or thermal systems.

R Infinitesimal calculus, combined with differential equations, allows us not only to understand instantaneous changes but also to approximate the evolution over intervals of time or other variables.

2.1.4 Single equations versus PDE systems

Definition 2.1.4 (Single equation)
A single PDE consists of determining one solution variable, e.g.,

$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}.$$

Typical examples are Poisson's problem, the heat equation, wave equation, Laplacian.

Definition 2.1.5 (PDE system)
A PDE system determines a solution vector

$$u = (u_1, ..., u_n) : \Omega^n \to \mathbb{R}^n.$$

For each $u_i, i = 1, ..., n$, a PDE must be solved. Inside these PDE, the solution variables may depend on each other or not.

2.2 Systems of differential equations

In this chapter we will consider simultaneous first-order differential equations in several variables, that is, equations of the form

$$\begin{cases} \frac{dy_1}{dt} = f_1(t, y_1, y_2 ..., y_n), \\ \frac{dy_2}{dt} = f_2(t, y_1, y_2 ..., y_n), \\ \vdots \\ \frac{dy_n}{dt} = f_n(t, y_1, y_2 ..., y_n), \end{cases} \tag{2.1}$$

A solution of (2.1) is n functions $x_1(t)$, ..., $x_n(t)$ such that

$$\frac{dx_j(t)}{dt} = f_j(t, x_1(t), ..., x_n(t)), \quad j = 1, 2, ..., n.$$

In addition to system (2.1), we will often impose initial conditions on the functions $x_1(t)$, ..., $x_n(t)$. These will be of the form

$$x_1(t_0) = x_1^0, \;\; x_2(t_0) = x_2^0, \;\; ..., x_n(t_0) = x_n^0. \tag{2.2}$$

Equation (2.1), together with the initial conditions (2.2), is referred to as an initial-value problem. A solution of this initial-value problem is n functions $x_1(t)$, ..., $x_n(t)$ which satisfy (2.1) and the initial conditions (2.2).

Equation (2.1) is usually referred to as a system of n first-order differential equations. Equations of this type arise quite often in biological and physical applications and frequently describe very complicated systems since the rate of change of the variable x_j depends not only on t and x_j, but on the value of all the other variables as well.

First-order systems of differential equations also arise from higher-order equations for a single variable $y(t)$. Every n^{th} order differential equation for the single variable y can be converted into a system of n first-order equations for the variables

$$x_1(t) = y, \;\; x_2(t) = \frac{dy}{dt}, ..., \;\; x_n(t) = \frac{d^{n-1}y}{dt^{n-1}}.$$

Convert the differential equation

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + ... + a_0 y = 0$$

into a system of n first-order equations.
Let $x_1(t) = y$, $x_2(t) = \frac{dy}{dt}$,..., $x_1(t) = \frac{d^{n-1}y}{dt^{n-1}}$, then

$$\frac{dx_1}{dt} = x_2, \quad , \frac{dx_2}{dt} = x_3, ... \quad , \frac{dx_{n-1}}{dt} = x_n,$$

and

$$\frac{dx_n}{dt} = \frac{a_{n-1}(t)x_n + a_{n-2}(t)x_{n-1} + ... + a_0(t)x_1}{a_n(t)}.$$

If each of the functions $f_1, ..., f_n$ in (2.1) is a linear function of the dependent variables $x_1, ..., x_n$, then the system of equations is said to be linear.
The most general system of n first-order linear equations has the form

$$\begin{cases} \frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + ... + a_{1n}x_n) + g_1(t) \\ \frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + ... + a_{2n}x_n(t) + g_2(t) \\ \vdots \\ \frac{dx_n(t)}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) + ... + a_{nn}x_n(t) + g_n(t) \end{cases} \tag{2.3}$$

lf each of the functions $g_1, ..., g_n$ is identically zero, then the system (2.3) is said to be homogeneous; otherwise it is nonhomogeneous. In this chapter, we only consider the case where the coefficients a_{ij} do not depend on t.
Now, even the homogeneous linear system with constant coefficients

$$\begin{cases} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n) \\ \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + ..., + a_{nn}x_n \end{cases} \tag{2.4}$$

is quite cumbersome to handle. This is especially true if n is large. There-fore, we seek to write these equations in as concise a manner as possible.
To this end we introduce the concepts of vectors and matrices.

Definition 2.2.1 A vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is a shorthand notation for the sequence of numbers $x_1, ..., x_n$. The numbers $x_1, ..., x_n$ are called the components of x. If $x_1 = x_1(t)$, ... , and $x_n = x_n(t)$, then

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

is called a vector-valued function. Its derivative $\frac{dx(t)}{dt}$ is the vector-valued function

$$x(t) = \begin{pmatrix} \frac{dx_1}{dt}(t) \\ \frac{dx_2}{dt}(t) \\ \vdots \\ \frac{dx_n}{dt}(t) \end{pmatrix}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

is a shorthand notation for the array of numbers a_{ij} arranged in m rows and n columns. The element lying in the i^{th} row and j^{th} column is denoted by a_{ij}, the first subscript identifying its row and the second subscript identifying its column. A is said to be a square matrix if $m = n$.
Next, we define the product of a matrix A with a vector x.

Definition 2.2.2 Let A be an $n \times n$ matrix with elements a_{ij} and let x be a vector with components $x_1, \ldots, x_n$.
We define the product of A with x, denoted by Ax, as the vector whose i^{th} component is $i = 1, 2, \ldots, n$.
In other words, the i^{th} component of Ax is the sum of the product of corresponding terms of the i^{th} row of A with the vector x. Thus,

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix}.$$

Finally, we observe that the left-hand sides of (2.4) are the components of the vector $\frac{dx_j}{dt}$, while the right-hand sides of (2.4) are the components of the vector Ax.
Hence, we can write (2.4) in the concise form

$$\dot{x} = \frac{dx}{dt} = Ax, \quad where \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad and \quad A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}. \tag{2.5}$$

Moreover, if $x_1(t), \ldots, x_n(t)$ satisfy the initial conditions $x_1(t_0) = x_1^0$, $x_2(t_0) = x_2^0$,..., $x_n(t_0) = x_n^0$, then $x(t)$ satisfies the initial-value problem

$$\dot{x} = \frac{dx}{dt} = Ax, x(t_0) = x^0 \quad where \quad x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}. \tag{2.6}$$

Example 2.1 For example, the system of equations

$$\begin{cases} \frac{dx_1}{dt} = 3x_1 - 7x_2 + 9x_3, \\ \frac{dx_2}{dt} = 15x_1 + x_2 - x_3, \\ \frac{dx_3}{dt} = 7x_1 + 6x_2 \end{cases}$$

can be written in the concise form

$$\dot{x} = \begin{pmatrix} 3 & -7 & 9 \\ 15 & 1 & -1 \\ 7 & 6 & 0 \end{pmatrix} x, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and the initial-value problem

$$\begin{cases} \frac{dx_1}{dt} = x_1 - x_2 + x_3, & x_1(0) = 1 \\ \frac{dx_2}{dt} = 3x_1 - x_3, & x_2(0) = 0 \\ \frac{dx_3}{dt} = x_1 + 7x_3, & x_3(0) = -1 \end{cases}$$

can be written in the concise form

$$\dot{x} = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 0 & -1 \\ 1 & 0 & 7 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

■

2.2.1 Applications of linear algebra to differential equations

Theorem 2.2.1 (**Existence-uniqueness theorem**).
There exists one, and only one, solution of the initial-value problem

$$\frac{dx}{dt} = Ax, \quad x(t_0) = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{pmatrix}. \tag{2.7}$$

Moreover, this solution exists for $-\infty < t < +\infty$.

Theorem 2.2.1 is an extremely powerful theorem, and has many implications. In particular, if $x(t)$ is a nontrivial solution, then $x(t) \neq 0$ for any t.
(If $x(t^*) = 0$ for some t^*, then $x(t)$ must be identically zero, since it, and the trivial solution, satisfy the same differential equation and have the same value at $t = t^*$.)

Theorem 2.2.2 (**Test for linear independence**).
Let $x^1, x^2, ..., x^k$ be k solutions of $\dot{x} = Ax$. Select a convenient t_0. Then, $x^l, x^2, ..., x^k$ are linear independent solutions if, and only if, $x^1(t_0), x^2(t_0), ..., x^k(t_0)$ are linearly independent vectors in $\mathbb{R}^n$.

Proof. Suppose that $x^1, x^2, ..., x^k$ are linearly dependent solutions. Then, there exist constants $c_1, c_2, ..., c_k$, not all zero, such that

$$c_1 x^1(t_0) + c_2 x^2(t_0) + ... + c_k x^k(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence $x^1(t_0), x^2(t_0), ..., x^k(t_0)$ are linearly dependent vectors in $\mathbb{R}^n$.
Conversely, suppose that the values of $x^1, x^2, ..., x^k$ at some time t_0 are linearly dependent vectors in $\mathbb{R}^n$. Then, there exist constants $c_l, c_2, ..., c_k$, not all zero, such that

$$c_1 x^1(t_0) + c_2 x^2(t_0) + ... + c_k x^k(t_0) = 0.$$

With this choice of constants $c_l, c_2, ..., c_k$, construct the vector-valued function

$$\phi(t) = c_1 x^1(t) + c_2 x^2(t) + ... + c_k x^k(t).$$

This function satisfies (2.4) since it is a linear combination of solutions.
Moreover, $\phi(t_0) = 0$. Hence, by Theorem 2.2.1, $\phi(t) = 0$ for all t. This implies that $x^1, x^2, ..., x^k$ are linearly dependent solutions. ■

Example 2.2 Consider the system of differential equations

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = -x_1 - 2x_2, \end{cases} \tag{2.8}$$

This system of equations arose from the single second-order equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \tag{2.9}$$

by setting $x_1 = y$ and $x_2 = \frac{dy}{dt}$. Since $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ are two solutions of (2.9), we see that

$$x^1(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad and \quad x^2(t) = \begin{pmatrix} te^{-t} \\ (1-t)e^{-t} \end{pmatrix}$$

are two solutions of (2.8). To determine whether x^1 and x^2 are linearly dependent or linearly independent, we check whether their initial values

$$x^1(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad and \quad x^2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are linearly dependent or linearly independent vectors in $\mathbb{R}^2$. Thus, we consider the equation

$$c_1 x^1(0) + c_2 x^2(0) = \begin{pmatrix} c_1 \\ -c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This equation implies that both c_1 and c_2 are zero. Hence, $x^1(0)$ and $x^2(0)$ are linearly independent vectors in $\mathbb{R}^2$. Consequently, by Theorem 2.2.2, $x^1(t)$ and $x^2(t)$ are linearly independent solutions of (2.8), and every solution $x(t)$ of (2.8) can be written in the form

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} te^{-t} \\ (1-t)e^{-t} \end{pmatrix} = \begin{pmatrix} (c_1 + c_2 t)e^{-t} \\ (c_2 - c_1 - c_2 t)e^{-t} \end{pmatrix}.$$

■

2.2.2 The eigenvalue-eigenvector method of finding solutions

We return to the first-order linear homogeneous differential equation (2.5).
The goal is to find n linearly independent solutions $x^1(t), x^2(t), ..., x^k(t)$. Now, recall that both the first-order and second-order linear homogeneous scalar equations have exponential functions as solutions. This suggests that we try $x(t) = e^{\lambda t}v$, where v is a constant vector, as a solution of (2.5). To this end, observe that

$$\frac{d}{dt}e^{\lambda t}v = \lambda e^{\lambda t}v$$

and

$$A(e^{\lambda t}v) = e^{\lambda t}Av.$$

Hence, $x(t) = e^{\lambda t}v$ is a solution of (2.5) if, and only if, $\lambda e^{\lambda t}v = e^{\lambda t}Av$. Dividing both sides of this equation by $e^{\lambda t}$ gives

$$Av = \lambda v. \tag{2.10}$$

Thus, $x(t) = e^{\lambda t}v$ is a solution of (2.5) if, and only if, λ and v satisfy (2.10).

Definition 2.2.3 *A nonzero vector v satisfying (2.10) is called an eigenvector of A with eigenvalue λ.*

R The vector $v = 0$ is excluded because it is uninteresting. Obviously, $A0 = \lambda\lambda 0$ for any number λ.

An eigenvector of a matrix A is a rather special vector: under the linear transformation $x \to Ax$, it goes into a multiple A of itself. Vectors which are transformed into multiples of themselves play an important role in many applications. To find such vectors, we rewrite Equation (2.10) in the form

$$0 = Av - \lambda v = (A - \mathbf{I})v. \tag{2.11}$$

But, Equation (2.11) has a nonzero solution v only if $det(A - \lambda\mathbf{I}) = 0$. Hence the eigenvalues λ of A are the roots of the equation

$$0 = det(A - \lambda\mathbf{I}) = \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix}$$

and the eigenvectors of A are then the nonzero solutions of the equations $(A - A\mathbf{I})v = 0$, for these values of λ.
The determinant of the matrix $A - A\mathbf{I}$ is clearly a polynomial in A of degree n, with leading term $(-l)^n\lambda^n$. It is customary to call this polynomial the characteristic polynomial of A and to denote it by $p(\lambda)$. For each root λ_j of $p(\lambda)$, that is, for each number λ_j such that $p(\lambda) = 0$, there exists at least one nonzero vector v^j such that $Av^j = \lambda_j v^j$. Now, every polynomial of degree $n \geq 1$ has at least one (possibly complex) root. Therefore, every matrix has at least one eigenvalue, and consequently, at least one eigenvector. On the other band, $p(\lambda)$ has at most n distinct roots. Therefore, every $n \times n$ matrix has at most n eigenvalues. Finally, observe that every $n \times n$ matrix has at most n linearly independent eigenvectors, since the space of all vectors

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

has dimension n.

R Let v be an eigenvector of A with eigenvalue λ. Observe that

$$A(cv) = cAv = C\lambda v = \lambda(cv)$$

for any constant c. Hence, any constant multiple $(c \neq 0)$ of an eigenvector of A is again an eigenvector of A, with the same eigenvalue.

For $_e acheigenvector v^j$ of A with eigenvalue λ_j we have a solution $x^j(t) = e^{\lambda_j t} v^j$ of (2.5). If A has n linearly independent eigenvectors $v^l, ..., y_n$ with eigenvalues $\lambda_1, ..., \lambda_n$ respectively ($\lambda_1, ..., \lambda_n$ need not be distinct), then $x^j(t) = e^{\lambda_j t} v^j$, $j = 1, ..., n$ are n linearly independent solutions of (2.5). This follows immediately from Theorem 2.10 and the fact that $x^j(0) = v^j$. In this case, then, every solution $x(t)$ of (2.5) is of the form

$$x(t) = c_1 e^{\lambda_1 t} v^1 + c_2 e^{\lambda_2 t} v^2 + ... + c_n e^{\lambda_n t} v^n. \tag{2.12}$$

This is sometimes called the "**general solution**" of (2.5).

The situation is simplest when A has n distinct real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ with eigenvectors $v^1, v^2, ..., v^n$ respectively, for in this case we are guaranteed that $v^1, v^2, ..., v^n$ are linearly independent.

Example 2.3 We solve the initial-value problem

$$\dot{x} = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} x = Ax, \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The characteristic polynomial of the matrix A is

$$p(\lambda) = det \begin{pmatrix} 1- & 12 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 36 = (\lambda - 7)(\lambda + 5).$$

Thus, the eigenvalues of A are $\lambda_1 = 7$ and $\lambda_2 = -5$.
(i)$lambda_1 = 7$: We seek a nonzero vector v such that

$$(A - 7\mathbf{I})v = det \begin{pmatrix} -6 & 12 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that $v_1 = 2v^2$. Consequently, every vector

$$v = c \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an eigenvector of A with eigenvalue 7. Therefore,

$$x^1(t) = e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is a solution of the differential equation. (ii) $lambda_1 = -5$: We seek a nonzero vector v such that

$$(A + 5\mathbf{I})v = det \begin{pmatrix} 6 & 12 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that $v_1 = -2v^2$. Consequently, every vector

$$v = c \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

is an eigenvector of A with eigenvalue 7. Therefore,

$$x^2(t) = e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

is a second solution of the differential equation. These solutions are linearly independent since A has distinct eigenvalues. Hence, $x(t) = c_1x^1(t) + c_2x^2(t)$. The constants c_1 and c_2 are determined from the initial condition

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1x^1(0) + c_2x^2(0) = \begin{pmatrix} 2c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -2c_2 \\ c_2 \end{pmatrix}.$$

Thus, $2c_1 - 2c_22 = 0$ and $c_1 + c_2 = 1$.
The solution of these two equations iS $c_1 = C_2 = \frac{1}{2}$. Consequently,

$$X(t) = \frac{1}{2}e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{7t} - e^{-5t} \\ \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t} \end{pmatrix} \blacksquare$$

2.2.3 Fundamental matrix solutions

If $x^1(t), ..., x^n(t)$ are n linearly independent solutions of the differential equation (2.5), then every solution x(t) can be written in the form

$$x(t) = c_1x^1(t) + c_2x^2(t) + ... + c_nx^n(t). \tag{2.13}$$

Let $X(t)$ be the matrix whose columns are $x_1(t), ..., x_n(t)$. Then, Equation (2.13) can be written in the concise form $x(t) = X(t)c$, where

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Definition 2.2.4 A matrix $X(t)$ is called a **fundamental matrix solution** of (2.5) if its columns form a set of n linearly independent solutions of (2.5).

Example 2.4 Find a fundamental matrix solution of the system of differential equations

$$\dot{x} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} x. \tag{2.14}$$

We have

$$e^t\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \quad e^{3t}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad and \quad e^{-2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

are three linearly independent solutions of (2.14). Hence

$$X(t) = \begin{pmatrix} -e^t & e^{3t} & -e^{-2t} \\ 4e^t & 2e^{3t} & e^{-2t} \\ e^t & e^{3t} & e^{-2t} \end{pmatrix}$$

is a fundamental matrix solution of (2.14). ∎

In this subsection we will show that the matrix e^At can be computed directly from any fundamental matrix solution of (2.5). This is rather remarkable since it does not appear possib1e to sum the infinite series

$$[\mathbf{I}+At+\frac{(At)^2}{2!}+...]$$

exactly, for an arbitrary matrix A. Specifically, we have the following theorem.

Theorem 2.2.3 Let $X(t)$ be a fundamental matrix solution of the differential equation $\dot{x}=Ax$. Then,

$$e^{At}=X(t)X^{-1}(0). \tag{2.15}$$

In other words, the product of any fundamental matrix solution of (2.5) with its inverse at $t=0$ must yield e^{At}.

Example 2.5 Find e^{At} if

$$A=\begin{pmatrix}1&1&1\\0&3&2\\0&0&5\end{pmatrix}$$

■

Our first step is to find 3 linearly independent solutions of the differential equation

$$\dot{x}=\begin{pmatrix}1&1&1\\0&3&2\\0&0&5\end{pmatrix}x$$

To this end we compute

$$p(\lambda)=det(A-\lambda\mathbf{I})=det\begin{pmatrix}1-\lambda&1&1\\0&3-\lambda&2\\0&0&5-\lambda\end{pmatrix}=(1-\lambda)(3-\lambda)(5-\lambda).$$

Thus, A has 3 distinct eigenvalues $\lambda=1$, $\lambda=3$, and $\lambda=5$.
(i) $\lambda=1$: Clearly,

$$v^1=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

3 Systems of differential equations is an eigenvector of A with eigenvalue one. Hence

$$x^1(1)=e^t\begin{pmatrix}1\\0\\0\end{pmatrix}$$

is one solution of $\dot{x}=Ax$.
(ii)$\lambda=3$: We seek a nonzero solution of the equation

$$(A-3\mathbf{I})v=\begin{pmatrix}-2&1&1\\0&0&2\\0&0&2\end{pmatrix}\begin{pmatrix}v_1\\v_3\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

This implies that $v_3=0$ and $v_2=2v_1$ Hence,

$$v^2=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

is an eigenvector of A with eigenvalue 3. Consequently,

$$x^2(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

is a second solution of $\dot{x} = Ax$.

(iii)$\lambda = 5$: We seek a nonzero solution of the equation

$$(A - 5\mathbf{I})v = \begin{pmatrix} -4 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This implies that $v_2 = v_3$ and $v_1 = \frac{v_3}{2}$ Hence,

$$v^3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

is an eigenvector of A with eigenvalue 5. Consequently

$$x^3(t) = e^{5t}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

is a third solution of $\dot{x} = Ax$. These solutions are clearly linearly independent. Therefore,

$$X(t) = \begin{pmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}$$

is a fundamental matrix solution.
We compute

$$X^{-1}(0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\exp\left[\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix} t\right] = \begin{pmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} e^t & -\frac{1}{2}(e^t + e^{3t}) & -\frac{1}{2}(e^{3t} + e^{5t}) \\ 0 & e^{3t} & -e^{3t} + e^{5t} \\ 0 & 0 & e^{5t} \end{pmatrix}.$$

2.2.4 The nonhomogeneous equation

Consider now the nonhomogeneous equation $\dot{x} = Ax + f(t)$. In this case, we can use our knowledge of the solutions of the homogeneous equation (2.5).
to help us find the solution of the initial-value problem

$$\dot{x} = Ax + f(t), \quad x(t_0) = x^0. \tag{2.16}$$

Let $x^1(t), ..., x^n(t)$ be n linearly independent solutions of the homogeneous equation (2.5).
Since the general solution of (2.5) is $c_1x^1(t) + ... + c_nx^n(t)$, it is natural to seek a solution of (2.16) of the form

$$x(t) = u_1x^1(t) + u_2x^2(t) + ... + u_nx^n(t). \tag{2.17}$$

This equation can be written concisely in the form $x(t) = X(t)u(t)$ where

$$X(t) = (x^1(t), \ldots, x^n(t)) \quad \textit{and} \quad u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}.$$

Plugging this expression into the differential equation $\dot{x} = Ax + f(t)$ gives

$$\dot{X}(t)u(t) + X(t)\dot{u}(t) = AX(t)u(t) + f(t). \tag{2.18}$$

The matrix $X(t)$ is a fundamental matrix solution of (2.5). Hence, $\dot{X}(t) = AX(t)$, and Equation (2.18) reduces to

$$X(t)\dot{u}(t) = f(t). \tag{2.19}$$

Recall that the columns of $X(t)$ are linearly independent vectors of $\mathbb{R}^n$ at every time t. Hence $X^1(t)$ exists, and

$$\dot{u}(t) = X^{-1}(t)f(t). \tag{2.20}$$

Integrating this expression between t_0 and t gives

$$u(t) = u(t_0) + \int_{t_0}^{t} X^{-1}(s)f(s)ds$$

$$u(t) = X^{-1}(t_0)x^0 + \int_{t_0}^{t} X^{-1}(s)f(s)ds.$$

Consequently,

$$x(t) = X(t)X^{-1}(t_0)x^0 + X(t)\int_{t_0}^{t} X^{-1}(s)f(s)ds. \tag{2.21}$$

If $X(t)$ is the fundamental matrix solution e^{At}, then Equation (2.21) simplifies considerably. To wit, if $X(t) = e^{At}$, then $X^{-1}(s) = e^{-As}$. Hence

$$x(t) = e^{At}e^{-At_0}X^0 + e^{At}\int_{t_0}^{t} e^{-As}f(s)ds$$

$$x(t) = e^{A(t-t_0)}X^0 + \int_{t_0}^{t} e^{A(t-s)}f(s)ds. \tag{2.22}$$

Example 2.6 Find all solutions of the differential equation

$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{ct}, \quad c \neg 1.$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$$

We "guess" a particular solution $\psi(t)$ of the form $\psi(t) = be^{ct}$. Plugging this expression into (**??**) gives

$$cbe^{ct} = Abe^{ct} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{ct},$$

or

$$(A - c\mathbf{I})b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

This implies that

$$b = \frac{-1}{1-c} = \begin{pmatrix} 1 \\ \frac{2(c-4)}{4+(1-c)^2} \\ \frac{1+3c}{4+(1-c)^2} \end{pmatrix}$$

Hence, every solution $x(t)$ of (**??**) is of the form

$$x(t) = e^t = \left[c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} - \frac{-e^{ct}}{1-c} = \begin{pmatrix} 1 \\ \frac{2(c-4)}{4+(1-c)^2} \\ \frac{1+3c}{4+(1-c)^2} \end{pmatrix} \right]$$

■

R We run into trouble when $c = 1$ because one is an eigenvalue of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$$

More generally, the differential equation $\dot{x} = Ax + ve^{ct}$ may not have a solution of the form be^{ct} if c is an eigenvalue of A.
In this case we have to guess a particular solution of the form

$$\psi(t) = e^{ct}[b_0 + b_1 t + \ldots + b_{k-1}t^{k-1}]$$

for some appropriate integer k.

2.3 Exercises

Exercise 2.1 Convert the given differential equation for the single variable y into a system of first-order equations.

1. $\frac{d^3y}{dt^3} + \cos y = e^t$;
2. $\frac{d^4y}{dt^4} + \frac{d^2y}{dt^2} = 1$.

Exercise 2.2 (i) Let $y(t)$ be a solution of the equation $y'' + y' + y = 0$.
Show that

$$x(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of the system of equations

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x.$$

(ii) Let

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

be a solution of the system of equations

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x.$$

Show that $y = x_1(t)$ is a solution of the equation $y'' + y' + y = 0$.

Exercise 2.3 For each of the differential equations

1.
$$\dot{x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} x, \quad x^1(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}, \quad x^2(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix},$$

2.
$$\dot{x} = \begin{pmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{pmatrix} x, \quad x^1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix}, \quad x^2(t) = \begin{pmatrix} 0 \\ e^{4t} \\ e^{4t} \end{pmatrix}, \quad x^3(t) = \begin{pmatrix} e^{6t} \\ 0 \\ e^{6t} \end{pmatrix}$$

determine whether the given solutions are a basis for the set of all solutions.

Exercise 2.4 Find all solutions of the equation

$$\dot{x} = Ax = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix} x.$$

Exercise 2.5 In each of problems 1-3 find all solutions of the given differential equation.

1. $\dot{x} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} x;$
2. $\dot{x} = \begin{pmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{pmatrix} x;$
3. $\dot{x} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} x.$

Exercise 2.6 In each of problems 1-3 solve the given initial-value problem.

1. $\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix};,$
2. $\dot{x} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix};$
3. $\dot{x} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$

Exercise 2.7 (a) Show that $e^{\lambda(t-t)}v$, t_0 constant, is a solution of $x = Ax$ if $Av = \lambda v$.

(b) Solve the initial-value problem

$$\dot{x} = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix} x, \quad x(1) = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}.$$

3. Introduction to partial differential equations

Introduction

Up to this point, the differential equations that we have studied have all been relations involving one or more functions of a single variable, and their derivatives. In this sense, these differential equations are **ordinary** differential equations. On the other hand, many important problems in applied mathematics give rise to **partial** differential equations (PDE). A partial differential equation is a relation involving one or more functions of **several** variables, and their partial derivatives.

3.1 Notations and generalities

We collect most parts of the notation and some important results from linear algebra, analysis, vector and matrixs, necessary for numerical mathematics.

3.1.1 Domains

We consider open, bounded domains $\Omega \in \mathbb{R}^n$ where $n = 1,2,3$ is the dimension. The boundary is denoted by $\partial\Omega$. We assume that Ω is sufficiently smooth (i.e., a Lipschitz domain or domain with Lipschitz boundary) such that the normal n can be defined. What also works for most are convex, polyhedral domains with finite corners.

3.1.2 Independent variables

A point in $\mathbb{R}^n$ is denoted by

$$x = (x_1, ..., x_n).$$

The variable for 'time' is denoted by t.

3.1.3 Function

We start with

Definition 3.1.1 Let x and y be variable. If x is given and there is a unique associated y, then y is a function and we write

$$y = f(x).$$

The variable x is the 'independent' variable or also called the argument. All x values form the set of definition of $f(x)$. The variable y is called 'dependent'. All y values form the image space of $f(x)$.

In these notes, functions are often denoted by

$$u := u(x)$$

if they only depend on the spatial variable $x = (x_1, ..., x_n)$. If they depend on time and space, they are denoted by

$$u = u(t, x).$$

A PDE is an equation whose unknown is a function and which relates the function to its partial derivatives. Typically, the function u sought is defined on an open Ω of $\mathbb{R}^n$ and has values in $\mathbb{R}^m$, Ω being assumed non-empty and $n \geq 2$; the case $n = 1$ is treated by the theory of ordinary differential equations. A PDE is therefore of the form

$$\forall x \in \Omega, \quad F(x, u(x), Du(x), ..., D^k u(x)) = 0. \tag{3.1}$$

Since F has values in $\mathbb{R}^m$ (we generally ask for as many equations as unknowns).
If $k = 1$, it's a scalar equation.
If $k \geq 2$, it's called a PDE system.
Let $\Omega \subset \mathbb{R}^n$ be an open. For a function $u \in \mathscr{C}^1(\Omega)$ we use the following notations for the partial for the partial derivative in $x \in \Omega$ with respect to x_i:

$$\frac{\partial u(x)}{\partial x_i} = u_{x_i}(x).$$

3.1.4 Partial derivatives

We frequently use:

$$\frac{\partial u}{\partial x} = u_x$$

and

$$\frac{\partial u}{\partial t} = u_t$$

and

$$\frac{\partial^2 u}{\partial x^2} = u_x x$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = u_{xy}.$$

3.1.5 Multiindex notation

For higher-order partial derivatives of a function $u \in \mathscr{C}^k(\Omega)$ we use multi-indices.
For a general description of ODEs and PDEs the multiindex notation is commonly used.

- A multiindex is a vector $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$. The order is

$$|\alpha| := \sum_{i=1}^{n} \alpha_i$$

and

$$\alpha! = \alpha_1 !...\alpha_n !.$$

- For a given multiindex we define the partial derivative:

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x).$$

- The integer k is called **order** of the PDE (k is the maximum order involved in the equation in a non-trivial way).

Example 3.1 Let the problem dimension $n = 3$. Then, $\alpha = (\alpha_1, \alpha_{2,\alpha_3})$. For instance, let $\alpha = (2,0,1)$.
Then $|\alpha| = 3$ and $D^\alpha u = \frac{\partial^3 u(x)}{\partial t^2 \partial x_1} = u_{ttx_1}$. ∎

3.2 Classification

Definition 3.2.1 The partial differential equation 3.1 **linear** if it has the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

for given function a_α ($|\alpha| \leq k$), f. This linear PDE is **homogeneous** if $f \equiv 0$.

A system of differential equations is, informally speaking, a collection of several PDE for several unknown functions.

Definition 3.2.2 The **order** of a PDE is the highest order of the partial derivatives involved in the PDE.

Example 3.2
1. (E_1) : $x^3 \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 4$: PDE of 2^{nd} order;
2. (E_2) : $\frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + u = 0$: PDE of 1^{st} order;
3. (E_3) : $(\frac{\partial^4 u}{\partial x^3 \partial y^1})^3 + x^5 y^2 = \text{x}$: PDE of 4^{th} order.

The PDE (E_1) is linear, but the PDEs (E_2) and (E_3) are not linear. ∎

Here we are supposing that the system comprises the same number m of scalar equations as unknowns $(u^1, ..., u^m)$. This is the most common circumstance, although other systems may have fewer or more equations than unknowns.

R We use "PDE" as an abbreviation for both "partial differential equation" and "partial differential equations".

3.2.1 Some classic models

There is no general theory available that addresses the solvability of all partial differential equations (PDE). Such a theory is highly unlikely to exist, given the vast range of physical, geometric, and probabilistic phenomena that can be modeled by PDE. Instead, research tends to focus on specific PDE that are significant for various applications both within mathematics and beyond. The hope is that understanding the origins and context of these equations can provide valuable insights into their solutions.

Below is a list of several key PDEs that are of interest in current research. This list is intended to introduce the reader to the names and forms of several well-known PDEs. To highlight their mathematical structure most clearly, we have generally set the relevant physical constants to unity.

1. **Laplace**'s equation
$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0.$$
2. **Helmholtz**'s (or eigenvalue) equation
$$-\Delta u = \lambda u$$
3. **Linear transport** equation
$$u_t + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$
4. **Liouvilles**'s equation
$$u_t - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$
5. **Heat** (or diffusion) equation
$$u_t - \Delta u = 0.$$
6. **Schrdinger**'s equation
$$iu_t + \Delta u = 0.$$
7. **Kolmogorov**'s equation
$$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$
8. **Wave** equation
$$u_{tt} - \Delta u = 0.$$

3.2.2 Boundary and initial value problems

The general solutions of PDE are usually not unique. We impose some additional conditions such as a domain, boundary values, or initial values to give solutions some additional nice properties.

Definition 3.2.3 A PDE is **well-posed** if it satisfies:
1. **Existence**: There is at least one solution $u(x)$ satisfying the PDE and the additional conditions.
2. **Unique**: There is at most one solution $u(x)$ satisfying the PDE and the additional conditions.
3. **Stability**: The solution depends continuously on the initial data. This means that if the conditions are changed a little, the corresponding solution changes only a little.

Example 3.3 The following second order PDE
$$u_{xy} + x = 0 \quad for \quad x, y > 0,$$

with boundary conditions

$$u|_{y=0} = 0, u|_{x=0} = 0$$

has the particular solution

$$u(x,y) = -\frac{yx^2}{2}.$$

If we give too few constraints, then we might not get a unique solution. If we give too many constraints, then a solution may fail to exist. ∎

Definition 3.2.4 We have the following terminology for these different types of constraints:
1. **Initial Value Problem** (IVP): A constraint on the time variable is put at $t = 0$,
e.g. $u|_{t=0} = f(x)$.

2. **Boundary Value Problem** (BVP): A constraint on the spacial variable is put on the boundary of the domain Ω,
e.g. $u|_{\partial\Omega} = f(x)$.
3. **Initial Boundary Value Problem** (IBVP): A constraint on both the time and space variables.

Temperature in a room

Let us compute the heat distribution in our room b302. The room volume is Ω. The window wall is a Dirichlet boundary $\partial_D\Omega$ and the remaining walls are Neumann boundaries $\partial_N\Omega$. Let K be the air viscosity.
We consider the heat equation: Find $T : \Omega \times I \to \mathbb{R}$ such that

$$\partial_t T + (v.\nabla)T - \nabla.(K\nabla T) = f \quad \Omega \times I,$$

$$T = 18^oC \quad \partial_D\Omega \times I,$$

$$K\nabla T.n = 0 \quad \partial_N\Omega \times I,$$

$$T(0) = 15^oC \quad \Omega \times \{0\}.$$

The homogeneous Neumann condition means that there is no heat exchange on the respective walls (thus neighboring rooms will have the same room temperature on the respective walls). The nonhomogeneous Dirichlet condition states that there is a given temperature of 18^oC, which is constant in time and space (but this condition may be also non-constant in time and space). Possible heaters in the room can be modeled via the right hand side f. The vector $v :\to \mathbb{R}^3$ denotes a given flow field yielding a convection of the heat, for instance wind. We can assume $v \approx 0$. Then the above equation is reduced to the original heat equation: $\partial_t T - \nabla.(K\nabla T) = f$.

3.2.3 Concepts in numerical mathematics

In introductory classes to numerical methods, we deal with concepts that are very characteristic for numerical modeling. We summarized them into seven points:
1. **Approximation**: since analytical solutions are not possible to achieve as we just learned in the previous section, solutions are obtained by numerical approximations.
2. **Convergence**: is a qualitative expression that tells us when members an of a sequence $(a_n)_{n\in\mathbb{N}}$ are sufficiently close to a limit a. In numerical mathematics this limit is often the solution that we are looking for.
3. **Order of convergence**: While in analysis, we are often interested in the convergence itself, in numerical mathematics we must pay attention how long it takes until a numerical solution has sufficient accuracy.

4. **Errors**: Numerical mathematics can be considered as the branch 'mathematics of errors'. What does this mean? Numerical modeling is not wrong, inexact or non-precise! Since we cut sequences after a final number of steps or accept sufficiently accurate solutions obtained from our software, we need to say how well the (unknown) exact solution by this numerical solution is approximated. In other words, we need to determine the error, which can arise in various forms.
5. **Error estimation:** This is one of the biggest branches in numerical mathematics. We need to derive error formulae to judge the outcome of our numerical simulations and to measure the difference of the numerical solution and the (unknown) exact solution in a certain norm.
6. **Efficiency:** In general we can say, the higher the convergence order of an algorithm is, the more efficient our algorithm is. Therefore, we obtain faster the numerical solution to a given problem. But numerical efficiency is not automatically related to resource-effective computing.
7. **Stability**: Despite being the last concept, in most developments, this is the very first step to check. How robust is our algorithm against different model and physical parameters? Is the algorithm stable with respect to different input data?

3.3 Exercises

Exercise 3.1 Let $x, y \in R$, $t \in R^+$ and c be a non-zero constant.
For each of the following equations, indicate its order, its linearity and whether it is homogeneous and there name:

- $u_t(t,x) + cu_x(t,x) = 0$;
- $u_t(t,x) + u(t,x)u_x(t,x) = 0$;
- $u_{tt}(t,x) = u_{xx}(t,x)$;
- $\Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = 0$.

Exercise 3.2 Let u be a fixed function of space variables.

1. Answer true or false:
   The characteristic curves of the EDP
   $$\frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = c, \quad c \in \mathbb{R}$$
   sont $\begin{cases} x^2 + y^2 = k, \\ x^2 - y^2 = k, \end{cases}$ where k is a constant
2. Choosing the right response, the EDP
   $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + u = 0$$
   is:
   - parabolic;
   - hyperbolic;
   - elliptic.
3. Give two methods of solving second-order linear partial differential equations, specifying the principle of each.

Justify your answers to the first two questions.

Exercise 3.3 Let $c \in \mathbb{R}$ be a constant. Consider the partial differential equation:

$$u_t + cu_x = 0 \tag{3.2}$$

1. Show that $(x,t) = f(x - ct)$ is a solution of the PDE (7.16).
2. What is the initial data $u|_{t=0} = u(.,0)$? How can we deduce u(., t) as the solution at time t from the initial data?
3. What happens when $c = 0$ (resp. $c > 0$, $c < 0$)?

4. Linear PDE of first and second order

Introduction

4.1 First-order linear partial differential equations

In the case of two variables, a PDE of **order** 1 is written:

$$\forall (x,y) \in \mathbb{R}^2, \;\; F(x,y,u(x,y),u_x(x,y),u_y(x,y)) = 0. \tag{4.1}$$

More generally, we can consider equations involving derivatives $\partial_x^{m_i}\partial_y^{n_i}u$. The order of a PDE is then the highest order of derivation $m_i + n_i$ which appears in the equation.

Here are a few examples of two-variable PDE, which are very simple at first sight. Some of these PDE model the evolution over time of certain systems, and it is It is customary to use the notation t for the time variable.

4.1.1 Equations with constant coefficients

We will solve PDE of the form

$$au_t(t,x) + bu_x(u,t) = 0. \tag{4.2}$$

where a and b are two real constants, at least one of which is non-zero.
As we saw in the first chapter, it is important to specify what we mean by 'solve'. We are looking here for all the functions u defined on $\Omega \subseteq \mathbb{R}^2$ of class $\mathscr{C}^1$ and such that for all (x,t) the equality (4.2) is verified. Let's start by examining the case of the equation ($a = 1$ and $b = 0$)

$$u_t(t,x) = 0$$

We can see immediately that u is a solution if and only if u does not depend on t. In other words, the solutions are the functions u that can be written as

$$u(t,x) = f(x)$$

for a function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^1$.
The first thing to note is that there are many solutions!
There is also a more geometrical remark: the solutions $(t,x) \to u(t,x)$ are exactly those functions which are constant along the horizontal lines of the plane (Otx), i.e. along the lines directed by the vector $(a,b) = (1,0)$. This phenomenon also occurs for all equations (4.2), and this is what we are going to demonstrate.

4.1.2 Initial conditions and boundary conditions

How many solutions does a partial differential equation have? In general, lots. Even ordinary differential equations have infinitely many solutions. Indeed, the general solution to a single n^{th} order ordinary differential equation depends on n arbitrary constants. The solutions to partial differential equations are yet more numerous, in that they depend on arbitrary functions. Very roughly, we can expect the solution to an n^{th} order partial differential equation involving m independent variables to depend on n arbitrary functions of $m-1$ variables. But this must be taken with a large grain of salt - only in a few special instances will we actually be able to express the solution in terms of arbitrary functions.
The solutions to dynamical ordinary differential equations are singled out by the imposition of initial conditions, resulting in an **initial value problem**. On the other hand, equations modeling equilibrium phenomena require boundary conditions to specify their solutions uniquely, resulting in a **boundary value problem**.

4.1.3 Notion of a well-posed problem

The above examples show that the number of solutions to a PDE can be very large. Recall the case of homogeneous linear differential equations with constant coefficients. For the equation

$$a_n u^{(n)}(x) + a_{n-1} u^{(n-1)}(x) + \ldots + a_1 u'(x) + a_0 u(x) = 0, \tag{4.3}$$

The general solution depends on n constants (n is the order of the equation). We obtain a unique solution when we fix n conditions conditions of the type

$$\begin{cases} u(0) = y_0, \\ u'(0) = y_1, \\ \vdots \\ u^{(n-1)}(0) = y_{n-1}, \quad . \end{cases} \tag{4.4}$$

where $y_0, y_1, ..., y_{n-1}$ are n fixed real numbers. The problem of solving the equation (4.3) under the condition (4.4) is called the Cauchy problem. The three previous examples are homogeneous linear PDEs of order 2, and their general solution depends on two arbitrary functions - instead of two constants for ODEs. The only point to remember is that the set of solutions of a PDE can be difficult to describe. However, when PDEs are derived from the modelling of a real-world phenomenon, the interesting solutions are those that satisfy certain additional conditions. Let's take an example. We want to describe the vertical vibrations of a string of length L, stretched between two fixed points A and B. Let $u(t,x)$ be the height at time t of the point on the string at distance x from A. Clearly, the only functions $u(t,x)$ we are interested in are those for which

$$\forall t, u(t,A) = u(t,B) = 0.$$

This type of condition is called "**condition at the edge**", but there are many other kinds of constraints that we come across very often, for example:

- **Conditions of regularity**: The solutions must be sufficiently differentiable, at least for the equation to make sense.It is in particular this kind of condition that is missing for the equation (5.8) to make precise sense.
- **Initial conditions**: We know the state of the system we want to describe at time $t = 0$ and we need to describe its evolution over time.
- Infinite behaviour conditions.
- Stationary conditions.

It is then possible that the problem considered "PDE + physical condition(s)" admits a single solution. When, moreover, the solution depends "continuously" on the physical data, in the sense that a small error in the data does not change the solution very much, we speak of a well-posed problem. The solution, we speak of a well-posed problem. Of course, all this will have to be defined more precisely mathematically.

4.2 Second-order linear partial differential equations

In the case of two variables, a PDE of **order** 2 equation is written as

$$\forall (x,y) \in \mathbb{R}^2, \ F(x,y,u(x,y),u_x(x,y),u_y(x,y),u_{xx}(x,y),u_{xy}(x,y),u_{yy}(x,y)) = 0. \tag{4.5}$$

Here are a few examples of two-variable PDE, which are very simple at first sight. Some of these PDE model the evolution over time of certain systems, and it is It is customary to use the notation t for the time variable.

4.2.1 Classification of linear second order partial differential equations

Let $u = u(x,y)$, and consider the second order PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = g(x,y) \tag{4.6}$$

where the coefficients $A, B, ..., F, g$ are functions of x, y.
There are 3 types of second order PDE:

1. **Elliptic**: If $B^2 - 4AC < 0$, then (4.6) can be reduced to the form
   $u_{xx} + u_{yy}$ + lower order terms = 0.

   Extends the derivation of the equation for heat flow in $1-D$, along a rod to $2-D$ (a slab of uniform thickness) and to $3-D$ objects. Finite-difference quotients are used to approximate

the derivatives, allowing one to set up a system of equations whose solution is the steady-state temperatures within the object. Ways to solve the equations more economically are described. Another form of elliptic equation, called Poisson's equation, is employed to find a quantity related to the torsion within a rod when subjected to a twisting force.

2. **Hyperbolic**: If $B^2 - 4AC > 0$, then (4.6) can be reduced to the form
   $u_{xx} - u_{yy}$ + lower order terms = 0.
   Begins with the derivation of the equation for determining the lateral displacements of a vibrating string. The equation is solved through finite difference approximations for the derivatives. Remarkably, the solution is found to match exactly to the analytical solution. Unfortunately, this is found to be not true for a vibrating drum head.
3. **Parabolic**: If $B^2 - 4AC = 0$, then (4.6) can be reduced to the form
   u_{xx}+ lower order terms = 0.
   Discusses how temperatures vary with time when heat flows along a rod (1-D) or within a slab (2-D) after deriving the equations for these cases.
   Beginning with a method that is not very accurate, it progresses to a better technique and then generalizes the procedure to show how these are related.

Example 4.1 The linear second order PDE

$$4u_{xx} - 6u_{xy} + 9u_{yy} + u_x + u_y + u = 0$$

is elliptic because $(-6)^2 - 4(4 \times 9) < 0$.

■

We focus on second-order equations in two variables, such as the wave equation

$$u_{xx} - \frac{1}{c^2}u_{tt} = f(x,t), c \neq 0 \quad (Hyperbolic)$$

Laplace or Poisson's equation

$$u_{xx} + u_{xx} = f(x,t) \quad (Elliptic)$$

or Fourier's heat equation

$$u_{xx} - \kappa u_t = f(x,t), \kappa \neq 0 \quad (Parabolic).$$

R The naming of these second order equations coincide with the conic sections. Consider the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \quad a,b,...,f \in \mathbb{R}.$$

The type of equation can be analyzed by looking at the discriminant $\Delta = b^2 - 4ac$:

$$(1)\ Elliptic: \ \Delta < 0 \quad (2)\ Hyperbolic: \ \Delta > 0 \quad (3)\ Parabolic: \ \Delta = 0.$$

4.2.2 Types of boundary conditions

The following are some of the types of boundary conditions.

Definition 4.2.1 Three well-known boundary conditions are
a) **Dirichlet boundary conditions**: The value of the dependent variable is specified on the boundary.
$u(0,t) = g(t)$ and $u(l,t) = h(t)$, where $0 < x < l$,
is used when u is given at the boundary.
b) **Neumann boundary conditions**: The normal derivative of the dependent variable is specified

on the boundary.
The Neumann boundary conditions with the form
$u_x(0,t) = f_1(t)$ and $u_x(l,t) = f_2(t)$, where $0 < x < l$,
is used when $\frac{\partial u}{\partial n}$ is given at the boundary.
c)**Robin boundary conditions**: Both the value and the normal derivative of the dependent variable are specified on the boundary.
Less commonly met are Robin boundary conditions, where the value of a linear combination of the dependent variable and the normal derivative of the dependent variable is specified on the boundary.

- For $1D$ problems, this term refers to the extremes of the interval.
- For $2D$ problems, it refers to the contour of the domain.
- For $3D$ problems, it refers to the boundary surfaces.

Each class of PDE's requires a different class of boundary conditions in order to have a unique, stable solution.

4.3 Exercises

Exercise 4.2 Let $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ a function with real variables $(x,y) \in D$ de classe $\mathscr{C}^2$. Let the partial differential equation:

$$\frac{\partial^2}{\partial x \partial y} u(x,y) = 0 \tag{4.7}$$

1. Give the order and type of the equation (4.7)
2. Show that the function u satisfies the equation (4.7) if only if $\exists f, g : \mathbb{R} \to \mathbb{R}$ class $\mathscr{C}^1$ such as

$$u(x,y) = f(x) + g(y)$$

Exercise 4.3 Choose the right answer, The PDE

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + u = 0$$

is:

- parabolic;
- hyperbolic;
- elliptic.

Exercise 4.4 Let u a fixed function of the space variables.

1. Give the general form of a linear PDE of order 2 in two variables.
2. Give the formula for the heat equation.
3. Indicate the order, the operator associated with the equation, whether or not it is linear and whether or not it is homogeneous for the partial differential equation below.

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^4 u}{\partial x \partial y^2 \partial z} + e^x = 0.$$

Exercise 4.5 Determine for which values (x,y) the following second-order linear equation is hyperbolic, parabolic and elliptic:

$$\frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} - xy \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = 0.$$

Exercise 4.6 Let $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ a function of class $\mathscr{C}^2$.
Using polar coordinates, determine the functions $f : \mathbb{R}_+^* \times \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^1$, solutions of the partial differential equation:

$$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 0.$$

5. Classical Solutions

Let us focus our attention on a single differential equation involving a single, scalar valued function u that depends on one or more independent variables. The function u is usually real-valued, although complex-valued functions can, and do, play a role in the analysis. Everything that we say in this section will, when suitably adapted, apply to systems of differential equations.
By a **solution** we mean a sufficiently smooth function u of the independent variables that satisfies the differential equation at every point of its domain of definition. We do not necessarily require that the solution be defined for all possible values of the independent variables. Indeed, usually the differential equation is imposed on some domain D contained in the space of independent variables, and we seek a solution defined only on D. In general, the domain D will be an open subset, usually connected and, particularly in equilibrium equations, often bounded, with a reasonably nice boundary, denoted by ∂D.
We will call a function smooth if it can be differentiated sufficiently often, at least so that all of the derivatives appearing in the equation are well defined on the domain of interest D. More specifically, if the differential equation has order n, then we require that the solution u be of class $\mathscr{C}^n$, which means that it and all its derivatives of order $\leq n$ are continuous functions in D, and such that the differential equation that relates the derivatives of u holds throughout D. Smooth solutions are often referred to as classic solutions.

5.1 First-order linear partial differential equations

5.1.1 Equations with constant coefficients

On will solve PDE of the form

$$au_t(t,x) + bu_x(u,t) = 0. \tag{5.1}$$

where a and b are two real constants, at least one of which is non-zero.
As we saw in the first chapter, it is important to specify what we mean by "solve". Here we are looking for all the functions u defined on $\Omega \subseteq \mathbb{R}^2$ of class $\mathscr{C}^1$ and such that for all $(x,t) \in \Omega$ the

equality (5.1) is verified.
Let's start by examining the case of the equation ($a = 1$ et $b = 0$)

$$u_t(t,x) = 0$$

We can see immediately that u is a solution if and only if u does not depend on t. In other words, the solutions are the functions u which can be written as

$$u(t,x) = f(x)$$

for a function $f : \mathbb{R} \to \mathbb{R}$ de classe $\mathscr{C}^1$.
The first thing to note is that there are lots of solutions!
There is also a more geometric observation to be made: the solutions $(t,x) \to u(t,x)$ are exactly those functions which are constant along the horizontal lines of the plane (Otx), i.e. along the lines directed by the vector $(a,b) = (1,0)$. This phenomenon also occurs for all equations (5.1), and this is what we are going to demonstrate.

5.1.2 Characteristics method

Let's go back to equation (5.1). Assume that $u : \mathbb{R}^2 \to \mathbb{R}$, of class $\mathscr{C}^1$, is a solution. In terms of the differential, (5.1) is given by

$$\forall (t,x) \in \mathbb{R}^2, d_{(t,x)}u(a,b) = 0. \tag{5.2}$$

In other words, the directional derivative $\partial_{(a,b)}u(t,x)$ of u in the direction of the vector (a,b) is zero at any point (t,x) of $\mathbb{R}^2$. We then have the following proposition.

Proposition 5.1.1 If u is a solution of (5.1), then u is constant along each line of direction (a,b).

Definition 5.1.1 The features of equation (5.1) are the lines with direction (a,b). These are all the lines $\mathscr{D}_c$ of equation $bt - ax = c$, where c traverses the set of real numbers.

Let us now denote $f : \mathbb{R} \to \mathbb{R}$ the function which associates a real c with the value of u on the straight line $\mathscr{D}_c$. Let (t_0,x_0) be a point in $\mathbb{R}^2$. There is one and only one characteristic which passes through (t_0,x_0): this is the line $\mathscr{D}_{c_0}$, where $c_0 = bt_0 - ax_0$. So we have

$$u(t_0,x_0) = f(c_0) = f(bt_0 - ax_0).$$

Since this reasoning is valid for all (t_0,x_0) of $\mathbb{R}^2$, we finally have

$$\forall (t,x) \in \mathbb{R}^2, u(t,x) = f(bt - ax). \tag{5.3}$$

Note in passing that, since u is $\mathscr{C}^1$, so is f. So far, we have reasoned by necessary condition. It remains to prove that any function u of the form (5.3) with $f : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^1$ is indeed a solution of (5.1). We then proved the following result.

Theorem 5.1.2 The functions $u : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathscr{C}^1$ which satisfy the equation (5.1) are all functions written as follows

$$u(t,x) = f(bt - ax)$$

for a certain function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^1$.

5.1.3 Change of variables method

We're going to find the previous result using another method, which turns out to be very practical. Rather than a completely different method, it's another formulation of the same idea. We have seen that the solutions to the equation (5.1) depend only on the variable $bt-ax$. So we set $t' = bt-ax$ and choose another independent coordinate x'. For example

$$\begin{cases} t' = bt-ax, \\ x' = bt+ax, \end{cases}$$

We then pose $v : (x',t') \to v(t',x') = u(t,x)$, and examine the equation verified by v when u is a solution of (5.1). First we calculate the partial derivatives of u as a function of those of v.

$$\begin{cases} u_t(t,x) = v_t(bt-ax,bt+ax), \\ u_x(t,x) = v_x(bt-ax,bt+ax), \end{cases}$$

which gives

$$\begin{cases} u_t(t,x) = bv_1(bt-ax,bt+ax)+av_2(bt-ax,bt+ax), \\ u_x(t,x) = -av_1(bt-ax,bt+ax)+bv_2(bt-ax,bt+ax). \end{cases}$$

So u, of class $\mathscr{C}^1$, is a solution of the equation (5.1) if and only if v satisfies the equation

$$(a^2+b^2)v_2(t',x') = 0.$$

. In other words, since $a^2+b^2 \neq 0$, u is a solution of equation (5.1) if and only if v does not depend on $x' : v(t',x') = f(t')$ for a certain function f, of class $\mathbb{C}^1$ since v is. Returning to u, we find Theorem (5.1.2)

$$u(t,x) = f(bt-ax).$$

5.1.4 Equations with variable coefficients

5.1.5 Vector fields

Consider, for example, the equation

$$u_x(x,y)+x\,u_y(x,y) = 0 \tag{5.4}$$

We want to apply the method of characteristics to this equation.
Let's read the equation: the directional derivative of u in the direction of the vector $X = (1,x)$ must be zero:

$$d_{(1,x)}u(x,y) = 0.$$

Obviously, the difficulty that arises is that the vector in question depends on the point (x,y) where we are. We need to adapt the notion of characteristic a little.

Definition 5.1.2 We call a vector field a (regular) application X of $\Omega \subset \mathbb{R}^2$, considered as a subset of the points of the plane, in $\mathbb{R}^2$ considered as a set of the vectors of the plane (i.e. the vector space $\mathbb{R}^2$).

Here regular means of class $\mathscr{C}^0$ or of class $\mathscr{C}^1$. For example, the equation (5.4) naturally leads us to consider the vector field $X(x,y) = (1,x)$. Another example is the vector field *grad* V, where V is a regular function defined on Ω:

$$gradV(x,y) = (V_x(x,y),V_y(x,y)).$$

For example, take

$$\Omega = \mathbb{R}^2 - \{(0,0)\}$$

and

$$V(x,y) = \frac{1}{\sqrt{x^2+y^2}}.$$

. The vector field is called central because it is parallel, at the point (x,y) to the vector (x,y).

Definition 5.1.3 Let X be a regular vector field on an open Ω of $\mathbb{R}^2$. An integral curve of X is a parametric curve $\gamma : I \subset \mathbb{R} \to \Omega$ such that, for all $t \in I$,

$$\gamma'(t) = X(\gamma(t)).$$

The integral curves of the vector field $X(x,y) = (1,x)$ are called the characteristics of the equation (5.4) (there is a little ambiguity here because integral curves are parametric curves!). This definition is motivated by the following proposition.

Proposition 5.1.3 If u is a solution of the equation (5.4), u is constant along the integral curves $t \to \gamma(t)$ of the field X:

$$\frac{d}{dt}(u(\gamma(t)) = 0.$$

This proposition generalises what we have seen in the case of equations with constant coefficients: in this case, the vector field X associated with the equation is the constant field $X(x,y) = (a,b)$, whose integral curves are the straight lines of direction (a,b).

5.1.6 A Cauchy problem for the equation

As an example, we will solve a Cauchy problem associated with the equation (5.4):

$$\begin{cases} u_x(x,y) + x\, u_y(x,y) = 0 \\ u(0,y) = \phi(y), \end{cases} \tag{5.5}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a given $\mathscr{C}^1$ function. We start by looking for the characteristic curves of the equation. These are the integral curves $t \to \gamma(t) = (x(t), y(t))$ of the vector field $X(x,y) = (1,x)$. By definition we have

$$\begin{cases} x'(t) = 1, \\ y'(t) = x(t), \end{cases}$$

which gives

$$\begin{cases} x(t) = t + x_0, \\ y(t) = \frac{t^2}{2} x_0 t + y_0, \end{cases}$$

where (x_0, y_0) is the point on γ corresponding to $t = 0$. If we prefer a Cartesian equation, we can see that the γ curve which passes through (x_0, y_0) (there is one and only one...), has the equation

$$\gamma : y = x^2/2 + + y_0 - x_0^2/2.$$

. We now want to determine the value of the solution of the Cauchy problem (5.5) at the point (x_0, y_0). We know that u is constant along the integral curve that passes through the point (x_0, y_0). This curve intersects the y-axis at the point $(x_1 = 0, y_1 = y_0 - x_0^2/2)$, and we know that

$$u(x_1, y_1) = u(0, y_1) = y_1 = y_0 - x_0^2/2$$

. We therefore obtain, for any (x_0, y_0) of $\mathbb{R}^2$,

$$u(x_,, y_0) = \phi(y_0 - x_0^2/2).$$

Reasoning by necessary conditions, we obtain

$$u(x, y) = \phi(y - x^2/2).$$

It's very easy to check that this function is indeed a solution of (5.5), and the method of characteristics has again enabled us to construct the only solution to this problem.

5.1.7 Solutions of some special PDE

As we pointed out in the introduction, it is generally unrealistic to want to know explicitly the solution(s) of a PDE. However, this is sometimes possible in special cases: here are three a priori very simple examples.

Example 1

Let $u : \mathbb{R}^2 \to \mathbb{R} \in \mathscr{C}^2$ and the PDE

$$u_{xx}(x, y) = 0. \tag{5.6}$$

We have $u_{xx}(x, y) = \frac{\partial^2}{\partial x^2} u(x, y) = \frac{\partial}{\partial x}(\frac{\partial}{\partial x} u(x, y))$. The equation (5.6) therefore means that the partial derivative with respect to the first variable of the partial derivative of u with respect to the first variable is zero: $\frac{\partial}{\partial x}(\frac{\partial}{\partial x} u(x, y))$. So let's start by positing $v(x, y) = \frac{\partial}{\partial x} u(x, y)$. For all $(x, y) we must have \in \mathbb{R}$.

$$\frac{\partial}{\partial x} v(x, y) = 0.$$

For any fixed y the partial application $x \to v(x, y)$ must therefore be constant. Of course this constant can depend on y. We ca

$$v(x, y) = C(y).$$

for a certain function C. We are brought back to the following problem: find u such that

$$\frac{\partial}{\partial x} u(x, y) = C(y).$$

Reasoning in the same way, we see that necessarily,

$$u(x, y) = C(y)x + D(y)$$

where D is again a certain function. Finally, it's easy to check that any function of this type satisfies equation (5.6), as long as it has partial derivatives. Note now that there are a huge number of solutions to equation (5.6), since no conditions on the functions C and D have appeared in the demonstration.

Example 2

We want to solve the equation

$$u_{xx}(x,y) + u(x,y) = 0. \tag{5.7}$$

Let's assume the variable y, and let $v(x) = u(x,y)$. We need to solve the differential equation

$$v'' + v = 0.$$

The solutions are

$$v(x) = A(y)\cos x + B(y)\sin s,$$

and returning to u, we obtain

$$u(x,y) = A(y)\cos x + B(y)\sin s,$$

where A and B are any two functions.

Example 3

We now turn to the equation

$$u_{xy}(x,y) = 0. \tag{5.8}$$

We shall see that the notation u_{xy} can be interpreted in two different - and both reasonable - ways, leading to different sets of solutions. This is of course very annoying, and we'll soon see how to remedy this kind of ambiguity.

Let's start by considering that u_{xy}. Equation (5.8) first gives

$$u_y(x,y) = C(y)$$

, where C is any function of y, then

$$u(x,y) = \int_{y_0}^{y} C(s)ds + D(x).$$

Note that the function D is arbitrary, but that C must have a primitive. In particular, the function $u(x,y)$ found is differentiable with respect to y. Now suppose that, following another convention, u_{xy} denotes $\partial_y(\partial_x u)$. We then find $\partial_x\, u(x,y) = E(x)$, then

$$u(x,y) = \int_{x_0}^{x} E(s)ds + F(y).$$

This time the function found is derivable by (x,y).

In other words, the set of solutions to e depends on the interpretation of the equation. Note that the difficulty disappears if we restrict ourselves to finding fairly regular solutions.

5.2 EDP du second ordre

5.2.1 Heat equation

We now describe the phenomenon of diffusion, such as heat diffusion through a body or the diffusion of a dye in a still liquid. In both cases, the underlying physical principle is the same and is called Fourier's law (or Fick's law): heat flows from the hot region to the cold region, and its intensity is proportional to the temperature gradient. Moreover, heat can only be lost through the walls of the container. We limit ourselves here to heat diffusion along a single axis. The temperature at position x at time t is denoted by $u(t,x)$.

$$u_t(t,x) - c\,u_{xx}(t,x) = 0, c > 0 \tag{5.9}$$

for $0 < x < L$ and $t > 0$ with additional conditions:

- initial condition:

$$u(0,x) = f(x); \tag{5.10}$$

- boundary conditions:

$$u(t,0) = u(t,L) = 0. \tag{5.11}$$

5.2.2 Solving the Heat Equation

In order to solve these equations for unsteady-state heat flow (and they apply as well to diffusion or to any problem where the potential is proportional to the gradient), we need to make the solution agree with specified conditions along the boundary of the region of interest. In addition, because the problems are time dependent, we must begin with specified initial conditions (at $t = 0$) at all points within the region. We might think of these problems as both boundary-value problems with respect to the space variables and as initial-value problems with respect to time.
We begin with the 1-D case,(but we can extend the treatment to 2-D and 3-D). For 1-D, we think of heat flowing along a rod. (If the temperatures do reach a steady state) No heat leaves or enters the rod through its circumference (it may be insulated) but flows only along the rod

5.2.3 Variable separation method

The equation (5.9) and the boundary conditions (5.11) are linear and homogeneous. The **separation of variables** then consists of looking for u in the form

$$u(t,x) = \varphi(t)\psi(x).$$

Then the equation (5.9) becomes

$$\varphi'(t)\psi(x) = c\varphi(t)\psi''(x) \Leftrightarrow \frac{1}{c}\frac{\varphi'(t)}{\varphi(t)} = \frac{\psi''(x)}{\psi(x)} = -\lambda$$

(is any constant)

Once again, the variables are separable because they are on either side of the equality. Once again, we introduce the separation constant λ which must be defined.
We then have to solve the following problems:

⋆ **1 eigenvalue problem (Sturm-Liouville) on the variable** x (because the homogeneous boundary conditions are on this variable):

$$\begin{cases} \psi''(x) + \lambda\psi(x) = 0, \\ \psi(0) = \psi(L) = 0, \end{cases}$$

where we need to find all the eigenvalues λ_n and the eigenfunctions $\psi_n(x)$.

⋆ **1 differential equation of** 1^{er} **order on the variable t**:

$$\varphi'(t) + \lambda_n\varphi(t) = 0.$$

We start by solving the eigenvalue problem. We study all the possibilities for the separation constant *lambda*:

1. **Solving the eigenvalue problem**:

$$\psi'' + \lambda\psi = 0$$

has the characteristic equation

$$r^2 + \lambda = 0$$

so

$$\Delta = -4\lambda.$$

- 1^{st} case:$\Delta = 0 \Rightarrow \lambda = 0$
  The solution is written:
  $$\psi(x) = Ax + B$$
  and $\psi(0) = 0 \Rightarrow B = 0$ et $\psi(L) = 0 \Rightarrow A = 0$ As a result, $\lambda = 0$ is not an eigenvalue of the problem.
- 2^{nd} cas:$\Delta > 0 \Rightarrow \lambda < 0$
  We pose
  $$\lambda = -\alpha^2, \ \alpha \neq 0$$
  let
  $$\Delta = 4\alpha^2 \Rightarrow r_1 = \alpha \ et \ r_2 = -\alpha$$
  The solution is written as:
  $$\varphi(x) = Ae^{\alpha x} + Be^{-\alpha x}$$
  with
  $$\varphi(0) = 0 \Rightarrow A + B = 0$$
  $$\varphi(L) = 0 \Rightarrow A + B = 0$$
  As a result, $\lambda < 0$ are not eigenvalues.

- 3^{rd} case:$\Delta < 0 \Rightarrow \lambda > 0$

  We pose

  $$\lambda = \beta^2, \ \alpha \neq 0$$

  let

  $$\Delta = -4\alpha^2 \Rightarrow r_1 = i\beta \ \ et \ \ r_2 = -i\beta$$

  The solution is written:

  $$\varphi(x) = A\cos(\beta x) + B\sin(\beta x)$$

  $$\varphi(0) = 0 \Rightarrow A = 0$$

  $$\varphi(L) = 0 \Rightarrow B\sin(\beta L) = 0$$

  A non-trivial solution to the equation $(B \neq 0)$ corresponds to:

  $$\beta = \frac{n\pi}{L}, \ n = 1,2,3,...$$

  The outcome,

  $$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

  are the eigenvalues of the boundary problem.

  The eigenfunctions associated with the eigenvalues λ_n can be written as follows:

  $$\varphi(x) = B_n \sin\frac{n\pi}{L}x\Big).$$

2. Now that we know the values of λ (eigenvalues), we can solve for the differential equation in t:

   $$\psi'(t) + \lambda_n c\psi(t) = 0.$$

   For $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, we get:

   $$\psi'(t) + \left(\frac{n\pi}{L}\right)^2 c\psi(t) = 0 \Rightarrow \psi(t) = A_n e^{-c(\frac{n\pi}{L})^2 t}$$

   where A_n is an arbitrary constant.

3. **General solution to the PDE:**

   The general solution of the partial differential equation is the superposition of the set of solutions (all the eigenvalues must be considered):

   $$u(x,t) = \sum_{i=1}^{\infty} \varphi_n(x)\psi_n(t)$$

   or

   $$u(x,t) = \sum_{i=1}^{\infty} A_n e^{-c(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

4. **Special solution**: The general solution must satisfy the initial condition:

$$T(x;0) = f(x) = \begin{cases} x, & 0 < x < L/2; \\ L-x, & L/2 < x < L. \end{cases}$$

From the general solution, we obtain:

$$f(x) = \sum_{i=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right).$$

It therefore remains to decompose $f(x)$ on the basis of the eigenfunctions by calculating the coefficient A_n:

$$A_n = \frac{2}{L}\int_0^{L/2} x \sin\left(\frac{n\pi}{L}x\right) dx + \frac{2}{L}\int_{L/2}^{L} (L-x)\sin\left(\frac{n\pi}{L}x\right) dx$$

Developing the calculations gives us:

$$A_n = \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right).$$

The temperature inside the bar changes according to the following law:

$$u(x,t) = \sum_{i=1}^{\infty} \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) e^{-c(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

Example 5.1 At time $t = O$, the temperature $u(x,0)$ in a thin copper rod ($\alpha^2 = 1.14$) of length one is $2\sin 3\pi x + 5\sin 8\pi x, 0 \leq x \leq l$. The ends of the rod are packed in ice, so as to maintain them at $0°C$. Find the temperature $u(x,t)$ in the rod at any time $t > 0$.
The temperature $u(x,t)$ satisfies the boundary-value problem

$$\begin{cases} u_t = u = 1.14u_{xx}, & 0 < x < 1, t > 0; \\ u(x,0) = 2\sin 3\pi x + 5\sin 8\pi x, & 0 < x < 1; \\ u(0,t) = u(1,t) \end{cases}$$

and this implies that

$$u(x,t = 2\sin 3\pi x e^{-9(1.14)\pi^2 t} + 5\sin 8\pi x e^{-64(1.14)\pi^2 t}.$$

■

5.3 Exercises

Exercise 5.1 Using a change of variables:

$$\begin{cases} u = \frac{x+y}{2} & ; \\ v = \frac{x-y}{2} & . \end{cases}$$

Determine the functions $f : (x,y) \mapsto f(x,y)$ of class $\mathscr{C}^2$ on $\mathbb{R}^2$ solutions of the PDE:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}.$$

Exercise 5.2 Solve in $\mathbb{R}^2$ the problem

$$\begin{cases} 4u_t(t,x) - 3u_x(t,x) = 0, \\ u(0,x) = x^3. \end{cases}$$

Exercise 5.3 Solve in $\mathbb{R}^2$ the problem

$$\begin{cases} 3u_t(t,x) + 5u_x(t,x) = 0, \\ u(t,0) = t^2. \end{cases}$$

Exercise 5.4 Solve in $\mathbb{R}^2$ the problem

$$\begin{cases} 2u_t(t,x) + 3u_x(t,x) = 0, \\ u(0,x) = sin(x). \end{cases}$$

Exercise 5.5 Consider the vector field $X(x,y) = (1,-xy)$.

1. Determine and plot its integral curves;
2. Show that the solutions (of class $\mathscr{C}^1$) of equation

   $$u_x(x,y) - xyu_y(x,y) = 0$$

   are necessarily written $u(x,y) = f(ye^{x^2/2})$ for a certain function f of class $\mathscr{C}^1$. all the solutions to Cauchy's problem.

   $$\begin{cases} u_x(x,y) - xyu_y(x,y) = 0, \\ u(0,y) = y^2. \end{cases}$$

3. The Cauchy problem

   $$\begin{cases} u_x(x,y) - xyu_y(x,y) = 0, \\ u(x,0) = x^2, \end{cases}$$

   are there any solutions?

Exercise 5.6 Consider the partial differential equation

$$u_t + (x^2+1)u_x = 0, \quad u(x,0) = f(x), \quad x \in \mathbb{R}, t \geq 0. \tag{5.12}$$

1. Determine the characteristic curves of equation 5.12.
2. Deduce the solution $u(x,t)$ of the PDE 5.12 obtained. Is it defined $\forall (x,t) \in \mathbb{R} \times [0,+\infty[$? Specify its domain of definition in $\mathbb{R}times[0,+\infty[$.

Exercise 5.7 A rope is stretched between its two ends A and B (let $AB = L$). It is moved away from its position and then let go. It then begins to vibrate in a plane passing through A and B. Let's give this plane with an orthonormal reference $(A,\vec{i},\vec{j})$. We assume that the point M_x of the chord with initial abscissa x $(0 \leq x \leq L)$ retains the same abscissa at all times t. Let u the function from $[0,L[\times[0,+\infty[$ in $\mathbb{R}$, which, to any pair (x,t), associates the ordinate of the point M_x at time t. We then show that u satisfies the vibrating string equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

(where c is a real non-zero constant depending on the characteristics of the string), with boundary conditions:

$$\forall t \geq 0 : u(0,t) = u(L,t) = 0,$$

which reflect the fact that the ends are fixed.

1. Write the boundary problem checked by u.
   We propose to determine the particular non-zero solutions of this problem which are of the form of separate variables", i.e. of the form

   $$u(x,t) = f(x)g(t),$$

   where f and g are functions of class $\mathscr{C}^2$ defined respectively on $[0,L[$ and $[0,+\infty[$ and with values real values.
2. Show that there is a constant λ such that

   $$\begin{cases} f'' = \lambda f & ; \\ g'' = \lambda c^2 g & . \end{cases}$$

3. Show successively that λ is non-zero, then that it cannot be positive, and that its only are of the form $-\frac{k^2\pi^2}{L^2}$ where $k \in \mathbb{N}^*$.
4. Deduce that the particular solutions sought are functions u of the form

   $$u(x,t) = A sin\left(\frac{k\pi}{L}x\right) sin\left(\frac{k\pi}{L}c(t-t_0)\right),$$

   where $k \in \mathbb{N}^*$, t_0 et A are real.

Exercise 5.8 Consider the partial derivative problem:

$$\begin{cases} \frac{\partial^2}{u}\partial t^2 = x^2\frac{\partial^2}{u}\partial x^2 + 3x\frac{\partial}{u}\partial x & x \in]1,e[, t \in \mathbb{R}^+; \quad (E) \\ u(1,t) = u(e,t) = 0, & t > 0; \\ u(x,0) = f(x), & 1 < x < e. \end{cases}$$

where e designates the base of the natural logarithm ($\ln e = 1$) and f is a continuous function on $]1,e[$.

1. Let

   $$u(x,t) = X(x)T(t)$$

   in (E).
   Show that for this separate form to satisfy (1), there must exist a real constant λ such that X is a solution to the following differential equation:

   $$\frac{x^2X'' + 3xX'}{X} = \lambda \tag{5.13}$$

2. Using the boundary conditions, give the values of $X(1)$ and $X(e)$.

3. To solve the differential equation (5.13), we use the change of variable $x = e^z$ and pose $Z(z) = X(x)$. Show that the equation becomes:

$$Z''(z) + 2Z'(z) - \lambda Z(z) = 0 \tag{5.14}$$

4. Give the values of $Z(0)$ and $Z(1)$.
5. (a) In the case where $1 + \lambda > 0$, let $1 + \lambda = \omega^2$, ωin R^+. Give the general form of the solutions of (5.14). Is this case compatible with the boundary conditions?

   (b) In the case where $1 + \lambda = 0$, give the general form of the solutions of (5.14). Is this case compatible with the boundary conditions?

   (c) Dans le cas où $1 + \lambda < 0$, on pose $1 + \lambda = -\omega^2$, $(\omega \in \mathbb{R}^+)$.

   (i) Give the general form of the solutions of (5.14);

   (ii) Can this case be compatible with the boundary conditions?

   Show then that $\omega = k\pi$, $k \in \mathbb{N}$, and give, for an integer k, the corresponding expression, denoted $Z_k(z)$ of $Z(z)$. , for an integer k, the corresponding expression for $X(x)$, denoted $X_k(x)$.

   What property does the family (X_k) satisfy?
6. Show that the associated (T_k) functions are solutions of decoupled differential equations:

$$T_k'' - \lambda_k T_k = 0. \tag{5.15}$$

   Give the expression for T_k.
7. Using the initial conditions, show that:

$$u(x,t) = \sum_{k=0}^{+\infty} \frac{c_k}{x} \sin(k\pi \ln x) \sin(\sqrt{1 + k^2\pi^2}t)$$

   where C_k are real constants to be determined.

II Numerical analysis of partial differential equations

6. Finite Difference solution of partial differential eq

Introduction

Finite difference methods are popular numerical techniques for solving science and engineering problems modeled by differential equations. The earliest application of the finite difference methods to option valuation was performed by Brennan and Schwartz (1978). Tavella and Randall (2000) presented a comprehensive survey of finite difference methods applied to numerical pricing of financial instruments. In the construction of finite difference schemes, we approximate the differential opera- tors in the governing differential equation of the option model by appropriate finite difference operators, hence the name of this approach.

To go from an exact continuous problem governed by a PDE to an approximate discrete problem, there are three main families of methods:

- **Finite differences**:

  The method consists in replacing partial derivatives by divided differences or combinations of point values of the function at a finite number of discrete points or mesh nodes.

  Advantages: easy to write and low computational cost.

  Disadvantages: limited to simple geometries, difficult to take into account Neumann-type boundary conditions.

- **Finite volumes**:

  This method integrates the equations written in the form of conservation laws onto elementary volumes of simple form. It thus naturally provides conservative discrete approximations. It is easy to implement with rectangular elementary volumes.

  Advantages: allows complex geometries to be treated with volumes of any shape, more natural determination of Neumann-type boundary conditions.

Disadvantage: few theoretical convergence results.

- **Finite elements**:

  The method consists in approximating, in a finite-dimensional subspace, a problem written in variational form (as energy minimization in general) in an infinite-dimensional space. In this case, the approximated solution is a function determined by a finite number of parameters. number of parameters such as, for example, its values at certain mesh points or nodes.
  Advantages: can handle complex geometries, numerous theoretical results on convergence.
  Disadvantages: complexity of implementation and high cost in terms of computing time and memory.

6.1 Fine-difference approach

Introduction

This is a solution method (ODE, PDE) generally of 2nd order, which transforms the initial boundary problem by replacing the derivatives (partial or otherwise) of the function to be sought by discrete difference approximations.

6.1.1 Principle - order of precision

The finite-difference method (FDM) consists in approximating the derivatives of equations by means of Taylor developments, and can be deduced directly from the definition of the derivative. It is due to the work of several 18th-century mathematicians (Euler, Taylor, Leibniz...). Let $u(x,y,t)$ be a function of space and time at least of class $\mathscr{C}^2(D)$. By definition of the derivative, we have:

$$u_x = \frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y,t)-u(x,y,t)}{h}.$$

If h is small, a **Taylor expansion** of $u(x,y,t)$ in the neighborhood of x yields:

$$u(x+h,y,t) = u(x,y,t) + h\frac{\partial u}{\partial x}(x,y,t) + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}(x,y,t) + \ldots$$

Truncating the series to first order at h, we obtain:

$$\frac{u(x+h,y,t)-u(x,y,t)}{h} = \frac{\partial u}{\partial x}(x,y,t) + \mathscr{O}(h).$$

The approximation of the derivative u_x is then of order 1, indicating that the **troncatures** error $\mathscr{O}(h)$ tends towards zero as the prime power of h.

The power of h with which the truncation error tends to zero is called the **order of the method**.

6.2 Fine-difference approach in dimension 1

Let's consider a one-dimensional case where we wish to determine a quantity $u(x)$ on the interval $[0,1]$. The search for a discrete solution of the quantity u leads to the creation of a mesh of the interval of definition. Consider a mesh (or calculation grid) composed of $N+1$ points x_i for

$i = 0, ..., N$ regularly spaced with a step h. The points $x_i = ih$ are called mesh nodes.
The initial continuous problem of determining a quantity on an infinite-dimensional set thus boils down to finding N discrete values of this quantity at the various nodes of the mesh.
Notation: Note u_i the discrete value of $u(x)$ at the point x_i, i.e. $u_i = u(x_i)$. Similarly for the derivative of $u(x)$ at node x_i, we note

$$u'_i = \left(\frac{\partial u}{x}\right)_{x=x_i} = \left(\frac{\partial u}{x}\right)_i.$$

This notation is used equivalently for all successive-order derivatives of the quantity u.
The 1st-order finite-difference scheme presented above can be written in index notation:

$$u'_i = \frac{u_{i+1} - u_i}{h} + \mathcal{O}(h).$$

This scheme is called "forward" or "off-center forward" or upwind or "progressive finite difference(PFD)". It is also possible to construct another 1st-order scheme, called "backward" or "regressive finite difference(RFD)":

$$u'_i = \frac{u_i - u_{i-1}}{h} + \mathcal{O}(h).$$

6.2.1 Higher-order scheme

Higher-order finite-difference schemes can be constructed by manipulating Taylor expansion in the neighborhood of x_i. We write:

$$u_{i+1} = u(x_i + h) = u_i + hu'_i + \frac{h^2}{2}u''_i + \mathcal{O}(\langle^\ni) \tag{6.1}$$

$$u_{i-1} = u(x_i - h) = u_i - hu'_i + \frac{h^2}{2}u''_i + \mathcal{O}(\langle^\ni) \tag{6.2}$$

Taking the difference between these two relationships yields:

$$u_{i+1} - u_{i-1} = 2hu'_i + \mathcal{O}(h^3).$$

This leads to the second-order scheme, known as the "centered" or "symmetric finite difference (SFD)" method, for approximating the first derivative of u:

$$u'_i = \frac{u_{i+1} - u_{i-1}}{2h} + \mathcal{O}(h^2).$$

6.2.2 Higher-order derivative

The principle is identical and is based on Taylor expansions in the neighborhood of x_i. For example, to construct a scheme for approximating the second derivative of u, we sum the two equalities 6.1 and 6.2 to arrive at:

$$u_{i+1} + u_{i-1} - 2u_i = h^2\mathcal{O}(h^4).$$

$$u_i'' = \frac{\partial^2}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{6.3}$$

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1. There is also a "forward" and "backward" formulation for the second derivative, both of order 1;
2. it is also possible to construct, by the same procedure, higher-order finite-difference schemes for the second, third, etc. derivatives.

Example 6.1 Consider the differential equation for the following **Dirichlet** boundary value problem:

$$(P) \begin{cases} -u''(x) = f(x), & x \in int([0,1]); \\ u(0) = \alpha, u(1) = \beta & . \end{cases}$$

where f is a continuous function.

The mesh is constructed by introducing $N+1$ nodes x_i with $i = 0, 1, ..., N$, regularly evenly spaced with a step h. The quantity u_i will denote the value of the function $u(x)$ at node x_i.

The equation to be solved is written in discrete form at each node x_i:

$$-u_i'' = f(x_i) = f_i$$

Approximating the second derivative of u by means of a scheme centered at order 2, using the 6.3 approximation, the discretized equation is as follows:

$$\frac{2u_i - u_{i+1} - u_{i-1}}{h^2} = f_i \ , pour \ \ i \ \ variant \ \ de \ \ 1 \quad N-1. \tag{6.4}$$

It's very convenient to use a matrix formulation, showing the vector of discrete unknowns:

$$\frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 + \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-2} \\ f_{N-2} + \frac{\beta}{h^2} \end{pmatrix}$$

■

Example 6.2 Consider the differential equation for the following **mixed (Dirichlet-Neumann)** boundary value problem:

$$(P) \begin{cases} -u''(x) = f(x), & x \in int([0,1]); \\ u(0) = \alpha, u'(1) = \beta. \end{cases}$$

where this time we have a Neumann condition in $x = 1$.

The modifications to the discretized problem compared with the previous case are as follows. Firstly, the number of unknowns has changed. There is an unknown at the edge at $x = 1$. On the

basis of the same mesh as before, the discrete problem now has N unknowns u_i for i varying from 1 to N.

On the other hand, we need to discretize the Neumann condition $u'(1) = \beta$. Several choices are possible to approximate this first derivative. This is one of the drawbacks of the finite-difference method: it doesn't naturally give a good approximation of the Neumann conditions.

In our case, let's use a first-order approximation: $u'(1) = \frac{u_N - u_{N-1}}{h}$.

In matrix form, we obtain:

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} f_1 + \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-2} \\ f_{N-1} \\ \frac{\beta}{h^2} \end{pmatrix}$$

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Exercise 6.1 Develop an algorithm for this method (SFD).

6.3 Finite difference approximations in two space dimensions

Consider the stationary two-dimensional problem of heat conduction in a rectangular rectangular domain $[0, L_x] \times [0, L_y]$. The temperature field $T(x,y)$ satisfies Laplace's equation:

$$(P) \begin{cases} \Delta T = T_{xx} + T_{yy} = 0, & (x,y) \in [0,L_x] \times [0,L_y]; \\ T(0,y) = T_g etT(L_x,y) = T_d, & 0 < y < L_y; \\ T(x,0) = T_b etT(x,L_y) = T_h, & 0 < x < L_x. \end{cases}$$

The computational domain is discretized into $(N+1) \times (P+1)$ nodes (x_i, y_j) (i varying from 0 to N and j varying from 0 to P). The space steps in each direction h and k are assumed to be constant. The discrete temperature at node (x_i, y_j) will be denoted

$$T_{ij} = T(x_i, y_j).$$

We use a centered scheme of order 2 to approximate the second derivatives in space:

$$(T_{xx})_{ij} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{h^2}, \tag{6.5}$$

$$(T_{yy})_{ij} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{k^2}. \tag{6.6}$$

The discretized formulation is then, for i varying from 1 to $N-1$ and j varying from 1 to $P-1$:

$$k^2(T_{i+1,j} + T_{i-1,j}) + h^2(T_{i,j+1} + T_{i,j-1}) - 2(h^2 + k^2)T_{i,j} = 0.$$

Consider the matrix form for $N = P = 4$, where $\delta = h^2 + k^2$:

$$\begin{pmatrix} -2\delta & k^2 & 0 & h^2 & 0 & 0 & 0 & 0 & 0 \\ k^2 & -2\delta & k^2 & 0 & h^2 & 0 & 0 & 0 & 0 \\ 0 & k^2 & -2\delta & k^2 & 0 & h^2 & 0 & 0 & 0 \\ h^2 & 0 & 0 & -2\delta & k^2 & 0 & h^2 & 0 & 0 \\ 0 & h^2 & 0 & k^2 & -2\delta & k^2 & 0 & h^2 & 0 \\ 0 & 0 & h^2 & 0 & k^2 & -2\delta & 0 & 0 & h^2 \\ 0 & 0 & 0 & h^2 & 0 & 0 & -2\delta & k^2 & 0 \\ 0 & 0 & 0 & 0 & h^2 & 0 & k^2 & -2\delta & k^2 \\ 0 & 0 & 0 & 0 & 0 & h^2 & 0 & k^2 & -2\delta \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{pmatrix} = - \begin{pmatrix} h^2 T_b + k^2 T_g \\ h^2 T_b \\ h^2 T_b + k^2 T_d \\ k^2 T_g \\ 0 \\ k^2 T_d \\ h^2 T_h + k^2 T_g \\ h^2 T_h \\ h^2 T_h + k^2 T_d \end{pmatrix}$$

The resulting matrix is tridiagonal and each of its blocks is tridiagonal. The system can be solved

by a matrix **Thomas** method or a matrix iterative method (**Gauss-Seidel** method).

6.3.1 Algorithme de Thomas matriciel (Adapted Gauss)

This algorithm is used to solve a system with a block tridiagonal matrix involving a vector of discrete unknowns X_i, of the form:

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i, \;\; i = 1, ..., N-1$$

where A_i, B_i, C_i are matrices and D_i is a vector.

In matrix form:

$$\frac{1}{h^2} \begin{pmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & B_{N-2} & C_{N-1} \\ 0 & 0 & \cdots & A_{N-2} & B_{N-1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-2} \\ X_{N-1} \end{pmatrix} = \begin{pmatrix} D_1 + A_1 X_0 \\ D_2 \\ \vdots \\ D_{N-2} \\ D_{N-1} + C_{N-1} X_N \end{pmatrix}$$

We introduce the matrix α_i and the vector β_i evaluated by the following recurrence relations:

$$\alpha_i = (B_i + \alpha_{i+1} C_i)^{-1} A_i$$

$$\beta_i = (B_i + \alpha_{i+1} C_i)^{-1} \times (D_i + \beta_{i+1} C_i)$$

for i varying from $N-1$ to 1 with $\alpha_N = 0$ and $\beta_N = X_N$ (where X_N expresses a boundary condition).

The second step determines the unknowns, for i varying from 1 to $N-1$:

$$X_i = \alpha_i X_{i-1} + \beta_i.$$

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With a Neumann condition on one of the edges of the domain, for example at $y = 0$, a heat flux equal to ϕ_b would have to be added to the previous formulation. a heat flux equal to ϕ_b, we would have to add to the previous formulation the discretization of this condition at the edge. this condition at the edge.
This results in the addition of $N-1$ additional unknowns, namely the temperature values at the edge ($j = 0$ and i varying from 1 to $N-1$).

Exercise 6.2 (i) Calculate $u_{i+1,j}$ et $u_{i-1,j}$;
(ii) Write the TSE in the neighborhood of point (i, j) considering $u_{i+1,j+1}$, $u_{i+1,j-1}$, $u_{i-1,j-1}$ et $u_{i-1,j+1}$;
(iii) Demonstrate that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(u_{i+1,j+1} + u_{i-1,j-1}) - (u_{i+1,j-1} + u_{i-1,j+1})}{4hk}.$$

6.4 Consistency, convergence and stability

6.4.1 Consistency

Let u be a PDE verifying a boundary problem with equation (E).
(E) is always written in the form

$$Au = f$$

where A is a partial derivative operator, written at node (i, j) as follows

$$(Au)_{i,j} = f_{i,j}.$$

During discretization, at node (i, j) of the mesh, (E) is transformed into a discretization scheme (DS) noted (E_d) which is written as follows

$$(\tilde{A}u)_{i,j} = f_{i,j}.$$

We note

$$[R(u)]_{i,j} = [(A - \tilde{A})(u)]_{i,j}.$$

R

$R(u)]_{i,j}$ depends on:Discretization step (h and k for $n = 2$); how A is approached by $\tilde{A}$ (SDF, RDF or PDF).

If

$$\lim_{(h,k)\to(0,0)} [R(u)]_{i,j} = 0,$$

we'll say that the DS of $tildeA$ is **consistent** (or **cohesive**) with A and (E_d) is consistent with (E) at each node (i, j).

General procedure for demonstrating DS consistency:

1. Assuming u of sufficiently differential class, we carry out the Taylor development (TD) in the vicinity of the node (i,j), of each of the terms of $(\tilde{A}u)_{i,j}$;
2. Write a sequence $[(A-\tilde{A})(u)]_{i,j}$ taking these TD into account;
3. Simplify, and make the steps tend towards 0;
4. Conclude on the consistency of DS.

Example 6.3 Let

$$(E): \quad \underbrace{(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2})}_{=A}u=0, \tag{6.7}$$

$$(E_d): \quad \frac{u_{i,j+1}-u_{i,j}}{k}-\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}=0. \tag{6.8}$$

TD:

$$u_{i,j+1}=u_{i,j}+k\frac{\partial u_{i,j}}{\partial t}+\frac{k^2}{2!}\frac{\partial^2 u_{i,j}}{\partial t^2}+\frac{k^3}{3!}\frac{\partial^3 u_{i,j}}{\partial t^3}+\mathcal{O}(k^4),$$

$$u_{i+1,j}=u_{i,j}+h\frac{\partial u_{i,j}}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{h^3}{3!}\frac{\partial^3 u_{i,j}}{\partial x^3}+\mathcal{O}(h^4),$$

$$u_{i-1,j}=u_{i,j}-h\frac{\partial u_{i,j}}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}-\frac{h^3}{3!}\frac{\partial^3 u_{i,j}}{\partial x^3}+\mathcal{O}(h^4).$$

Replace in equation (6.8), we obtain

$$(6.8)\Leftrightarrow \frac{\partial u_{i,j}}{\partial t}-\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{k}{2}\frac{\partial^2 u_{i,j}}{\partial t^2}+\mathcal{O}(h^2)+\mathcal{O}(k^2)=0.$$

$$\lim_{(h,k)\to(0,0)}[R(u)]_{i,j}\approx\lim_{(h,k)\to(0,0)}\frac{k}{2}\frac{\partial^2 u_{i,j}}{\partial t^2}=0,$$

so $\tilde{A}$ is consistent with A.

If we change $u_{i-1,j}$ by $u_{i-1,j+1}$ in (6.8) we find:

$$u_{i-1,j+}=u_{i,j}-h\frac{\partial u_{i,j}}{\partial x}+k\frac{\partial u_{i,j}}{\partial t}+\frac{h^2}{2!}\frac{\partial^2 u_{i,j}}{\partial x^2}+\frac{k^2}{2!}\frac{\partial^2 u_{i,j}}{\partial t^2}-hk\frac{\partial^2 u_{i,j}}{\partial x\partial t}+\mathcal{O}(h^s,k^l),\quad s\wedge l\geq 3.$$

0n will

$$\lim_{(h,k)\to(0,0)}[R(u)]_{i,j}=\infty,$$

DS is not consistent with the original equation. ∎

R Consistency is a necessary condition for convergence.

6.4.2 Stability

We'll say that the proposed SD is **stable** if:
At each node (i, j) inside the mesh we

$$\lim_{(h,k)\to(0,0)} [R(u)]_{i,j} = 0$$

(the ideal), but in reality it remains bounded at each node.

6.4.3 Convergence

The notions of consistency and stability do not involve (directly) the exact solution u of (E) and that of u_d given by the scheme (E_d).
We'll say that the (E_d) schema is **convergent** if:
At each node (i, j) of the mesh, we have $(u_d)_{i,j}$ tends towards $(u)_{i,j}$ when the discretization steps tend towards 0.

u exact solution of the PDE	u_d exact solution of the SD	$\hat{u_d}$ approximate solution
$Au = f$ continuous case	$(\tilde{A}u)_{i,j} = f_{i,j}$ discretized case	numerical solution of DS (on computer)

6.4.4 Relation between the three notions

Let (P) be a well-posed PDE problem (the solution exists and is unique and depends continuously on L. C). Let A be the DP operator associated with (P).
Let $tildeA$ be the PD operator associated with DS, and hence the following theorem:

Theorem 6.4.1 (Lax-Milgram equivalence theorem)
$\tilde{A}$ **consistent** and **stable** if only $\tilde{A}$ **convergent**.

Conclusion:
Before starting to work out the numerical results of a PDE problem, we need to make sure that the chosen DS is consistent and stable (i.e. convergent).

6.5 Exercises

Exercise 6.3 A homogeneous metal bar $I = [0,2]$ is at time $t = 0$, at temperature $f = f(x)$ continuous over I such that $f(0) = f(2) = 0$. Both ends of the bar are constantly held at zero temperature for $t > 0$. Assuming $\forall (x,t) \in]0,2[\times]0,+\infty[$, u (assumed sufficiently differentiable) is a solution of the PDE

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}.$$

1. Write the boundary problem verified by u;
2. Using the variable separation method $(u(x,t) = g(x).h(t))$, find u when

$$f(x) = \sin(\frac{k\pi x}{2}), \quad k \in \mathbb{N}^*;$$

3. Can u be found if $u(x,0) = x$ for $x \in I$?
4. Take $k = h = \frac{1}{10}$ and approximate u to the nœud of the mesh ($j = 0,...,5$) using the explicit scheme, then compare with the theoretical values.

Solve the following boundary problem using the DF method (explicit scheme):

$$(P): \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x^2}, & 0 < x < 1, t > 0; \\ u(x,0) = f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}; \\ 2(1-x), & \frac{1}{2} < x \le 1 \end{cases} & \text{(C.I)}; \\ u(0,t) = u(1,t) = 0, & \text{t>0} \quad \text{(C.L)}. \end{cases}$$

For $h = \frac{1}{10}$ and $k = \frac{1}{1000}$, perform the calculations at each point on the bar, for $t = 1 \times 10^{-3}$, $t = 2 \times 10^{-3}$, ...,$t = 9 \times 10^{-3}$,$t = 10 \times 10^{-3}$.

Exercise 6.4 Let a and b be two strictly positive real constants. Consider the partial differential equation

$$u_t + au - bu_{xx}, \quad u(x,0) = f(x), x \in \mathbb{R}, t \ge 0. \tag{6.9}$$

Let be a time step $\triangle t$ and a space step $\triangle x$.
We define $x_j = j\triangle x$ and $t_n = n\triangle t$ for $j \in \mathbb{Z}$, $n \in \mathbb{N}$.
Let j be an approximate value of $u(x_j, t_n)$ calculated by the following explicit numerical scheme:

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + au_j^n - b\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\triangle x^2} = 0.$$

In the following,

$$r = \frac{b\triangle t}{\triangle x^2}, \quad s = a\triangle t.$$

1. Show that the scheme can be written as

$$u_j^{n+1} = ru_{j-1}^n + (1 - 2r - s)u_j^n + ru_{j+1}^n.$$

2. Show that the consistency error of the scheme is $\mathcal{O}(\triangle t^2) + \mathcal{O}(\triangle t \triangle x^2))$. $k \in \mathbb{R}$ and the initial $u(x,0) = \exp(ikx)$. Assuming a $j = (G(k))^n u_j^0$, calculate the amplification factor $G(k)$.
3. Assume $2r + (s/2) \leqslant 1$. Show that the scheme is stable.
   Indication: We can pose $\theta = k\triangle x$ and write $G(k)$ in the following form G(k) = 1 - s - 2r(1 -costheta).
4. Show that if $2r + s \le 1$ then $max_j|u_j^n| \le (1-s)^n max_j|u_j^0|$. Let

$$v(x,t) = e^{at}u(x,t).$$

   Show that $v(x,t)$ satisfies the diffusion equation

$$v_t - bv_{xx} = 0, \quad v(x,0) = f(x), \quad x \in \mathbb{R}, t \ge 0. \tag{6.10}$$

   Assuming that $max_x|v(x,t)| = max_x|v(x,0)|$, can you explain the majoration obtained in question 5?

Exercise 6.5 Let $\alpha > 0$ a strictly positive real constant. Consider the partial differential equation

$$u_t - \alpha u_{xx} = 0, \quad u(x,0) = f(x), \quad x \in \mathbb{R}, t \geq 0 \tag{6.11}$$

Let's consider a time step δt and a space step δx. We define $x_j = j\delta x$ et $t_n = n\delta t$ for $j \in \mathbb{Z}, n \in \mathbb{N}$. Let u_j^n be an approximate value of $u(x_j, t_n)$ calculated by the following explicit numerical scheme:

$$\frac{u_j^{n+1} - u_j^n}{\delta t} - \alpha \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\delta x^2} = 0$$

In the following, we put

$$r = \alpha \frac{\delta t}{\delta x^2}.$$

1. Show that the diagram can be written as

$$u_j^{n+1} = ru_{j-1}^n + (1-2r)u_j^n + ru_{j+1}^n.$$

2. Show that if $r \leq \frac{1}{2}$, than $\forall n \in \mathbb{N}, \inf_j u_j^0 \leq u_j^n \leq = \sup_j u_j^0$.
3. Show that the consistency error of the scheme $varepsilon(\delta t, \delta x)$ checks:

$$\varepsilon(\delta t, \delta x) = \mathscr{O}(\delta t^2) + \mathscr{O}(\delta t.\delta x^2).$$

4. Let $E_j^n = u(x_j, t^n) - u_j^n$ be the approximation error. Show that

$$E_j^{n+1} = rE_{j-1}^n + (1-2r)E_j^n + rE_{j+1}^n + \varepsilon(\delta t, \delta x).$$

5. It is now assumed that $r \leq \frac{1}{2}$. We note $||E_n||_\infty = \sup_{j \in \mathbb{Z}} |E_n^j|$. Demonstrate the increase

$$||E_{n+1}||_\infty \leq ||E_n||_\infty + C\delta t(\delta t + \delta x^2),$$

   where C is a suitable constant.
6. Let $N \geq 1$ be a fixed integer. Let $T = N\delta t$. Assume that $u_j^0 = f(x_j) = u(x_j, 0)$. Show that

$$\forall n \leq N, \ ||E^n||_\infty \leq CT(\delta t + \delta x^2).$$

   This shows that the scheme is convergent on $\mathbb{R} \times [0,T]$ and that it is of order one in time and of order two in space.

7. Black-Scholes equation: Numerical analysis

Introduction

In previous chapters, we obtained closed form price formulas for a variety of option models. However, option models that lend themselves to analytic solutions are limited. In most cases, option valuation must be relegated to numerical procedures. The classes of numerical methods employed in option valuation include the lattice tree methods, finite difference algorithms and Monte Carlo simulation.

The finite difference approach seeks the discretization of the differential operators in the Black-Scholes equation. The numerical schemes arising from the discretization procedure can be broadly classified as either implicit or explicit schemes. Each class of schemes has its merits and limitations. The explicit schemes have better computational efficiency, but they may be susceptible to numerical instabilities to round-off errors if the time steps in the numerical computation are not chosen to be sufficiently small. Interestingly, the lattice tree schemes are seen to have the same analytic forms as those of the explicit finite difference schemes, though the two classes of numerical schemes are derived using quite different approaches.

7.1 Partial differential equations for options

7.1.1 Black-Scholes partial differential equation

There are many types of options. A European call option is a contract that entitles the holder of the option to buy a given stock at price $K > 0$, called the strike price, at a future time T. If the option is exercised, the seller of the option is obligated to sell the stock at price K regardless of the market price of the stock. At the expiry time T (in the future), the market price of the stock might be greater than or less than K. Therefore, if the price of the stock at time T is greater than K, then the option holder can buy the stock at K (from the option seller) and immediately sell the stock

at the higher price (on the market), making a profit. On the other hand, if the market price of the stock at time T is less than$K$, then at time T the holder of the option gains nothing from exercising the option. So, no action is taken; the option is worthless at this point. The question that Black, Scholes, and Merton were trying to answer in their seminal papers (1973) is: "How should one set the price of this option contract, given that the stock price now at time $t < T$ is S"?

Theorem 7.1.1 (Black-Scholes Equation)
Assume that the asset price S follows a geometric Brownian motion as in [**BS-eq**]. Under the assumptions of Black-Scholes framework, the call or put option price $u(t,S)$ satisfies the parabolic partial differential equation

$$\frac{\partial u(t,s)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u(t,s)}{\partial S^2} + rS\frac{\partial u(t,s)}{\partial S} - ru(t,S) = 0 \tag{7.1}$$

Proof. Exercise ■

This is the famous Black, Scholes and Merton partial differential equation (PDE). It is a relationship between V, S, t and certain partial derivatives of V.
Two points are worth raising immediately.

1. The drift parameter μ in the asset model does not appear in the PDE.
2. We have not yet specified what type of option is being valued. The PDE must be satisfied for any option on S whose value can be expressed as some smooth function $V(S,t)$.

The function $V(S,t)$ models the fair price of a contingent claim or "option" based on the market price of an underlying security. The variable S represents the market price of the security, and the variable t represents time. The fixed time T is called the expiry or time of maturity. The solution $u(S,t)$ depends on the current time, the current stock price, the expiry time, and the boundary condition which is determined by the type of option. The solution also depends on the parameters σ and r which represent the volatility of the underlying asset and the risk-free interest rate, respectively.

Regarding point (2), to determine $V(S,t)$ uniquely we must specify other conditions that involve information about the particular option. As is typical with many differential equations, these will apply somewhere along the edges of the domain $0 \leq S$, $0 \leq t \leq T$ on which the problem is posed. We will use $C(S,t)$ to denote the European call option value. In this case, we know for certain that at the expiry time, $t = T$, the payoff is$max(S(T) - E, 0)$. This must be the value of the option at time T , otherwise an obvious arbitrage opportunity exists. So

$$C(S,T) = max(S(T) - E, 0). \tag{7.2}$$

Now if the asset price is ever zero, then it is clear that S(t) remains zero for all time and hence the payoff will be zero at expiry. So, in this case, the value of the option must be zero at all times. Hence,

$$C(0,t) = 0, \quad for\ all 0 \leq t \leq T. \tag{7.3}$$

Conversely, if the asset price is ever extremely large, then it is very likely to remain extremely large and swamp the exercise price, so that,

$$C(S,t) \approx S, \quad for\ large\ \ S. \tag{7.4}$$

The constraint 7.2 is called a final condition, as it applies at the final time $t = T$. It is much more common to come across initial conditions, specified at $t = 0$, and we will see in the next Chapter, that the PDE is easily transformed into such a problem. The other constraints, (7.3) and (7.4), are known as boundary conditions.

7.2 Finite difference algorithms

This chapter introduces finite-difference methods, which represent the most popular computational approach. Here, we develop three widely used finite difference schemes for the basic heat equation and discuss their main properties. The next chapter is devoted to the use of finite difference technology for option pricing.

Finite difference methods are popular numerical techniques for solving science and engineering problems modeled by differential equations.

In the construction of finite difference schemes, we approximate the differential operators in the governing differential equation of the option model by appropriate finite difference operators, hence the name of this approach.

7.2.1 FTCS, BTCS and Crank-Nicolson for Black–Scholes

The Black-Scholes PDE (7.1) is typically augmented with a final time condition. Since convention (and every book on numerical PDEs) dictates that problems should be specified in initial time condition form, we make the change of variable $\tau = T - t$.

In this way τ represents the time to expiry and runs from T to 0 when t runs from 0 to T. Under this transformation the Black–Scholes PDE (7.1) becomes

$$\frac{\partial u(\tau,s)}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u(\tau,s)}{\partial S^2} - rS\frac{\partial u(\tau,s)}{\partial S} + ru(\tau,S) = 0. \tag{7.5}$$

In this section we focus on European calls and puts. The $t = T$ condition for a European call, (7.2), becomes the $\tau = 0$ condition

$$C(S,0) = max(S(0) - E, 0). \tag{7.6}$$

Similarly, the European put condition is

$$P(S,0) = max(E - S(0),0). \tag{7.7}$$

Turning to boundary conditions, the European call and put involve the PDE on the domain $S \in [0,\infty]$. This presents a difficulty. We must represent this range by a finite set of points. A reasonable fix is to truncate the domain to $S \in [0,L]$, where L is some suitably large value. Using (7.3) and (**??**), this gives call boundary conditions

$$C(0,\tau) = 0 \quad and \quad C(L,\tau) = L \tag{7.8}$$

and

$$P(0,\tau) = Ee^{-r\tau} \quad and \quad P(L,\tau) = 0 \tag{7.9}$$

for a European put.

We are now able to use a grid $\{jh, ik\}_{j=0,i=0}^{N_x,N_t}$. Letting

$$V^i = \begin{pmatrix} V_1^i \\ V_2^i \\ \vdots \\ \vdots \\ V_{N_x-1}^i \end{pmatrix} \in \mathbb{R}^{N_x-1}$$

denote the numerical solution at time level i, we have V0 specified by the initial data (7.6) or (7.7) and the boundary values V_0^i and $V_{N_x}^i$ for all $1 \le i \le N_t$ specified by the boundary conditions (7.8) or (7.9).

To obtain a generalized version of FTCS for the PDE (7.5) we use the full central difference operator for the $\frac{\partial V}{\partial S}$ term and evaluate the V term at (jh, ik) to get the difference equation

$$\frac{V_j^{i+1} - V_j^i}{k} - \frac{1}{2}\sigma^2 S^2 (jh)^2 \frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{h^2} - rSjh\frac{V_{j+1}^i - V_{j-1}^i}{2h} + rV_j^i = 0. \tag{7.10}$$

$$\frac{V_j^{i+1} - V_j^i}{k} - \frac{1}{2}\sigma^2 S^2 (jh)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{h^2} - rSjh\frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2h} + rV_j^{i+1} = 0. \tag{7.11}$$

The matrix–vector representation of FTCS in (23.9) remains valid if we redefine

$$F = (1 - rk)\mathbf{I} + \frac{1}{2}k\sigma^2 D_2 T_2 + \frac{1}{2}krD_1 T_1$$

and

$$p^i = \begin{pmatrix} \frac{1}{2}k(\sigma^2 - r)V_0^i \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x - 1)(\sigma^2(N_x - 1) + r)V^i N_x \end{pmatrix},$$

Where

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & N_x - 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1^2 & 0 & 0 & \cdots & 0 \\ 0 & 2^2 & 0 & \cdots & 0 \\ 0 & 0 & 3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & (N_x - 1)^2 \end{pmatrix}$$

and

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots\cdots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & -1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}, \quad T2 = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \cdots\cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots\cdots & & 1 & -2 & 1 \\ 0 & \cdots\cdots & & 0 & 1 & -2 \end{pmatrix}$$

Similarly, BTCS has the form (23.11) with

$$B = (1 + rk)\mathbf{I} - \frac{1}{2}k\sigma^2 D_2 T_2 - \frac{1}{2}krD_1 T_1$$

and

$$q^i = \begin{pmatrix} \frac{1}{2}k(\sigma^2 - r)V_0^{i+1} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x - 1)(\sigma^2(N_x - 1) + r)V^{i+1}N_x \end{pmatrix},$$

One way to generalize the Crank-Nicolson scheme (7.12) is to take the average of the FTCS and BTCS formulas (7.13) to give

$$\frac{1}{2}(\mathbf{I} + B)V^{i+1} = \frac{1}{2}(\mathbf{I} + F)V^i + \frac{1}{2}(p^i + q^i). \tag{7.12}$$

Example 7.1 We used our three finite difference methods to value a European put option with parameters $E = 4$, $\sigma = 0.3$, $r = 0.03$ and $T = 1$.
We truncated the asset range at $L = 10$. Since the exact value is known from the Black–Scholes formula (7.12), we may check the error. We focused on the maximum error at time zero:

$$Err^0 = max_{1 \le j \le N_x - 1} |V_j^{N_t} - V(jh, \tau = T)|. \tag{7.13}$$

With $N_x = 50$ and $N_t = 500$, so $k = 2 \times 10^{-3}$ and $h = 0.2$, we found that

$$err^0 = 1.5 \times 10^{-3}$$

for FTCS and

$$err^0 = 1.7 \times 10^{-3}$$

for BTCS. With Crank-Nicolson we were able to reduce N_t to 50, so $k = 2 \times 10^{-2}$, and still get a comparable error,

$$err^0 = 1.6 \times 10^{-3}.$$

■

Our treatment of stability and convergence of finite difference methods does not carry through directly to this section, since the PDE (7.5) has nonconstant coefficients and includes a first order spatial derivative. However, similar conclusions may be drawn.

7.2.2 Down-and-out call example

To illustrate the flexibility of finite difference methods, we turn to the down-and out call. We know that the PDE holds for $B \leq S$. Hence, we may truncate this to B = S = L and use a grid of the form $B + jh, ik_{j=0,i=0}^{N_x,N_t}$, where $h = (L-B)/N_x$. The FTCS scheme (7.10) becomes

$$\frac{V_j^{i+1} - V_j^i}{k} - \frac{1}{2}\sigma^2(B+jh)^2\frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{h^2} - r(B+jh)\frac{V_{j+1}^i - V_{j-1}^i}{2h} + rV_j^i = 0. \qquad (7.14)$$

$$\frac{V_j^{i+1} - V_j^i}{k} - \frac{1}{2}\sigma^2(B+jh)^2\frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{h^2} - r(B+jh)\frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2h} + rV_j^{i+1} = 0. \quad (7.15)$$

As before, these may be written in the matrix–vector forms (23.9) and (23.11), and the Crank-Nicolson method is given by (7.12).

The $\tau = 0$ condition (19.2) specifies $V_j^0 = max(B + jh - E, 0)$ and the left-hand boundary condition (19.1) gives $V_0^i = 0$. At the right-hand boundary, a reasonable approach is to argue that, since S is large, the asset is very unlikely to hit the out barrier, so $V_{N_x}^i = C(L,\tau)$ may be imposed, where $C(S,t)$ denotes the European call value.

Computational example For the case $B = 2$, $E = 4$, $\sigma = 0.3$, $r = 0.03$ and $T = 1$ we used Crank-Nicolson to value a down-and-out call. In this case the exact solution (19.3) may be used to check the error. With the asset domain truncated at $L = 10$, and with $N_x = N_t = 50$, we found the maximum time-zero error (7.13) to be

$$err^0 = 1.1 \times 10^{-3}.$$

7.3 Exercises

Exercise 7.1 Consider the following form of the Black–Scholes equation:

$$\frac{\partial w}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial x^2} + (r - q - \frac{\sigma^2}{2})\frac{\partial w}{\partial x} - ru(t,S) = 0, \quad w = e^{-rt}V \quad and \quad x = \ln S,$$

where $V(S,t)$ is the option price and S is the asset price. The two-level six point implicit compact scheme takes the form:

$$a_1 w_{j+1}^{n+1} + a_0 w_j^{n+1} + a_{-1} w_{j-1}^{n+1} = b_1 w_{j+1}^n + b_0 w_j^n + b_{-1} w_{j-1}^n$$

where

$$c = \left(r - q - \frac{\sigma^2}{2}\right)\frac{\Delta t}{\Delta x}, \quad \mu = \sigma^2 \frac{\Delta t}{\Delta x^2},$$

$$a_1 = 1 - 3\mu - 3c - \frac{c^2}{\mu} + \frac{c}{\mu}, a_0 = 10 + 6\mu + \frac{2c^2}{\mu}, a_{-1} = 1 - 3\mu + 3c - \frac{c^2}{\mu} - \frac{c}{\mu},$$

$$b_1 = 1 + 3\mu + 3c + \frac{c^2}{\mu} + \frac{c}{\mu}, b_0 = 10 - 6\mu - \frac{2c^2}{\mu}, b_{-1} = 1 + 3\mu - 3c + \frac{c^2}{\mu} - \frac{c}{\mu}$$

.

Show that the compact scheme is second-order time accurate and fourth order space accurate.

Exercise 7.2 A European call option $C(S,t)$ with expiry date T and strike price K is a contract that gives the holder the right (but not the obligation) to buy a share (or risky asset) on date T at price K (set when the contract is signed). This contract has a price (premium), settled by the buyer when the contract is signed.

Write an algorithm (finite difference method) for the heat equation that allows you to go from j to $j+1$ for the boundary problem.

$$(P)\begin{cases} \text{u: R}\times[0,T] \to \mathbb{R}, & \\ \text{u}_t = u_{xx}, & \text{t}\in[0,T], \quad x \in \mathbb{R}; \\ \text{u(x,0)}=\left(e^{\frac{1}{2}(2\frac{r}{\sigma^2}+1)x} - e^{\frac{1}{2}(2\frac{r}{\sigma^2}-1)x}\right), & x \in \mathbb{R}. \end{cases} \tag{7.16}$$

avec

$$C(S,T) = Ke^{\alpha x+\beta\tau}u(x,\tau),$$

où $\tau = T-t$, $S = Ke^x$, $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$, $\alpha = -\frac{1}{2}(\frac{2r}{\sigma^2} - 1)$ et $\beta = -\frac{1}{4}(\frac{2r}{\sigma^2} + 1)^2$.

Application to a real case

Let's calculate the price of a European call based on the maturity of the CAC40 index
$T = 1$ days.
The value of the underlying asset is: S =3,850 .
The strike is: $K = 4100$.
The constant rate is that of the European Central Bank.: $r = 1,25\%$.
The present value of the call is: $S_0 = 0.025$.
The volatility: $\sigma = 0.0168$.

III Appendix

8. Numerical analysis

8.1 Notions about errors

Introduction

In general, solving scientific problems involves representing the phenomena mathematically. However, these phenomena are often complex and varied, requiring the neglect of certain parameters and the simplification of others. Even after these simplifications, the resulting equations are often unsolvable using traditional analytical methods. For example, it is impossible to analytically find the solutions to equations such as $x^5 + 3x^4 + 7x + 8 = 0, x = e^{-x}, \sin x + ex = 0, \ldots$.

This is where numerical analysis comes in, distinguishing itself from other fields of mathematics. It allows problems to be solved using various algorithms, the effectiveness of which depends on certain parameters affecting the accuracy of the results. Moreover, these algorithms often involve approximations of varying precision, such as replacing a derivative with a finite difference, thus transforming a differential equation into an algebraic equation. The final result and its degree of precision depend on the choices made when applying these methods.

An essential part of numerical analysis, therefore, is to control the errors introduced by these approximations, which arise primarily from three sources.

- modelling errors;
- les erreurs de représentation sur ordinateurcomputer representation errors;
- truncation errors.

8.2 Absolute and relative errors

$$\text{Exact numbers} \begin{cases} \text{dans } \mathbb{N} : 1,3,9; \\ \text{dans } \mathbb{Q} : \frac{2}{3}, \frac{1}{7}, \frac{10}{3}; \\ \text{dans } \mathbb{R} : \sqrt{5}, \pi, e. \end{cases}$$

Let x be an exact number and x^* an approximate value of x, we write

$$x \simeq x^* \text{ ou } x \approx x^*.$$

8.2.1 Erreur absolue

Definition 8.2.1 The absolute error of the approximate number is calledx^* of x la quantity positive, denote $\Delta(x)$, defined by

$$\Delta(x) = |x - x^*|.$$

Example 8.1 For the exact value $x = \frac{2}{3}$, the approximate value $x_1^* = 0.666667$ est 1000 times more accurate than the approximate value $x_2^* = 0.667$. ∎

8.2.2 Relative error

Definition 8.2.2 The relative error of the approximate numberx^* of x the positive real quantity, denoted$r(x)$, define by

$$r(x) = \frac{|x - x^*|}{|x|} = \frac{\Delta(x)}{|x|}$$

$$r\% = r(x) \times 100$$

8.3 Derivation numerical

8.3.1 Position of the problem

As with the integral, we would like to be able to evaluate numerically the derivative of a function that is difficult to manipulate or that is only known at a certain number of points. This problem of numerical derivation is very common in engineering and analysis. analysis (it is the basis of finite difference methods).

Consider a function $f : [a,b] \longrightarrow \mathbb{R}$ of sufficiently high class, $x \in]a,b[$ fixé.
We want to approximate (as best we can!) the derivatives of the function f at the point x..

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

with $h > 0$ (small), we obtain

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h}.$$

We can see that this approximation of $f'(x)$ involves the value of f en x et $x+h$, which leads us to approximate the numbers $f'(x), f''(x), \ldots, f^{(n)}(x)$ using a discrete set of points.

One of the oldest methods used to obtain numerical derivation formulae is to construct differential quotients using the developments of Taylor [1].

8.3.2 Approximation of the first derivative

Approximation of the first derivative

$$\begin{array}{rcl} \Delta_h f(x) & = & f(x+h)-f(x), \\ \nabla_h f(x) & = & f(x)-f(x-h), \\ \delta_{2h} f(x) & = & f(x+h)-f(x-h). \end{array}$$

8.3.3 Two-point formulas

Let's do a first order Taylor expansion of $f.$ around x :

$$f(x+h) = f(x)+hf'(x)+\frac{h^2}{2}f''(\xi),\ \xi \in [x,x+h].$$

We obtain

$$\begin{cases} f'(x) \simeq f'_{hd}(x) = \frac{f(x+h)-f(x)}{h} = \frac{\Delta f(x)}{h}, \text{ c'est la formule de différences finies progressives (\textbf{DFP})} \\ E = -\frac{h}{2}f''(\xi),\ \xi \in [x,x+h] \text{ c'est l'erreur commise} \end{cases}$$

Let's do a second Taylor expansion of order 1 of f around x :

$$f(x-h) = f(x)-hf'(x)+\frac{h^2}{2}f''(\xi),\ \xi \in [x-h,x].$$

We obtain

$$\begin{cases} f'(x) \simeq f'_{hg}(x) = \frac{f(x)-f(x-h)}{h} = \frac{\nabla f(x)}{h}, \text{ c'est la formule de différences finies régressive (\textbf{DFR}),} \\ E = \frac{h}{2}f''(\xi),\ \xi \in [x-h,x] \text{ c'est l'erreur commise.} \end{cases}$$

The formulae obtained are therefore two approximations of order 1 of the first derivative, and the error in each case tends to be towards 0 when h tends towards 0..

Let's increase the order of the Taylor expansion of f around x :.

$$\begin{array}{rcl} f(x+h) & = & f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(\xi_1),\ \xi_1 \in [x,x+h], \\ f(x-h) & = & f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(\xi_2),\ \xi_2 \in [x-h,x]. \end{array}$$

By subtracting member by member, we obtain

$$f(x+h)-f(x-h) = 2hf'(x)+\frac{h^3}{6}(f'''(\xi_1)+f'''(\xi_2)).$$

[1] Brook Taylor, English,1685-1731

This gives

$$\begin{cases} f'(x) \simeq f'_{hc}(x) = \frac{f(x+h)-f(x-h)}{2h} = \frac{\delta_{2h}f(x)}{2h}, \text{ c'est la formule de différences finies centrées (\textbf{DFC}),} \\ E = -\frac{h^2}{12}(f'''(\xi_1)+f'''(\xi_2)) = -\frac{h^2}{6}(f'''(\xi), \ \xi \in [x-h, x+h], \text{ c'est l'erreur commise.} \end{cases}$$

The centred (symmetrical) formula is an approximation of order 2 and is therefore more accurate than the first two formulae, even though it requires knowledge of f at the same number of points. even though it requires knowledge of f at the same number of points. It should be noted, however, that the points used are arranged symmetrically with respect to the point at which the derivative is calculated.

R Assuming that the interval $[a,b]$ is divided into N intervals, we pose $h = \frac{b-a}{N}$ and introduce **lgrid points** x_i so that $x_i = a+ih, i = 0,\ldots,N$. Assume that f is known only at the points grid points x_i. Then to approximate $f'(x_i),\ i = 0,\ldots,N$:

1. The progressive finite difference formula gives:

$$f'(x_i) \simeq f'_{hd}(x_i) = \frac{f(x_{i+1})-f(x_i)}{h},\ i = 0,\ldots,N-1. \tag{8.1}$$

   Note that the relationship (8.1) is not defined for $i = N$, so we can't use this relationship to approximate $f'(x_N)$.
2. The regressive finite difference formula gives:

$$f'(x_i) \simeq f'_{hg}(x_i) = \frac{f(x_i)-f(x_{i-1})}{h},\ i = 1,\ldots,N. \tag{8.2}$$

   Note that the relationship(8.2) nis not defined for $i = 0$, so we cannot use this relationship to approximate $f'(x_0)$.
3. The centred finite difference formula gives:

$$f'(x_i) \simeq f'_{hc}(x_i) = \frac{f(x_{i+1})-f(x_{i-1})}{2h},\ i = 1,\ldots,N-1. \tag{8.3}$$

   Note that the relationship (8.3) is not defined for $i = 0$ and $i = N$, so we cannot use this relationship to approximate $f'(x_0)$ et $f'(x_N)$. Constatons aussi que

$$f'_{hc}(x_i) = \frac{f'_{hg}(x_i)+f'_{hd}(x_i)}{2},\ i = 1,\ldots,N-1.$$

Example 8.2 To illustrate the three previous formulas, let's consider the function $x \mapsto f(x) = 2^x,; x \in [1,5]$ passing through the points $(x_0,y_0) = (1,2),\ (x_1,y_1) = (2,4),\ (x_2,y_2) = (3,8),\ (x_3,y_3) = (4,16)$ et $(x_4,y_4) = (5,32)$.

We want to approach the number $f'(x_2)$:

1. La formula of PFD: $f'(x_2) \simeq f'_{hd}(x_2) = \frac{f(x_3)-f(x_2)}{h} = y_3 - y_2 = 8.$
2. La formula of RFD:$f'(x_2) \simeq f'_{hg}(x_2) = \frac{f(x_2)-f(x_1)}{h} = y_2 - y_1 = 4.$
3. La formula of CFD:$f'(x_2) \simeq f'_{hc}(x_2) = \frac{f(x_3)-f(x_1)}{2h} = \frac{y_3-y_1}{2} = 6.$

Let's evaluate the approximation errors, knowing that $f'(x) = \ln 2\, 2^x, x \in [1,5]$:

$$\begin{array}{rcl} E_1 & = & |f'(x_2) - f'_{hd}(x_2)| = |8\ln 2 - 8| \simeq 2,454. \\ E_2 & = & |f'(x_2) - f'_{hg}(x_2)| = |8\ln 2 - 4| \simeq 1,545. \\ E_3 & = & |f'(x_2) - f'_{hc}(x_2)| = |8\ln 2 - 6| \simeq 0,454. \end{array}$$

Hence, the best formula for approaching $f'(3)$ is that of CFD. ∎

8.3.4 Three-point formulas

It is possible to develop other formulae, simply by performing a Taylor expansion of f around x with a step size of $2h$, for example:

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4h^3}{3} f'''(\xi_1),\ \xi_1 \in [x, x+2h].$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(\xi_2),\ \xi_2 \in [x, x+h].$$

By combining these two equations in such a way as to eliminate the second derivative, we obtain

$$\begin{cases} f'_{hd}(x) & = & \frac{4f(x+h)-3f(x)-f(x+2h)}{2h} \text{ is an approximation of the first derivative of } f \text{ on} x; \\ E & = & \frac{h^2}{3} f''(\xi),\ \xi \in [x, x+2h] \text{ It is the error committed.} \end{cases}$$

8.3.5 Approximation of the second derivative

To obtain an approximation of the second derivative, we proceed in the same way, but using Taylor expansions of order 3:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(\xi_1),\ \xi_1 \in [x, x+h].$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(\xi_2),\ \xi_2 \in [x-h, x].$$

To obtain an approximation of the second derivative, we proceed in the same way, but using Taylor expansions of orderTo obtain an approximation of the second derivative, we proceed in the same way, but using Taylor expansions of order

$$\begin{cases} f''_{hd}(x) & = & \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \frac{\delta_h^2 f(x)}{h^2} \text{ is an approximation of the second derivative of } f \text{ on} x; \\ E & = & -\frac{h^2}{12} f^{(4)}(\xi),\ \xi \in [x-h, x+h] \text{ It is the error committed.} \end{cases}$$

This formula is therefore of order 2 (the error term tends towards 0 when h tends towards 0) and is very important in practice.

8.3.6 Approximation of higher-order derivatives

We can of course generalise this approach and determine approximations to higher-order derivatives.

We have $\frac{\Delta_h f}{h}$, $\frac{\nabla_h f}{h}$ and $\frac{\delta_{2h} f}{2h}$ approximate the first derivative of f (on x) with an accuracy proportional to h, h et h^2 respectively. These three operators are linear with respect to and are called **finite difference operators**. For $n \in \mathbb{N}^*$, we can generalise these operator concepts to approximate the n^{th} derivative of f at a point $x \in]a,b[$. To do this, we'll define it recursively:

$$\Delta_h^n f(x) = \Delta_h(\Delta_h^{n-1} f)(x), \quad \nabla_h^n f(x) = \nabla_h(\nabla_h^{n-1} f)(x), \quad \delta_h^n f(x) = \delta_h(\delta_h^{n-1} f)(x).$$

Analogously to the case where $n = 1$, we can show that

$$\frac{\Delta_h^n f}{h^n}, \quad \frac{\nabla_h^n f}{h^n} \quad \text{and} \quad \frac{\delta_h^n f}{h^n}$$

are approximations of $f^{(n)}$ (en x) with an error proportional to h, h et h^2, respectively, as soon as f is of class C^{n+1}, C^{n+1} and C^{n+2}, respectively.

R Numerical derivation is a highly unstable operation, i.e. it is very sensitive to rounding errors (subtraction between neighbouring terms). For example

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2),$$

où $f(x-h) = f^*(x-h) \pm e_1$ et $f(x+h) = f^*(x+h) \pm e_2$, then

$$f'(x) = \frac{f^*(x+h) - f^*(x-h)}{2h} \pm \frac{e_2 + e_1}{2h} + O(h^2).$$

If the h step is too small, there will be many errors.

8.4 Calculating the eigenvalues and eigenvectors of a matrix

8.4.1 Position of the problem

Let A be a square matrix with coefficients in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ ou $\mathbb{C}$). We are looking for the eigenvalues and eigenvectors$\lambda \in \mathbb{C}$, $X \in (\mathbb{K}^n)^*$ (respectively) of A such that:

$$\lambda \text{ valeur propre de A} \iff \exists X \in (\mathbb{K}^n)^* : AX = \lambda X.$$

$$X \text{ eigenvector of A} \iff \exists \lambda \in \mathbb{C} : AX = \lambda X.$$

Let P be the characteristic polynomial of A. Then

$$\lambda \text{ eigenvalue of A} \iff P(\lambda) = \det(A - \lambda I) = 0,$$

i.e. λ is a root of the characteristic polynomial P.

There are two types of eigenvalue and eigenvector search methods:

Direct methods generally involve three main stages:

- **i)** search for the coefficients of P,
- **ii)** calculation of the roots of P,
- **iii)** obtaining eigenvectors.

Iterative methods: these are methods that do not involve finding the characteristic polynomial P..

8.4.2 Direct methods

1. Direct calculation of $P(\lambda) = \det(A - \lambda I)$: For $n = 2, 3, 4$, the determinant of a matrix is relatively easy to calculate manually. For n (quite) large, you need to use a computer, which evaluates the determinant of a matrix of order n at a cost of $O(n!)$ (exponential complexity). For example, if $n = 20$, a very powerful computer will take years to calculate $det(A)$.

2. Krylov's method: First, let's recall the following important theorem:

Theorem 8.4.1 (Cayley-Hamiltion theorem) If we denote by $P(\lambda)$ the characteristic polynomial of a martice square martice of order n A. Alors $P(A) = 0_{M_n(\mathbb{K})}$.

We have

$$\begin{aligned} P(\lambda) &= (-1)^n \left(\lambda^n - Tr(A)\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \alpha_{n-3}\lambda^{n-3} \ldots + \det(A)\right) \\ &= (-1)^n \left(\lambda^n + \sum_{k=0}^{n-1} \alpha_k \lambda^k\right) \text{ où } \alpha_0 = \det(A),\ \alpha_{n-1} = -Tr(A). \end{aligned}$$

This method calculates the coefficients of the polynomial P as follows:
On the one hand, according to the previous theorem, we have

$$P(A) = (-1)^n[A^n + \sum_{k=0}^{n-1} \alpha_k A^k] = 0_{M_n(\mathbb{K})} \Longrightarrow A^n = \sum_{k=0}^{n-1} \beta_k A^k, \text{ where} \beta_k = -\alpha_k.$$

On the other hand, let Y_0 be a non-zero vector of $\mathbb{R}^n$, B a square matrix of order n defined by:

$B = [Y_0, AY_0, A^2Y_0, \ldots, A^{n-1}Y_0]$, and $X = \begin{pmatrix} \beta_0 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{pmatrix}$. On trouve,

$$BX = \alpha_0 Y_0 + \beta_1 A Y_0 + \ldots + \beta_{n-1} A^{n-1} Y_0 = \left(\sum_{k=0}^{n-1} \beta_k A^k\right) Y_0 = A^n Y_0.$$

In other words, the vector X which contains the symmetries of the coefficients of the polynomial P satisfies the linear system $BX = b$ where $b = A^n Y_0$.

R

1. We need to choose Y_0 so that B is regular.
2. The linear system obtained can be solved by one of the methods studied in the previous chapter.

Example 8.3 Soient $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $Y_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Alors, $B = [Y_0, Ay_0] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}\right]$ et $b = A^2Y_0 = \begin{pmatrix} 17 \\ 37 \end{pmatrix}$. So the vector $X = (\beta_0, \beta_1)^t$ satisfies the linear system:

$$\begin{pmatrix} 1 & 3 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 17 \\ 37 \end{pmatrix} \Longrightarrow \begin{cases} \beta_0 = 2 \\ \beta_1 = 5 \end{cases} \Longrightarrow \begin{cases} \alpha_0 = -2 \\ \alpha_1 = -5 \end{cases}$$

d'où $P(\lambda) = \lambda^2 - 5\lambda - 2$. ∎

R Other methods for finding the coefficients of P exist (Faddeev, Leverrier,...), but the disadvantage of direct methods is that the roots of P, have to be found, which requires the use of methods for solving non-linear equations.

8.4.3 Iterative methods

Iterated power method

This method allows us to iteratively obtain the eigenvalue with the largest modulus of an A matrix and the corresponding eigenvector..

Let A be a real matrix of order n. It is assumed, for simplicity, that all the eigenvalues $lambda_1, \ldots, \lambda_n$ of A are distinct, so that the corresponding eigenvectors $V_1, \ldots, V_n$ form a basis for $\mathbb{R}^n$. We put

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n| \quad (i.e.\ \lambda_1 \in \mathbb{R}).$$

Let $X^{(0)}$ any vector in $\mathbb{R}^n_*$, then

$$\exists\, \alpha_i \in \mathbb{R},\ i = \overline{1,n} \text{ tel que } X^{(0)} = \sum_{i=1}^{n} \alpha_i V_i.$$

Let's apply the matrix A to the vector $X^{(0)}$, we obtain

$$AX^{(0)} = \sum_{i=1}^{n} \alpha_i A V_i = \sum_{i=1}^{n} \alpha_i \lambda_i V_i.$$

Let's apply again A

$$A^2X^{(0)} = \sum_{i=1}^{n} \alpha_i A^2 V_i = \sum_{i=1}^{n} \alpha_i \lambda_i^2 V_i,$$

and so on, we get

$$\begin{aligned} A^kX^{(0)} = \sum_{i=1}^{n} \alpha_i \lambda_i^k V_i &= \alpha_1 \lambda_1^k V_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^k V_i \\ &= \lambda_1^k \left(\alpha_1 V_1 + \sum_{i=2}^{n} \alpha_i \left(\tfrac{\lambda_i}{\lambda_1}\right)^k V_i \right). \end{aligned}$$

We can see that whene k tend vers $+\infty$, $\left(\alpha_1 V_1 + \sum_{i=2}^{n} \alpha_i \left(\tfrac{\lambda_i}{\lambda_1}\right)^k V_i \right)$ tendancieux $\alpha_1 V_1$.

Consequently, after a sufficiently large number of iterations, we can write

$$X^{(k)} = A^kX^{(0)} \simeq \lambda_1^k \alpha_1 V_1,$$

or to the next iteration:

$$X^{(k+1)} = AX^{(k)} = A^{k+1}X^{(0)} \simeq \lambda_1^{k+1} \alpha_1 V_1 \simeq \lambda_1 X^{(k)}.$$

As a result,

$$|\lambda_1| \simeq \frac{\|X^{(k+1)}\|}{\|X^{(k)}\|}.$$

In practice, this procedure is used as follows: In order to avoid the (possible) manipulation of quite large large numbers, it is a good idea to normalise each vector $X^{(k)}$; for $k \geq 1$, i.e. don't apply the matrix A to the vector $X^{(k)}$, but to the vector $Y^{(k)} = \frac{X^{(k)}}{\|X^{(k)}\|}$.. So if we suppose that $|x_p^{(k)}| = \max_i |x_i^{(k)}|$,

- la $p^{ième}$ component of $Y^{(k)}$ is 1,
- the eigenvalue $lambda_1$ estimated at $k^{ième}$. iterations is the $p^{iè}$ component of the vector $X^{(k)}$.
  Indeed; when k is large enough and the $p^{ième}$ component of V_1 is non-zero, we have

$$\frac{x_p^{(k+1)}}{x_p^{(k)}} \simeq \frac{\lambda_1^{k+1}\alpha_1 v_{1,p}}{\lambda_1^{k}\alpha_1 v_{1,p}} = \lambda_1.$$

The algorithm for the iterated power method is as follows as follows:

1. Given A, $X^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ arbitrary vector in $\mathbb{R}_*^n$, ε details onλ.
2. We pose $Y^{(0)} = X^{(0)}$, for $(k \geq 1)$, we calculate the vector$X^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$ which

$$X^{(k)} = AY^{(k-1)},$$

then the vector

$$Y^{(k)} = \left(\frac{x_1^{(k)}}{x_p^{(k)}}, \ldots, \frac{x_n^{(k)}}{x_p^{(k)}} \right), \quad \text{où} \quad |x_p^{(k)}| = \max_i |x_i^{(k)}|.$$

Then,

$$\lambda_1^{(k)} = x_p^{(k)} \quad \text{et} \quad V_1^{(k)} = Y^{(k)}.$$

3. Repeat step (2) for as long as

$$|\lambda_1^{(k)} - \lambda_1^{(k-1)}| \geq \varepsilon.$$

4.

$$\lambda_1 \simeq \tilde{x}_p^{(k)} \quad \text{et} \quad V_1 \simeq \tilde{Y}^{(k)}.$$

R When the iterated power method algorithm is applied to the matrix A^{-1}, it determines the smallest eigenvalue of A. eigenvalue of A, this is the iterated power method. power method.

Example 8.4 Soit

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

The eigenvalues of matrix A are $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$ and the corresponding eigenvectors are $V_1 = (0,0,1)$, $V_2 = (1,1,1)$ and $V_3 = (1,0,0)$.$powermethodcanbeusedtorecalculatelambda_1$ and V_1.$If we put \mathrm{X}^{(0)} = (0,1,0)$ and $\mathrm{Y}^{(0)} = (0,1,0)$ then

$$X^{(1)} = AY^{(0)} = (1,2,-1) \Longrightarrow p = 2,\ \lambda^{(1)} = 2 \text{ et } Y^{(1)} = (\frac{1}{2}, 1, \frac{-1}{2})$$

$$X^{(2)} = AY^{(1)} = (\frac{3}{2}, 2, \frac{-5}{2}) \Longrightarrow p = 3,\ \lambda^{(2)} = \frac{-5}{2} \text{ et } Y^{(2)} = (\frac{-3}{5}, \frac{-4}{5}, 1)$$

$$X^{(3)} = AY^{(2)} = (\frac{-7}{5}, \frac{-8}{5}, \frac{19}{5}) \Longrightarrow p = 3,\ \lambda^{(3)} = \frac{19}{5} \text{ et } Y^{(3)} = (\frac{-7}{19}, \frac{-8}{19}, 1)$$

$$X^{(4)} = AY^{(3)} = (\frac{-15}{19}, \frac{-16}{19}, \frac{65}{19}) \Longrightarrow p = 3,\ \lambda^{(4)} = \frac{65}{19} \text{ et } Y^{(4)} = (\frac{-15}{65}, \frac{-16}{65}, 1)$$

$$X^{(5)} = AY^{(4)} = (\frac{-31}{65}, \frac{-32}{65}, \frac{211}{65}) \Longrightarrow p = 3,\ \lambda^{(5)} = \frac{211}{65} \text{ et } Y^{(4)} = (\frac{-31}{211}, \frac{-32}{211}, 1)$$

$$X^{(6)} = AY^{(5)} = (\frac{-63}{211}, \frac{-64}{211}, \frac{665}{211}) \Longrightarrow p = 3,\ \lambda^{(6)} = \frac{665}{211} \text{ et } Y^{(4)} = (\frac{-63}{665}, \frac{-64}{665}, 1).$$

Note that the sequence $(\lambda^{(k)})$ converges to $\lambda_1 = 3$ and $(Y^{(k)})$. converges to $V_1 = (0,0,1)$. ∎

R If we choose an eigenvector of the matrix A as the initial vector initial vector $X^{(0)}$ of the method, we run the risk of not having the most of the eigenvalues.

8.5 Solving ordinary differential equations

8.5.1 Position of the problem

A differential equation is an equation relating a function and its successive derivatives. If the equation involves only the function and its derivative, it is called a first-order equation. Let's take as our starting point a first-order differential equation with an initial condition.

Let $f : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ a sufficiently differentiable function and $y_0 \in \mathbb{R}$.
The task is to determine a function$y : [t_0, T] \to \mathbb{R}$ solution du **Cauchy problem:**

$$\begin{cases} y'(t) &= f(t, y(t)), \quad t \in [t_0, T] \\ y(t_0) &= y_0 (\textbf{ initial condition or Cauchy condition }). \end{cases} \tag{8.4}$$

The numerical solution of differential equations is probably the area of numerical analysis with the most numerous applications. Whether in fluid mechanics, heat transfer, or structural analysis, it often leads to solving differential equations, systems of differential equations, or more generally, partial differential equations. Among their advantages, numerical methods allow the study of complex problems for which no analytical solutions are known, but which are of great practical importance.

The accuracy of the various solution methods presented in this course is proportional to the order of these methods. We begin the course with relatively simple methods that have a geometric interpretation. These will gradually lead us to more complex methods, such as the fourth-order Runge-Kutta methods, which provide highly accurate results.

R The independent variable t very often (but not always) represents time. time. The initial condition $y(t_0) = y_0$ is, in a sense the state of the solution at the moment when we start to take an interest in it. The aim is to obtain $y(t)$ for $tin[t_0, T]$ if we are looking for an analytical analytical solution, or an approximation of $y(t)$, if we are using numerical method.

Let's start with a few examples of Cauchy problems problems that can be solved analytically:

Example 8.5 Consider the following Cauchy problem:

$$\begin{cases} y'(t) &= t, \quad t \in [0, 1] \\ y(0) &= 1. \end{cases}$$

This example is one of the simplest you can imagine. imaginable. By integrating on each side, we get $y(t) = \frac{t^2}{2} + c$ où c is a real constant, to determine this constant, it suffices to impose the initial impose the initial codition, $y(0) = 1$. The result is

$$y(t) = \frac{t^2}{2} + 1, \ t \in [0, 1].$$

■

Example 8.6 Let the Cauchy problem:

$$\begin{cases} y'(t) &= \sqrt{y}, \quad t \in [0,1] \\ y(0) &= 0. \end{cases}$$

In this case, it is not enough to integrate the two sides of the equation to obtain the solution. First you have to separate the variables by writing, for example

$$\frac{dy}{\sqrt{y}} = dt, \quad y \neq 0$$

we can now integrate two ribs and get

$$2\sqrt{y} = t + c \Rightarrow y(t) = \left(\frac{t+c}{2}\right)^2$$

then, $y(0) = 0 \Rightarrow c = 0$. Donc $y(t) = \frac{t^2}{4}$ est a solution to the given problem.
Note also that $y \equiv 0$ is a solution to our problem. ∎

Example 8.7 Consider the Cauchy problem:

$$\begin{cases} y' &= -\frac{y}{t\ln t} + \frac{1}{\ln t}, \quad t \in [e,5] \\ y(e) &= e. \end{cases}$$

The differential equation for this problem is first-order linear. order. By applying the elementary method for solving this type of equation, we obtain

$$y(t) = \frac{t+c}{\ln t} \quad \text{où} \quad c \text{ is an arbitrary real constant,}$$

then the initial coding gives $c = 1$. The solution to the problem is then $y(t) = \frac{t}{\ln t}$. ∎

The following theorem gives us sufficient conditions that ensure the existence and uniqueness of the (theoretical) solution of the problem (8.4).

Theorem 8.5.1 Soit $f : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ a function such that
1. f is continuous on $[t_0, T] \times \mathbb{R}$.
2. f is Lipschitzian with respect to the second variable, i.e. there is a constant $L > 0$ such that for any $t \in [t_0, T]$ et $y_1, y_2 \in \mathbb{R}$, we have:

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|.$$

Then the Cauchy problem[a] (8.4) has a unique solution $y \in C^1([t_0, T], \mathbb{R})$.

[a] Augustin Louis Cauchy, French, 1789-1857

R 1. Note that the function $f(t,y) = -\frac{y}{t\ln t} + \frac{1}{\ln t}$ is continuous on $]0, +\infty[\times\mathbb{R}$, so it is continuous on the interval $[e,5] \times \mathbb{R}$. Lipschitzian with respect to y on $[e,5]$ with the Lipschitz constant

Lipschitz constant $L = frac1e$. Indeed, for allt $t \in [e,5]$ et $y_1, y_2 \in \mathbb{R}$ we have

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= |-\tfrac{y_1}{t\ln t} + \tfrac{1}{\ln t} + \tfrac{y_2}{t\ln t} - \tfrac{1}{\ln t}| \\ &= \tfrac{1}{t\ln t}|y_1 - y_2| \\ &\leq \tfrac{1}{e}|y_1 - y_2|, \end{aligned}$$

which justifies the uniqueness of the solution to the third problem found analytically.

2. However, the second member of the second problem $f(t,y) = \sqrt{y}$ is not Lipschitzian with respect to y. For any $y_1 \in R$ and $y_2 = 0$ we have

$$|f(t,y_1) - f(t,y_2)| = |\sqrt{y_1} - 0| = \sqrt{y_1} \geq y_1.$$

Hence the possibility of having more than one solution to this problem, (analytically we have found two solutions $y_1(t) = 0$ and $y_2(t) = \frac{t^2}{4}$ on $[0,1]$).

R Only a few types of differential equations can be solved analytically. differential equations (e.g. equations with separable variables, linear variables, linear, Bernoulli, Riccati...) but there is a large class of class of differential equations that cannot be solved analytically. The numerical resolution is essential here.

Systems of first-order differential equations

Let $t \in [a,b]$,

$$\begin{aligned} f : [a,b] \times \mathbb{R}^n &\to \mathbb{R}^n \\ (t,y) &\mapsto f(t,y) = \begin{pmatrix} f_1(t,y_1,y_2,...,y_n) \\ f_2(t,y_1,y_2,...,y_n) \\ . \\ . \\ . \\ f_n(t,y_1,y_2,...,y_n) \end{pmatrix}. \end{aligned}$$

We are looking for a function

$$\begin{aligned} y : [a,b] &\to \mathbb{R}^n \\ t &\mapsto y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ . \\ . \\ . \\ y_n(t) \end{pmatrix}. \end{aligned}$$

checking the problem:

$$\begin{cases} y'(t) = f(t,y(t)), \quad t \in [a,b] \\ y(a) = \alpha, \ \alpha \in \mathbb{R}^n. \end{cases} \tag{8.5}$$

où

$$y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ . \\ . \\ . \\ y_n'(t) \end{pmatrix}, \qquad \alpha = (\alpha_1, \alpha_2, ..., \alpha_n).$$

R The existence and uniqueness of the solution of the system (8.5) are ensured by the same conditions of the theorem 8.5.1, by replacing only the absolute value by a norm in $mathbbR^n$.

Differential equations of order greater than 1

Second-order equations

$$\begin{cases} y''(t) &= f(t,y,y'), \quad t \in [a,b] \\ y(a) &= \alpha \\ y'(a) &= \beta. \end{cases} \tag{8.6}$$

Equations of order $n \geq 2$

$$\begin{cases} y^{(n)}(t) &= f(t,y,y',y'',...,y^{(n-1)}), \quad t \in [a,b] \\ y(a) &= \alpha_1 \\ y'(a) &= \alpha_2 \\ . & \\ . & \\ . & \\ y^{(n-1)}(a) &= \alpha_n. \end{cases} \tag{8.7}$$

$\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$. We can reduce a Cauchy problem of order greater than 1 to a system of equations of order 1.

Example 8.8 (Second-order problem)

$$\begin{cases} y''(t) &= h(t,y,y'), \quad t \in [a,b] \\ y(a) &= \alpha \\ y'(a) &= \beta. \end{cases}$$

$$\Leftrightarrow \begin{cases} u_1'(t) &= u_2(t) \\ u_2'(t) &= u_1''(t) = h(t,u_1,u_2), \quad t \in [a,b] \\ u_1(a) &= \alpha \\ u_2(a) &= \beta. \end{cases} \quad \text{est un système d'ordre 1.}$$

■

Problems at the limits

$$\begin{cases} y''(t) &= f(t,y,y'), \quad t \in [a,b] \\ y(a) &= \alpha \\ y(b) &= \beta. \end{cases} \quad \text{is not a Cauchy problem} \tag{8.8}$$

This type of problem is more complex than the Cauchy problem and requires more complex numerical methods? In the following, we will use the conditions of the theorem 8.5.1.

R

1. With digital tools for solving differential it is not possible to obtain a solution for all values of the variable t. solution for all values of the variable t.*anapproximationtotheanalyticalsolutiononlyforcertai* certain values of t denoted t_i and separated by a value $h_i = t_{i+1} - t_i$. In the methods presented, this distance is constant for all i and is denoted h. It is called the time step. $y(t_i)$ be the analytical solution of the differential equation (8.4) in $t = t_i$, and y_i the approximate approximate solution in $t = t_i$ using a method numerical method.

One-step numerical methods

8.6 Euler method

Let's take the differential equation of (8.4) and the initial condition condition $y(t_0) = y_0$. The aim is to obtain an approximation of the solution in $t = t_1 = t_0 + h$. Before performing the first iteration, you need to determine the direction in which you need to move forward from the point of departure. (t_0, y_0) to obtain the point (t_1, y_1), which is an approximation of the point $(t_1, y(t_1))$.

The differential equation (8.4) ensures that:

$$y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0).$$

We can therefore follow the line through (t_0, y_0) with slope slope $f(t_0, y_0)$. The equation of this line, given by $d_0(t)$, is:

$$d_0(t) = f(t_0, y_0)(t - t_0) + y_0$$

At $t = t_1$, we have:

$$d_0(t_1) = f(t_0, y_0)(t_1 - t_0) + y_0 = y_0 + hf(t_0, y_0)) = y_1.$$

In other words, $d_0(t_1)$ is close to the solution solution $y(t_1)$, i.e.

$$y(t_1) \simeq y_1 = d_0(t_1) = y_0 + hf(t_0, y_0).$$

It is important to note that, in most cases, $y_1 \neq y(t_1)$. So if we want to do a second iteration and get an approximation of approximation of $y(t_2)$, we can repeat the previous analysis from the point starting from the point (t_1, y_1), but note that the slope of the analytical solution at $t = t_1$ is:

$$y'(t_1) = f(t_1, y(t_1)).$$

We don't know exactly $y(t_1)$, but we do know the approximation y_1 of $y(t_1)$.*the*y_1 approximation of $y(t_1)$. expression:

$$y\prime(t_1) = f(t_1, y(t_1)) \simeq f(t_1, y_1)$$

et construire la droite

$$d_1(t) = f(t_1, y_1)(t - t_1) + y_1,$$

which will allow us to estimate $y(t_2)$. We then have

$$y(t_2) \simeq y_2 = y_1 + hf(t_1, y_1).$$

Note that the error made on the first iteration is reintroduced into the calculations for the second iteration.

R Euler's method is by far the simplest method of numerically solving numerical solution of ordinary differential equations. It has a beautiful geometric interpretation and is easy to use. easy to use. However, it is relatively little used because of its precision.

Example 8.9 Let the differential equation be

$$\begin{cases} y'(t) = -y(t) + t + 1 \\ y(0) = 1. \end{cases}$$

Take $h = 0.1$ and $f(t,y) = -y+t+1$. The following table shows the results of the first ten iterations. It can be shown that the analytical solution of this equation is:

$$y(t) = e^{-t} + t,$$

which allows us to compare the results of the first ten iterations. numerical and analytical solutions and note the increase in error.

t_i	$y(t_i)$	y_i	$\lvert y(t_i) - y_i \rvert$
0,0	1,000000	1,000000	0,000000
0,1	1,004837	1,000000	0,004837
0,2	1,018731	1,010000	0,008731
0,3	1,040818	1,029000	0,011818
0,4	1,070302	1,056100	0,014220
0,5	1,106531	1,090490	0,016041
0,6	1,148812	1,131441	0,017371
0,7	1,196580	1,178297	0,018288
0,8	1,249329	1,230467	0,018862
0,9	1,306570	1,287420	0,019150
1,0	1,367879	1,348678	0,019201

■

8.6.1 Taylor methods

Taylor expansion allows an immediate generalisation of Euler's method Euler's method, which reduces the approximation error. However, we restrict ourselves to the second-order Taylor method. At time $t = t_n$, we are looking for an approximation to the solution at $t = t_{n+1}$. We have immediately:

$$\begin{aligned} y(t_{n+1}) &= y(t_n + h) \\ &= y(t_n) + y'(t_n)h + \tfrac{y''(t_n)}{2}h^2 + o(h^2). \end{aligned}$$

Using the differential equation(8.4), we find:

$$y(t_{n+1}) = y(t_n) + f(t_n, y(t_n))h + f'(t_n, y(t_n))h^2 + o(h^2)$$

and we have:

$$f'(t, y(t)) = \frac{\partial f(t, y(t))}{\partial t} + \frac{\partial f(t, y(t))}{\partial y} y'(t),$$

than

$$f'(t, y(t)) = \frac{\partial f(t, y(t))}{\partial t} + \frac{\partial f(t, y(t))}{\partial y} f(t, y(t)).$$

The result is

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2}\left(\frac{\partial f(t_n, y(t_n))}{\partial t} + \frac{\partial f(t_n, y(t_n))}{\partial y} f(t_n, y(t_n))\right) + o(h^2), \quad (8.9)$$

neglecting terms of orders greater than or equal to 3. From

$$y(t_{n+1}) \simeq y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2}\left(\frac{\partial f(t_n, y(t_n))}{\partial t} + \frac{\partial f(t_n, y(t_n))}{\partial y} f(t_n, y(t_n))\right). \quad (8.10)$$

This relationship will form the basis of the Taylor method.

Comment

In fact, Taylor's method consists in approximating the solution of the equation equation (8.4) by arcs of parabolas instead of the line segments (tangents) segments (tangents) used in the Taylor method.

Taylor algorithm of order 2

(1) Given a time step h, an initial condition (t_0, y_0), and a maximum number of iterations N.

(2)

Pour $0 \leq n \leq N$

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2}\left(\frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n)\right)$$

et $t_{n+1} = t_n + h$

write t_{n+1} et y_{n+1}.

(3) Stop.

In this algorithm, we have replaced the analytical solution $y(t_n)$ by its approximation y_n in the relation (refeq6). by its approximation y_n in the relation (8.10). This gives that errors propagate from one iteration to the next.

L'algorithme suivant (avec Matlab) permet de trouver l'approximation y_i de $y(t_i)$, avec la méthode de Taylor

```
function [T,Y] = taylor(f,n,T,Y,h)
syms t y
for i=2 : n+1
  T(i)=T(1)+(i-1)*h;
Y(i)=Y(i-1)+h*subs(f,{t,y},{[T(i-1)],[Y(i-1)]})+((h^2)/2)*(subs(diff(f,t,1),{t,y},{[T(i-1)],[Y(i-1)]}) + (subs(diff(f,y,1),{t,y},{[T(i-1)],[Y(i-1)]}))*subs(f,{t,y},{[T(i-1)],[Y(i-1)]}));
end
```

Example 8.10 Consider the differential equation already solved by Euler's method

$$\begin{cases} y'(t) = -y(t) + t + 1 \\ y(0) = 1. \end{cases}$$

Let $h = 0,1$. In this case $f(t,y) = -y + t + 1$ and $\frac{\partial f}{\partial t}(t,y) = 1$, $\frac{\partial f}{\partial ty}(t,y) = -1$. The following table summarises the results of the first ten iterations which allows the numerical and analytical solutions to be compared and to see the growth of the error and the compare with the Euler method

t_i	$y(t_i)$	y_i	$\lvert y(t_i) - y_i \rvert$
0,0	1,000000	1,000000	0,000000
0,1	1,004837	1,005000	0,000163
0,2	1,018731	1,019025	0,000400
0,3	1,040818	1,041218	0,000482
0,4	1,070302	1,070802	0,000482
0,5	1,106531	1,107075	0,000544
0,6	1,148812	1,149404	0,000592
0,7	1,196580	1,197210	0,000625
0,8	1,249329	1,249975	0,000646
0,9	1,306570	1,307228	0,000658
1,0	1,367879	1,368541	0,000662

■

R Note that the error is smaller with the Taylor method method of order 2 than with Euler's method.

R If we want to reduce the margin of error even further, we can continue with the Taylor expansion in (8.9) up to high-order terms. order. We must then evaluate the derivatives of the function $f(t,y(t))$ of increasingly higher order, which requires the calculation of:

$$\frac{\partial^2 f}{\partial t^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial t \partial y}, \ldots, \frac{\partial^{i+j} f}{\partial t^i \partial^j}.$$

For this reason, the methods obtained are difficult to use. Runge-Kutta methods.

$$y^{(k)}(t) = f^{[k-1]}(t, y(t)) \text{ avec } f^{[1]} = f'_t + f'_y.f.$$

The Taylor expansion of order p leads to l'**Taylor's Algorithme**

$$\begin{cases} y_{n+1} = y_n + f(t_n, y_n)h + \sum_{k=1}^{p} \frac{h^k}{k!} f^{[k-1]}(t_n, y_n) \\ t_{n+1} = t_n + h, \end{cases}$$

which is of order p (in the sense of precision).

9. General information on financial models

Introduction

The past few decades have witnessed a revolution in the trading of derivative securities in world financial markets. A financial derivative may be defined as a security whose value depends on the values of more basic underlying variables, like the prices of other traded securities, interest rates, commodity prices or stock indices. The three most basic derivative securities are forwards, options and swaps. A forward contract (called a futures contract if traded on an exchange) is an agreement between two parties that one party will purchase an asset from the counterparty on a certain date in the future for a predetermined price.

9.1 Financial concepts

9.1.1 Financial market

A financial market, commonly referred to as a "stock exchange," is a marketplace where buyers and sellers engage in the buying and selling of financial assets. The prices at which these transactions take place are determined by the balance of supply and demand. Financial markets are crucial for the liquidity and valuation of various securities, including stocks, bonds, and other financial instruments. They provide a platform for investors to trade these assets, facilitating the efficient allocation of capital in the economy. Additionally, these markets play a key role in price discovery, enabling participants to assess the current value of securities based on real-time trading activity.

9.1.2 Derivative product

A derivative is a financial instrument whose value is derived from the performance of an underlying asset, such as stocks, commodities, or currencies. These instruments were originally developed to help companies manage and mitigate various types of risks, including market risk, liquidity risk,

counterparty risk, and political risk. By using derivatives, businesses can protect themselves against potential adverse price movements or other uncertainties in the financial markets.

9.1.3 Financial assets

A financial asset represents a security or contractual agreement that is generally transferable and tradable, intended to provide income or potential capital appreciation to the holder in return for taking on a specific degree of risk. These assets can include stocks, bonds, derivatives, or other investment instruments. Initially, their primary purpose was to serve as a hedging tool, enabling companies to mitigate exposure to various financial risks, such as market fluctuations, currency volatility, or interest rate changes. Over time, financial assets have evolved to also serve as investment vehicles for individuals and institutions seeking returns.

9.1.4 Underlying assets

An underlying asset is the financial instrument upon which a derivative product is built. It can encompass any type of financial asset, including other derivatives, as there are now derivatives based on derivative assets. Essentially, the underlying asset serves as the reference or foundation for the derivative, as the value and price of the derivative are directly influenced by the performance and price movements of the underlying asset.

9.1.5 Strike price

A strike price, also known as an exercise price, is the fixed price specified in an options contract at which the holder of the option can buy (in the case of a call option) or sell (in the case of a put option) the underlying asset. This price is agreed upon when the option is purchased and remains constant throughout the life of the contract. The strike price is a key factor in determining the profitability of the option, as it serves as the benchmark against which the market price of the underlying asset is compared when the option is exercised.

9.1.6 Option

In finance, an option is a derivative that establishes a contract between a buyer and a seller. The buyer of the option obtains the right, not the obligation to buy (call) or sell (put) an underlying asset at a price fixed in advance (strike), for a given time or on a fixed date. The contract may be entered into speculative or insurance purposes. If, in the financial markets, the goods traded are financial assets (shares, bonds), the option gives the buyer the right to buy (call) or sell (put) the financial asset defined in the contract.

The prices set in advance and the period of validity of the option are defined in the contract. The seller undertakes to respect the terms of the contract if the buyer decides to exercise his option, in return for which the buyer gives him money. If the option is not exercised, the seller has earned an amount equal to the option price. The term stock option generally refers to remuneration paid by a company to its employees under a stock option plan. in the form of a call option on the company's

shares.

There are two types of options traded on the markets: calls and puts.

Call: The buyer of a call option has the right to buy an underlying asset for a period of period at an agreed price.

Put: The buyer of a call option has the right to sell an underlying asset during a given period at an agreed price.
There are several types of option depending on how they are exercised, for example: an option is said to be American if it can be exercised at any time until it expires, and European if it can only be exercised at expiry.

9.1.7 Financial portfolio

A Portfolio is a group of financial assets held by an investor.
Ces actifs peuvent provenir de différentes classes: actions, obligations, produits dérivés, matières premières, fonds, cash, etc.
To reduce risk, investors often diversify their assets. Each asset has its own volatility and is correlated to a greater or lesser extent. Holding several different assets therefore generally tends to reduce the overall volatility of the portfolio.

9.2 Mathematical concepts

9.2.1 Option value

Like the price of a share, the price of an option (the premium), which is determined by supply and demand, will also depend on expectations of the share's performance. at expiration.
The value of an option is made up of two parts: the intrinsic value and the time value.

$$\text{Option price} = \text{intrinsic value} + \text{time value}.$$

Intrinsic value: The intrinsic value represents the profit that would be obtained immediately if the option were exercised. It is the positive or zero difference between the price of the underlying asset and the strike price.

$$\text{Intrinsic value of a Call} = \max\{S_t - K, 0\} = (S_t - K, 0)_+,$$

where S_t is the price of the underlying asset over time and K is the strike price.
In the case of a put, it is the positive or zero difference between the strike price and the price of the underlying asset.

$$\text{Valeur intrinsèque d'un Put} = \max\{K - S_t, 0\} = (K - S_t, 0)_+.$$

Time value: Time value is measured as the difference between the option's market price and its intrinsic value. intrinsic value. It is zero at expiry for a European option.

9.2.2 Important features of the options

Underlying assets:

This is the asset to which the put or call option relates. The underlying asset of an option contract may be a physical asset (commodities or agricultural products), a financial financial instrument (shares, bonds, interest rates, exchange rates) or even a stock market or weather index.

Expiration date:

The deadline, or expiration date, is the specific date on which a contract becomes void and no longer enforceable. In the context of option contracts, these expiration dates are usually standardized, meaning they follow predetermined schedules set by the exchange on which they are traded. Once the deadline is reached, the holder of the option must either exercise their right to buy or sell the underlying asset or let the option expire worthless. This date is crucial as it determines the timeframe within which the option can be utilized.

Anticipation

- The buyer of a call generally expects the underlying asset to rise. He will exercise his call if the price of the underlying asset rises in line with his expectations and exceeds the strike price;
- the buyer of an option generally expects the underlying asset to fall. He will exercise his call if the price of the underlying asset moves in a direction favourable to his expectations and exceeds the strike price.

Exercise

Exercise refers to the action where the buyer of an option chooses to utilize their right to either buy (in the case of a call option) or sell (in the case of a put option) the underlying asset. The decision to exercise the option lies solely with the buyer, who will act based on whether the market conditions align with their expectations and if it is financially advantageous to do so. **Exercise price** The exercise price, also known as the strike price (STRIKE), is the predetermined price at which the seller of the option is required to either deliver (in the case of a call option) or purchase (in the case of a put option) the underlying asset if the buyer decides to exercise their right. This price is established when the option is initially negotiated and remains fixed throughout the life of the option, until its expiration. Whether the option is exercised or not depends on the relationship between the strike price and the market price of the underlying asset, which fluctuates over time. If the strike price is favorable compared to the market price, the buyer may choose to exercise the option; otherwise, they may let it expire.

118 **At the Money (ATM)** if the strike price is equal to the current price of the underlying asset. You can do what you like, but the profit is zero,

118 **Out The Money (OTM)** if the price of the underlying above the strike. It is in the option holder's interest not to exercise his option and the profit is nil.

118 **In the Money (ITM)** if the underlying price is below the strike, it is in the option holder's interest to exercise the option. The profit is the difference between the two prices.

The higher the strike price of an option, the cheaper the call, the cheaper the put and vice versa, since the holder of the call pays for it and the holder of the put receives the proceeds.

Expiry date

The further in the future the option matures, the greater the chance of anticipation, and the more expensive the call premium than the put premium.

Risk-free interest rate

Buying a call is like buying an asset and paying for it later if it is exercised at a risk-free interest rate. So the higher the interest rate, the more expensive the call, but the cheaper the put.

Underlying price volatility

It is measured by the standard deviation of the distribution of the asset's rate of return. The more volatile the asset price, the more likely it is to rise above the strike price (which favours the call) or fall below it (which favours the put). (which favours the call) or fall below it (which favours the put). So, the greater the volatility, the more expensive the option.

9.2.3 European call option price

In all that follows , we will use the following notations:

- S_0 : current price of the underlying asset;
- K: option exercise price;
- T: option expiry date;
- r: risk-free interest rate;
- σ: asset price volatility.

The theoretical value of a call option with an exercise price$K$, until an expiry date T, is given by its cash flow (Payoff)

$$max(S_t - K, 0) = (S_t - K, 0)^+$$

9.3 The Black-Sholes model and formula

9.3.1 Price trends

The model proposed by Black and Scholes to describe price movements is a continuous-time model with a risky asset with a risky asset (a share with price S_t at time t) and a risk-free asset (with price S_t^0). (with price S_t^0). The evolution of S_t^0 is assumed to be governed by the following (ordinary) differential equation:

$$dS_t^0 = rS_t^0 dt$$

where r is a positive constant. This means that the interest rate on the market for risk-free investments is constant and equal to r. (Note that r here is an instantaneous interest rate, not to be confused with the rate over a period in discrete models). Let us assume $S_0^0 = 1$, so that $S_t^0 = e^{rt}$, for $t \geq 0$ It is assumed that the evolution of the share price is governed by the following stochastic differential equation:

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

where μ and σ are two constants and (W_t) is standard Brownian motion The model is studied over the interval

$$[0, T]$$

where T is the option expiry date.

$$S_t = S_0 \exp(\mu t - \tfrac{\sigma^2}{2} t + \sigma W t)$$

where S_0 is the price observed at date 0. It follows in particular that, according to this model, the distribution of S_t is a lognormal distribution (i.e. its logarithm follows a normal distribution). More precisely, we see that the process (S_t) satisfies one of the above equations if and only if the process $(log(S_t))$ is a Brownian motion (not necessarily standard). the process (S_t) satisfies the following properties:

- continuity of trajectories;
- independence of relative increments:if $\mu \leq t$, S_t/S_u or (which amounts to the same thing) , the relative increase $(S_t - S_u)/S_u$ is independent of the tribeσ $(S_v, v \leq u)$
- stationarity of relative increases: if $u \leq t$, the law of $(S_t - S_u)/S_u$ the law ofthe law of $(S_{t-u} - S_0)/S_0$ These three properties give concrete expression to the Black Scholes hypotheses on the evolution of the share price.

9.3.2 Black-Scholes formula

Within the framework of the Black Scholes model, for certain pay-offs, there are explicit formulae which give their price in t. This is particularly the case for calls and puts. In the Black-Scholes model, the price of a call with maturity T and strike K is :

$$C_t = S_t \mathcal{N}(d_1) - Ke^{-r(T-t)} \mathcal{N}(d_2) \text{ , } t \in [0, \text{T}]$$

with $\mathcal{N}$ the distribution function of a normal distribution $\mathcal{N}(0,1)$, d_1 et d_2 donated by:

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The Call Put parity formula is written :

$$C_t - Pt = S_t - Ke^{-r(T-t)}, t \in [0,T]$$

And so the Put price is given by :

$$P_t = Ke^{-r(T-t)} \mathcal{N}(-d_2) - S_t \mathcal{N}(-d_1), t \in [0,T]$$

The Black-Scholes formula, using a numerical example

Now that we know the Black Scholes formula, we can apply it to the valuation of a European call, for example. of a European call, for example Let's take a call on Arcelourd, a company specialising in steel trading, whose future results look promising. future results look promising. The share is currently worth 80. Or a call with strike 90 and maturity one quarter. The risk-free interest rates for this period are *** Translated with www.DeepL.com/Translator (free version) ***

$\mathcal{N}(d_1) = 0.30$

$\mathcal{N}(d_2) = 0.25$

$call = 80\mathcal{N}(d_1) - 90e^{-0.05?0.25}\mathcal{N}(d_2) = 2.48 euros$ The implied volatility is estimated at 35.

Where : S = Share price , K = Strike of the option or "Exercise price" , r = risk-free rate, T = option maturity (in years), *sigma* = implied volatility of the underlying asset, $\mathcal{N}(x)$ = distribution function of the normal distribution.

The result is $S = 80, K = 90, T = 0,25, r = 0,05, \sigma = 0,35$

$$d_1 = \frac{\ln(\frac{80}{90}) + (0,05 + \frac{035^2}{2})0,25}{0,35\sqrt{0,25}}$$

$$d_2 = d_1 - 0.35\sqrt{0,25}$$

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