The democratic and Popular Republic of Algeria
Ministry of Higher Education and Scientific Research

University of Abderahmane Mira of Bejaia
Faculty of Technology
Automatic, Telecommunication and Electronic Department

COURSE HANDOUTS OF

CONTROL OF LINEAR DISCRET TIME SYSTEMS

Realized by

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Concerned students are
The 3^{rd} year and 6^{th} semester of
Automatic and Intelligent Systems Engineer
-ASI-

University year
2025-2026

Contents

CHAPTER ONE
STRUCTURE OF DIGITAL CONTROL SYSTEM

1.1 Introduction

Rapid advances in digital system technology have radically altered the control system design It has become routinely practicable to design very complicated digital controllers and to carry out the extensive calculations required for their design. These advances in implementation and design capability can be obtained at low cost because of the widespread availability of inexpensive and powerful digital computers and their related devices.

A *digital control system* uses digital hardware, usually in the form of a programmed digital computer, as the heart of the controller. A typical digital controller has analog components at its periphery to interface with the plant. It is the processing of the controller equations that distinguishes analog from digital control.

The signals used in the description of digital control systems are termed *discrete-time signals.* Discrete time signals are defined only for discrete instants of time, usually at evenly spaced time steps. Discrete-time computer-generated signals have discrete (or *quantized*) amplitudes and thus attain only discrete values. Figure 1.1 shows a continuous amplitude signal that is represented by a 3-bit binary code at evenly spaced time instants. In general, an *n*-bit binary code can represent only 2*n* different values. Because of

the complexity of dealing with quantized signals, digital control system design proceeds as if the signals involved are not of discrete amplitude. Further analysis usually must be performed to determine whether the proposed level of quantization is acceptable.

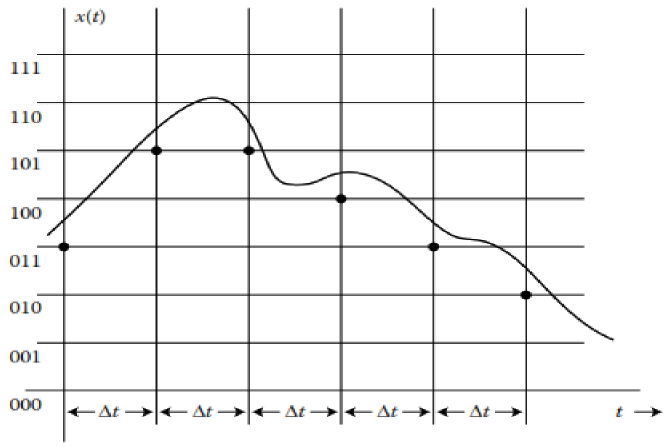


Figure 1.1: An example of a 3-bit quantized signal

A discrete-time system as a continuous time system is said to be *linear* if it satisfies the principle of *superposition.* Any linear combination of inputs produces the same linear combination of corresponding output components. If a system is not linear, then it is termed *nonlinear.* A discrete-time system is *invariant* if its properties do not change with time. Any time shift of the inputs produces an equal time shift of every corresponding output signal.

1.2 The structure of a digital control system

Figure 1.2 shows a block diagram of a typical digital control system for a continuous-time plant. The system has two reference inputs and five outputs, two of which are measured directly by analog sensors. The *analog-to-digital converters* (A/D) *sample* the analog sensor signals and produce equivalent binary representations of these signals. The sampled sensor signals are then modified by the digital controller algorithms, which are designed to produce the necessary digital control inputs $u_1(k)$ and $u_2(k)$. Consequently, the control inputs $u_1(k)$ and $u_2(k)$ are converted to analog signals $u_1(t)$ and $u_2(t)$ using *digital-to-analog converters* (D/A). The D/A transforms the digital codes to signal *samples* and then produces *step reconstruction* from the signal samples by transforming the binary-coded digital input to voltages. These voltages are held constant during the *sampling period T* until the next sample arrives. This process of holding each of the samples is termed *sample and hold.* Then the analog signals $u_1(t)$ and $u_2(t)$ are applied to control the behavior of the plant.

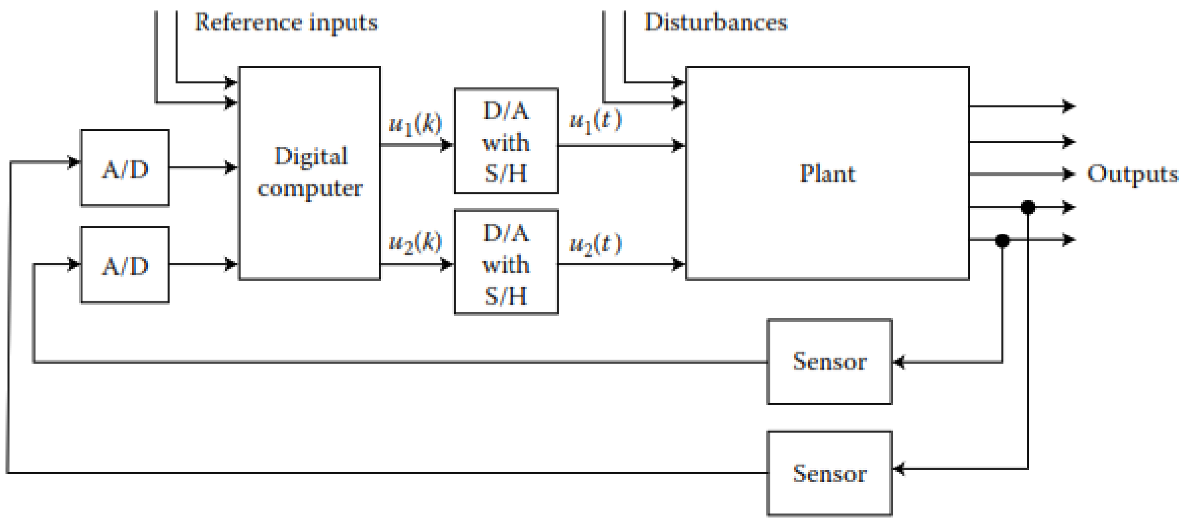


Figure 1.2: A digital control system controlling a continuous-time plant.

In Figure 1.2 is a real-time clock that synchronizes the actions of the A/D, D/A, and shift registers is not shown. Of course, there are many variations on this basic theme, including situations where the signals of the analog sensors are sampled at different sampling periods and where the system has many controllers with different sampling periods. Other examples include circumstances where:

- the A/D and D/A are not synchronized;
- the sampling rate is not fixed;
- the sensors produce digital signals directly;
- the A/D conversion is different from sample and hold;
- the actuators accept digital commands.

1.3 Advantages of digital control

Digital control offers distinct advantages over analog control such as:

Accuracy: Digital signals are represented in terms of zeros and ones with typically 12 bits or more to represent a single number. This involves a very small error as compared to continuous signals, where noise and power supply drift are always present.

Implementation errors: Digital processing of control signals involves addition and multiplication by stored numerical values. The errors that result from digital representation and arithmetic are negligible. By contrast, the processing of analog signals is performed using components such as resistors and capacitors with actual values that vary significantly from the nominal design values.

Flexibility: An analog controller is difficult to modify or redesign once implemented in hardware. A digital controller is implemented in firmware or software and its modification is possible without a complete replacement of the original controller. Furthermore, the structure of the digital controller need not follow one of the simple forms that are typically used in analog control. More complex controller structures involve a few extra arithmetic operations and are easily realizable.

Speed: The speed of computer hardware has increased exponentially since the 1980s. This increase in processing speed has made it possible to sample and process control signals at very high speeds. Because the interval between samples, the sampling period, can be made very small, digital controllers achieve performance that is essentially the same as that based on continuous monitoring of the controlled variable.

Cost: Although the prices of most goods and services have steadily increased, the cost of digital circuitry continues to decrease. Advances in semiconductor technology have made it possible to manufacture better, faster, and more reliable integrated circuits and to offer them to the consumer at a lower price. This has made the use of digital controllers more economical even for small, low-cost applications.

1.4 Examples of digital control systems

In this section, we briefly discuss examples of control systems where digital implementation is now the norm. There are many other examples of industrial processes that are digitally controlled, and the reader is encouraged to seek other examples from the literature.

1.4.1 Closed-loop drug delivery system

Several chronic diseases require the regulation of the patient's blood levels of a specific drug or hormone. For example, some diseases involve the failure of the body's natural closed-loop control of blood levels of nutrients. Most prominent among these is the disease diabetes, where the production of the hormone insulin that controls blood glucose levels is impaired. To design a closed-loop drug delivery system, a sensor is utilized to measure the levels of the regulated drug or nutrient in the blood. This measurement is converted to digital form and fed to the control computer, which drives a pump that injects the drug into the patient's blood. A block diagram of the drug delivery system is shown in Figure 1.3. Se

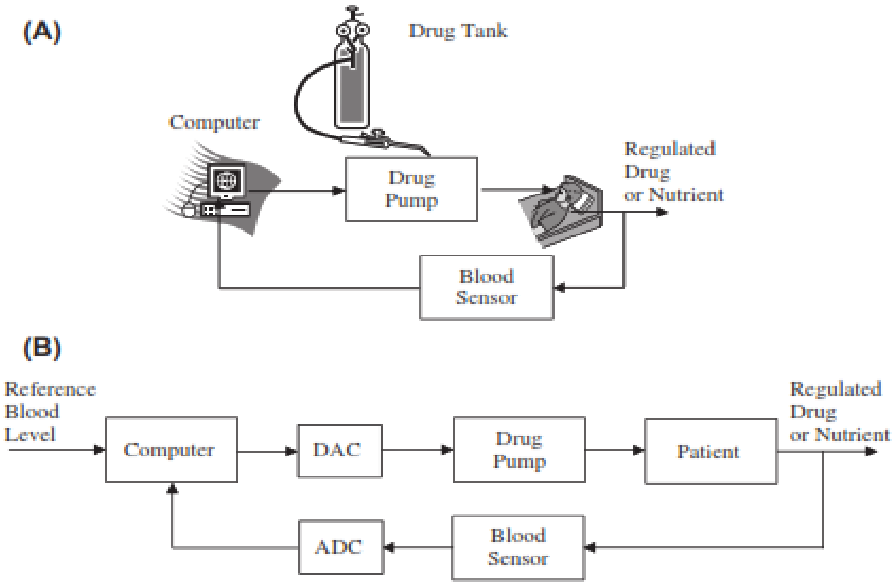


Figure 1.3: Drug delivery digital control system. (A) Schematic of a drug delivery system. (B) Block diagram

1.4.2 Computer control of an aircraft turbojet engine

To achieve the high performance required for today's aircraft, turbojet engines employ sophisticated computer control strategies. A simplified block diagram for turbojet computer control is shown in Figure 1.4. The control requires feedback of the engine state (speed, temperature, and pressure), measurements of the aircraft state (speed and direction), and pilot command.

(A)

(B)

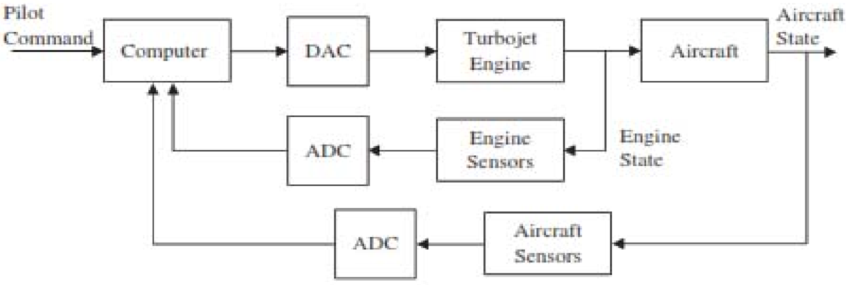


Figure 1.4: Turbojet engine control system. (A) F-22 military fighter aircraft. (B) Block diagram of an engine

1.5 A/D and D/A conversions

Although compensators are becoming digital, most plants are still analog. In order to connect digital controllers and analog plants, analog signals must be converted into digital signals and vice versa. These conversions can be achieved by using analog-to-digital (A/D) and digital-to analog (D/A) converters. We discuss how these conversions are achieved.

Consider the operational amplifier circuit shown in figure 1.5. It is essentially the output of this circuit is given by

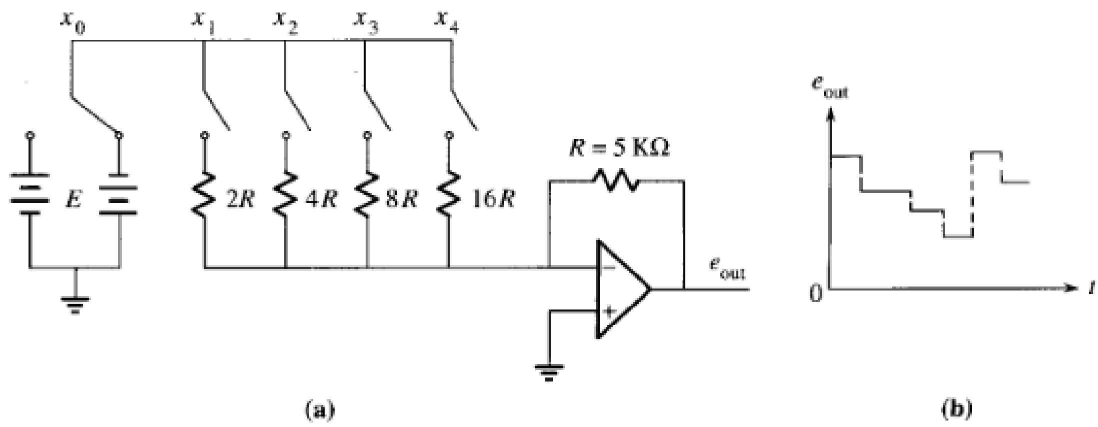


Figure 1.5: digital to analog converter (DAC)

$$v_o = -\left(x_1 \frac{R}{2R} + x_2 \frac{R}{4R} + x_3 \frac{R}{8R} + x_4 \frac{R}{16R}\right) E$$
$$= -(x_1 2^{-1} + x_2 2^{-2} + x_3 2^{-3} + x_4 2^{-4})E$$

$$v_o = -(1 \cdot 2^{-1} + 1 \cdot 2^{-3} + 1 \cdot 2^{-4}) \cdot (-10) = 0.6875$$

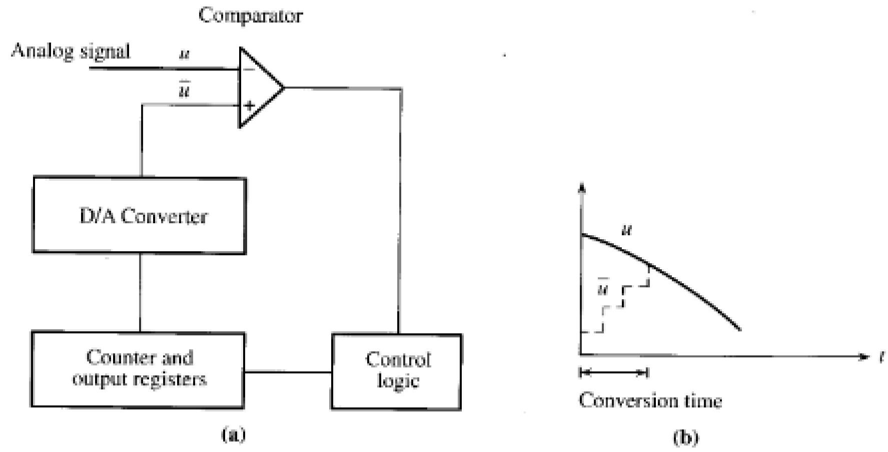


Figure 1.6: Analog-to-digital converter (ADC) and conversion time

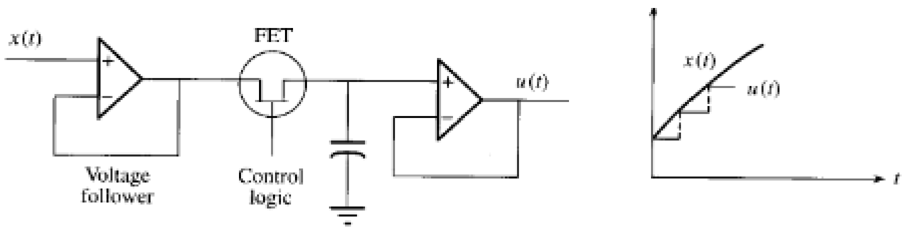


Figure 1.7: sample and hold circuit

CHAPTER TWO
SAMPLING OF SIGNALS

2.1 Introduction

In this chapter, we introduce the analysis of the sampling process and describe both a time domain and a frequency domain representation. From the companion process of data extrapolation or data holding, the continuous time signal can be represented. As part of this analysis we show that a sample-data system is made time varying by the introduction of sampling. However, a continuous time signal is recovered by the hold process and we can approximate the sinusoidal response of a sampler and hold by fitting another sinusoidal of the same frequency to the complete response. The section 2.2 the analysis of the sample and hold operation is considered and the section 2.3 the frequency analysis of the sample signal is given. Here the important phenomenon of signal aliasing caused by sampling is introduced. Section 2.4 the zero order hold and some of its generalizations are considered. Analysis of sampled-data systems in the frequency domain is introduced.

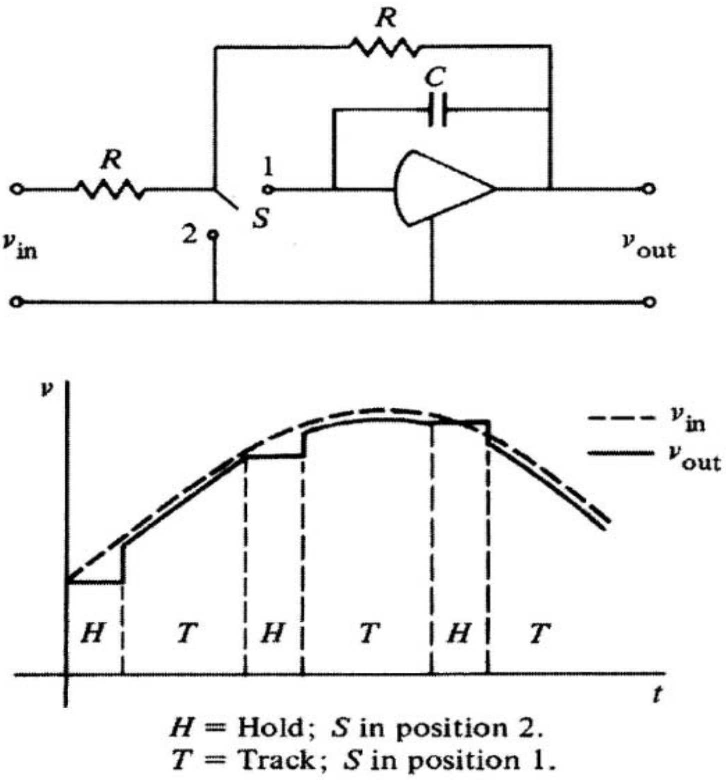


Figure 2.1: Analog-to-Digital Converter with Sample and Hold

2.2 Analysis of the Sample and Hold

To get sample of physical signal such as a position or a velocity into digital form, we typically have a sensor that produce a voltage proportional to the physical variable and an **analog-to-digital converter**, commonly called **A/D converter** or **ADC**, that transform the voltage into a digital number. The physical conversion takes a non-zero time, and in many instances this time is significant with respect to the sample period of the control or with respect to the rate of change of the signal to be sampled. In order to give the computer an accurate representation of the signal exactly at the sampling instants kT, the A/D converter is typically preceded by a **sample-and-hold circuit (SHC)** as sketched in figure 2.1, its operation described as follows.

With the switch, S in position 1, the amplifier output $v_{out}(t)$ tracks the input $v_{in}(t)$ through the transfer function $1/(1+RCs)$. The circuit bandwidth of the SHC, $1/RC$, is selected to be high compared to the input signal bandwidth. Typical values $R = 1000\, ohms, C = 30 \times 10^{-12} Farads$ for a bandwidth of $f = \frac{1}{2\pi RC} = 5.3\ MHz$. When a sample is to be taken at $t = kT$ the switch S is to position 2 and the capacitor C holds the output of the amplifier frozen from that time at $v_{out}(kT) = v_{in}(kT)$ is now signaled to begin conversion of the constant input from the SHC into a digital number which will be a true representation of input voltage at a sample instant. The same operation continued for all sample instants.

For the purpose of analysis, we separate the sample and the hold into two mathematical operations: a sampling operation represented by impulse modulation and a hold operation represented as a linear filter.

2.2.1 Impulse modulation

The schematic of the ideal sampler is shown in figure 2.2.

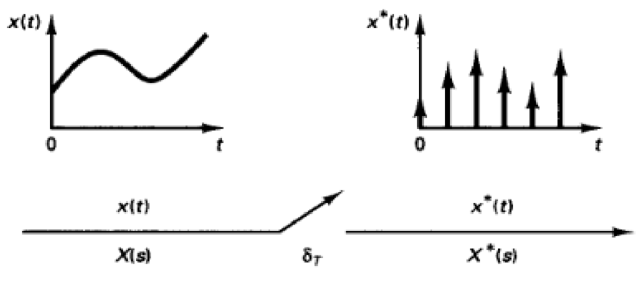


Figure 2.2 : the impulse sampler

We shall consider a fictitious sample commonly called an impulse sampler. The output of this simpler is considered to be a train of impulses that begins with $t = 0$ with the sampling period equal to T and the strength of each impulse equal to sampled value of continuous-time signal at the corresponding sampling instants. The sampled signal $x^*(t)$, a train impulses, can thus be represented by the infinite summation.

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\,\delta(t-kT) \tag{2.1}$$

Or

$$x^*(t) = x(0)\delta(0) + x(T)\delta(t-T) + \cdots + x(kT)\delta(t-kT) + \cdots$$

We shall define a train of unit impulses as $\delta_T(t)$, or

$$\delta_T(t) = \sum_{k=0}^{\infty} \delta(t-kT) \tag{2.2}$$

The sampler may be considered as a modulator with the input $x(t)$ as a modulating signal and the train of unit impulses $\delta_T(t)$ as the carrier as shown in figure 2.3.

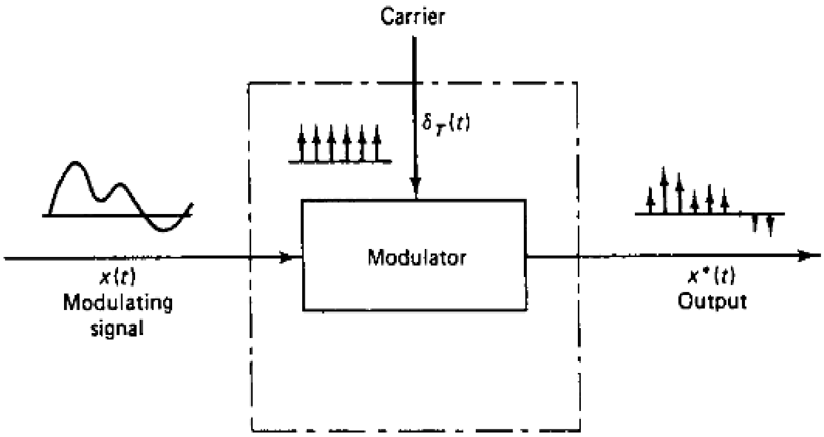


Figure 2.3 : Impulse sampler as a modulator

Next, we consider the Laplace transform of equation (2.1):

$$X^*(s) = \mathcal{L}\big(x^*(t)\big) = x(0)\mathcal{L}\big(\delta(t)\big) + x(T)\mathcal{L}\big(\delta(t-T)\big) + \cdots + x(kT)\mathcal{L}\big(\delta(t-kT)\big) + \cdots$$

$$= x(0) + x(T)e^{-Ts} + \cdots + x(kT)e^{-kTs} + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs} \tag{2.3}$$

Notice that if we define

$$e^{Ts} = z \;\Rightarrow s = \frac{1}{T}\ln(z)$$

Then equation (2.3) becomes

$$X^*(s)\big|_{s=\frac{1}{T}\ln(z)} = \sum_{k=0}^{\infty} x(kT)z^{-k} \tag{2.4}$$

It is the z transform of the sequence $x(0)$, $x(T)$, …, $x(kT)$,…generated from $x(t)$ at $t = kT$.

$$X^*(s)\big|_{s=\frac{1}{T}\ln(z)} = X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} \tag{2.5}$$

- Note that $\delta(t-kT_s) = 0$ unless $t = kT_s$
- $\mathcal{L}(\,\delta_T(t)) = 1/(1-e^{-Ts})$

Note that the variable z is a complex variable and T is the sampling period. It should be stressed that the notation $X(z)$ does not signify $x(s)$ with s replaced by z, but rather $X^*\left(s=\frac{1}{T}\ln(z)\right)$.

2.2.2 Data-Hold Circuit

In a conventional sampler, a switch closes to admit an input signal every sampling period. In practice, the sampling duration is very short in comparison with the most significant time constant of the plant. A sampler converts a continuous-time signal into a train of pulses occurring at the instants $t=0, T, \dots kT, \dots$ where T is the sampling period. (Note that between any two consecutive sampling instants the sampler transmits no information).

Data-Hold is a process of generating a continuous-time signal $h(t)$ from the discrete-time sequence $x(kT)$. A hold circuit converts the sampled signal into a continuous-time signal, which approximately reproduces the signal applied to the sampler. The signal $h(t)$ during the interval $kT \leq t < (k+1)T$ may be approximated by a polynomial in τ as follows.

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \dots + a_1\tau + a_0 \qquad (2.6)$$

where $0 \leq \tau < T$. Note that signal $h(kT)$ must equal $x(kT)$, or

$$h(kT) = x(kT)$$

Hence, equation (2.6) can be written as follows:

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \dots + a_1\tau + x(kT) \qquad (2.7)$$

If the data-hold circuit is an n^{th}order polynomial extrapolator, it is called an n^{th}order hold. Thus, if $n=1$ it is called a first order hold. The data-hold circuit is n^{th}order hold uses the past $(n+1)$ discrete data $x\big((k-n)T\big), \dots, x(kT)$ to generate a signal $h(kT+\tau)$.

Because a higher-order hold uses samples to extrapolate a continuous-time signal between the present sampling instant and the next sampling instant, the accuracy of the approximation is improved as the number of past samples used is increased. However, this better accuracy is obtained at the cost of greater time delay. In closed loop control systems, any added time delay in the loop will decrease the stability of the system and in some cases may even make the system instable.

The simplest data-hold is obtained when $n=0$ in equation (2.7), that is, when

$$h(kT+\tau) = x(kT) \qquad (2.8)$$

Figure 2.4 shows a sampler and zero-order hold. In this course, unless, otherwise stated, we consider the zero-order hold circuit, because is the simplest and is the most used in practice. It will be seen later in chapter 3, the transfer function.

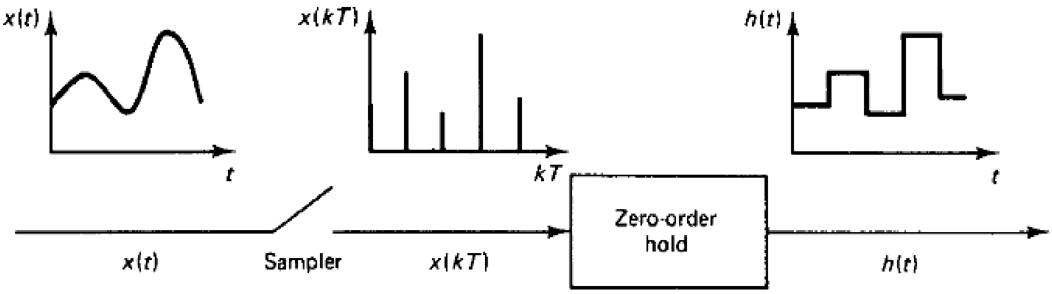


Figure 2.4: sampler and zero-order hold

$$G_{h0} = \frac{1 - e^{-sT}}{s} \tag{2.9}$$

2.2.3 Zero-order Hold

Figure 2.4 shows a sampler and zero-order hold. The input signal $x(t)$ is sampled at discrete instants and the sampled signal is passed through the zero-order hold circuit smooths the sampled signals to produce the signal $h(t)$, which is constant from the last sampled value until the next sample is available. That is,

$$h(kT + \tau) = x(kT), \qquad for\ 0 \le \tau < T \tag{2.10}$$

consider the sampled signal and zero-order hold, assume that the signal $x(t)$ is causal (is zero for $t < 0$) then the output signal is related to $x(t)$ as follows:

$$\mathcal{L}\big(h(t)\big) = H(s) = \frac{(1 - e^{-kTs})}{s} X^*(s) = G_{h0}(s)X^*(s) \tag{2.11}$$

Where $X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$.

2.3 Reconstruction of original signals from sampled signal

2.3.1 Sampling theorem

If the sampling frequency is sufficiently high compared with the highest-frequency component involved in the continuous-time signal, the amplitude characteristics of the continuous time signal may be preserved in the envelope of sampled signal.

To reconstruct the original signal from a sampled signal there is a certain minimum frequency that the sampling operation must satisfy. Such a minimum frequency is specified in *the sampling theorem*. We shall assume that a continuous time signal $x(t)$ has a frequency spectrum as shown in figure 2.5, this signal does not contain any frequency component above ω_1 radian per second.

Sampling theorem: If ω_s defined as $2\pi f_s = 2\pi/T$, where T is the sampling period, is greater than $2\omega_1$, or $\omega_s > 2\omega_1$, where $\omega_1$is the highest frequency component present in continuous time signal $x(t)$, then the signal can be reconstructed completely from the sampled signal $x^*(t)$.

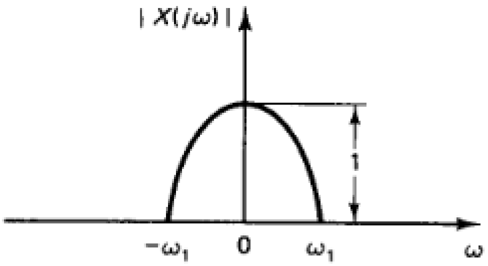


Figure 2.5: A frequency spectrum

The Laplace transform $x^*(t)$ is given by

$$X^*(s) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(s + jk\omega_s) \quad (2.12)$$

We substitute $j\omega$ for s in equation (2.12), thus,

$$X^*(s) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j\omega + jk\omega_s)$$

$$= \cdots + \frac{1}{T}X\big(j(\omega - \omega_s)\big) + \frac{1}{T}X(j\omega) + \frac{1}{T}X\big(j(\omega + \omega_s)\big) + \cdots \quad (2.13)$$

This equation gives the frequency spectrum of the sampled signal $x^*(t)$. We see that the frequency spectrum of the impulse response signal is reproduced an infinite number of times and is attenuated by the factor $1/T$. Since $X^*(s)$ is periodic with period $2\pi/\omega_s$ we have,

$$X^*(s) = X^*(s \mp j\omega_s k)\,,\ k = 0, 1, 2, \ldots$$

If a function $X(s)$ has root at $s = s_1$, then $X^*(s)$ has roots at $s = s_1 \mp j\omega_s k$ for $(k = 0, 1, 2, \ldots)$. Figure 2.6 shows the frequency spectra $X^*(s)$ versus ω.

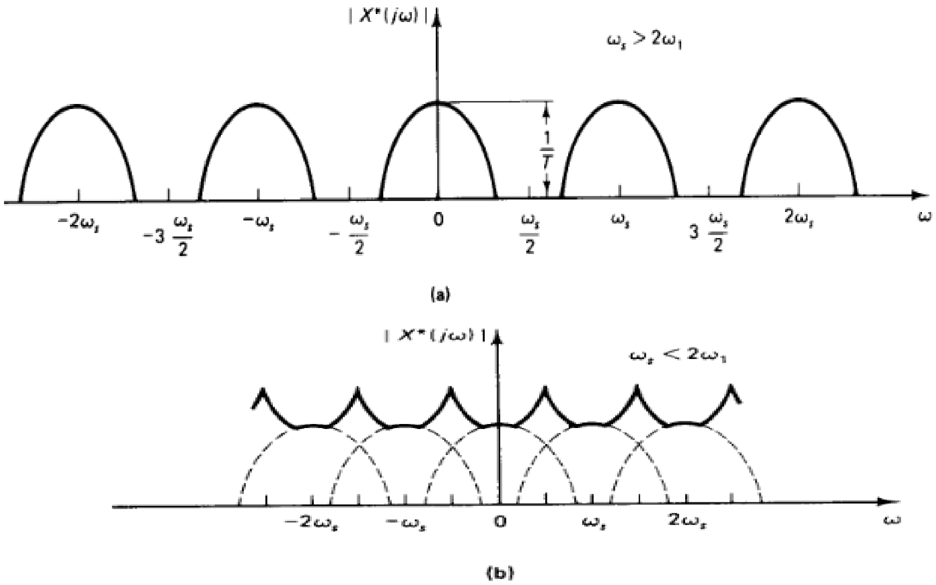


Figure 2.6: plot of the frequency spectra $|X^*(j\omega)|$ versus ω for two values of sampling frequency ω_s a) $\omega_s > 2\omega_1$ b) $\omega_s < 2\omega_1$

For two values of sampling frequencies ω_s. Figure 2.6 (a) corresponds to $\omega_s > 2\omega_1$, while figure 2.6 (b) corresponds to $\omega_s < 2\omega_1$. Each plot of $|X^*(j\omega)|$ versus ω consists of $|X(j\omega)|/T$ repeated every $\omega_s = 2\pi/T$ rad/sec. In the frequency spectrum of $|X^*(j\omega)|$ the component $|X(j\omega)|/T$ is called the primary component and the other components $|X(j(\omega \mp \omega_s k))|/T$ are called complementary components.

If $\omega_s > 2\omega_1$, no two components of $|X^*(j\omega)|$ will overlap, and the sampled frequency spectrum will be repeated every ω_s rad/sec.

If $\omega_s < 2\omega_1$, the original shape of $|X(j\omega)|$ no longer appears in the plot of $|X^*(j\omega)|$versus ω because of the superposition of the spectra. Therefore, we see that the continuous time signal $x(t)$ can be reconstructed from the impulse-sampled signal $x^*(t)$ by filtering if and only if $\omega_s > 2\omega_1$, this condition is the requirement on the minimum sampling frequency, which is specified by the sampling theorem as $\omega_s > 2\omega_1$, where $\omega_1$is the highest frequency-component presented in the signal, *practical consideration on the stability of the closed-loop system and other design considerations may make it necessary to sample at a frequency much higher than this theoretical minimum (frequently ω_s is chosen to be $10\omega_1$ to $20\omega_1$.*

An ideal filter will be introduced, in order to attenuate all such complementary components to zero and will pass only the primary component, provided the sampling frequency ω_s is greater than twice the highest frequency component of the continuous-time signal. Such an ideal filter reconstructs the continuous signal represented by the samples. Figure 2.7 shows the frequency spectra of the signals before and after ideal filtering.

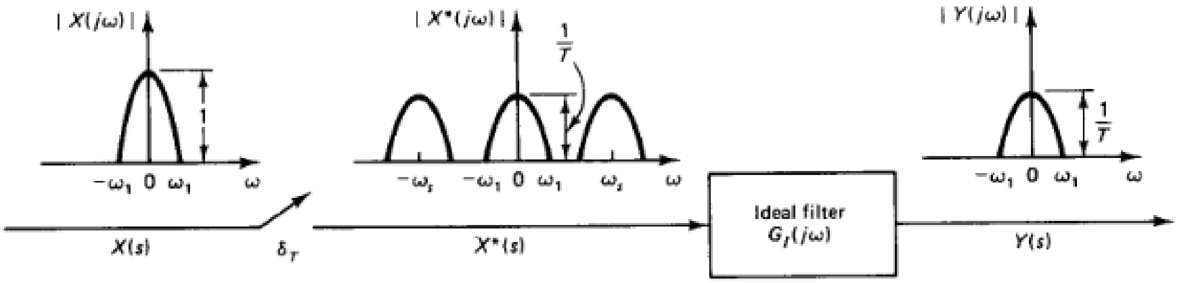


Figure 2.7: Frequency spectra of the signals before and after ideal filtering

2.3.2 Impulse response of the ideal filter

It will be shown that the output of ideal filter is required prior to the application of the output filter. Thus, it is not physically realizable.

Since the frequency spectrum of the ideal filter is given by as shown in figure 2.8.

$$G_I(j\omega) = \begin{cases} 1 & -\frac{1}{2}\omega_s \le \omega \le \frac{1}{2}\omega_s \\ 0 & elsewhere \end{cases} \tag{2.14}$$

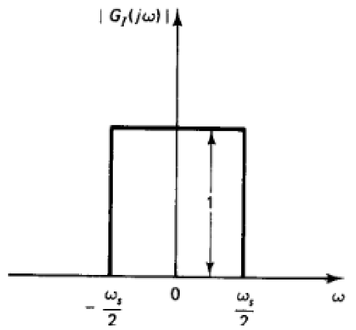


Figure 2.8: Amplitude frequency spectrum of the ideal low-pass filter

The inverse Fourier transform of the frequency spectrum gives

$$g_l(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G_l(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} e^{j\omega t}d\omega = \frac{1}{2\pi jt}\left(e^{\frac{j\omega_s}{2}t} - e^{-\frac{j\omega_s}{2}t}\right) = \frac{1}{\pi t}\sin\left(\frac{\omega_s t}{2}\right)$$

$$g_l(t) = \frac{1}{T}\frac{\sin\left(\frac{\omega_s t}{2}\right)}{\frac{\omega_s t}{2}} \tag{2.15}$$

The figure 2.9 shows a plot of $g_l(t)$ versus t.

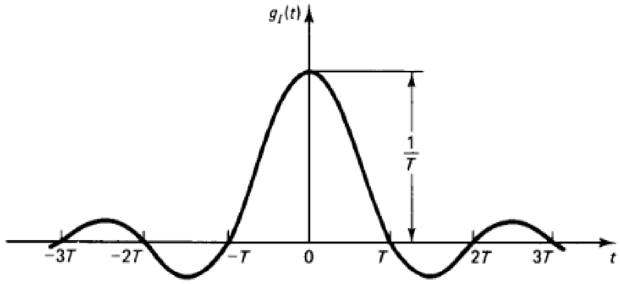


Figure 2.9: impulse response $g_l(t)$ of ideal filter

The approximation of this ideal filter $g_l(t)$ closely by adding a phase lag, which means adding a delay to the filter. In feedback control systems, increasing phase lag is not desirable from the viewpoint of stability. Therefore, we avoid to adding the phase lag to approximate the ideal filter.

2.3.3 Frequency response characteristics of the Zero-Order Hold

The transfer function of a zero-order hold is

$$G_{h0}(s) = \frac{(1 - e^{-Ts})}{s} \tag{2.16}$$

To compare the zero-order hold with the ideal filter, we shall obtain the frequency response characteristics of the transfer function (2.16) by substituting $j\omega$ for s, we obtain.

$$G_{h0}(j\omega) = \frac{\left(1 - e^{-Tj\omega}\right)}{j\omega} = \frac{Te^{-\frac{1}{2}T\omega j}\left(e^{\frac{1}{2}T\omega j} - e^{-\frac{1}{2}T\omega j}\right)}{\frac{2j(T\omega)}{2}} = \frac{Tsin\left(\frac{T\omega}{2}\right)}{\frac{T\omega}{2}}e^{-\frac{1}{2}T\omega j} \tag{2.17}$$

The amplitude of the frequency spectrum of $G_{h0}(j\omega)$ is

$$G_{h0}(j\omega) = T\left|\frac{sin\left(\frac{T\omega}{2}\right)}{\frac{T\omega}{2}}\right| \tag{2.18}$$

$$\angle\, G_{h0}(j\omega) = \angle\frac{Tsin\left(\frac{T\omega}{2}\right)}{\frac{T\omega}{2}}\angle e^{-\frac{1}{2}T\omega j} = \angle sin\left(\frac{T\omega}{2}\right) - \frac{T\omega}{2} \tag{2.19}$$

$$\angle sin\left(\frac{T\omega}{2}\right) = 0\ or\ \pm 180$$

The magnitude becomes zero at the frequency equal to the sampling frequency as in figure.2.10. Since the magnitude characteristics of the zero-order hold are not constant, if the is connected to the sampler and zero-order hold, distortion of the frequency occurs in the system.

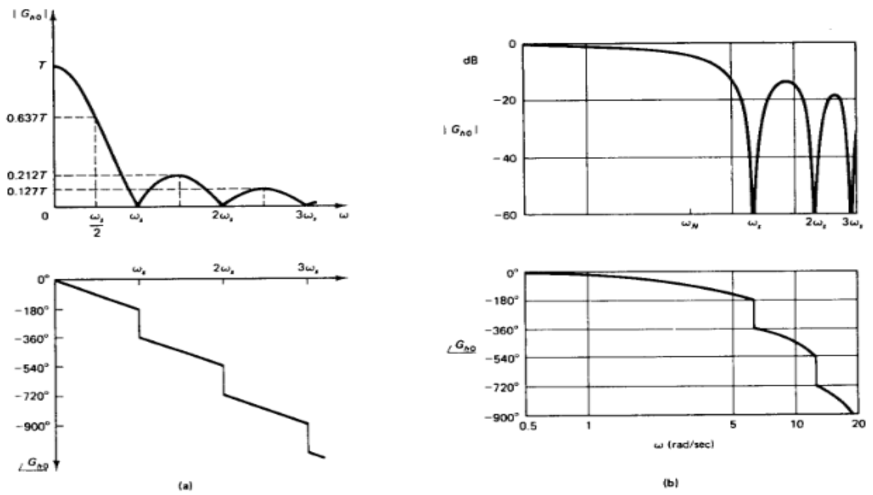


Figure 2.10: a) frequency-response curves for the zero-order hold; b) equivalent Bode diagram for $T = 1\ sec$

Figure 2.11 shows the comparison of the ideal filter and the zero-order hold for the sake of comparison. We see that the zero-order hold can be considered as a low pass filter, its accuracy depends on the choice of sampling time T. The output of the hold may be made as close as possible, when the sampling time is small enough as practically possible.

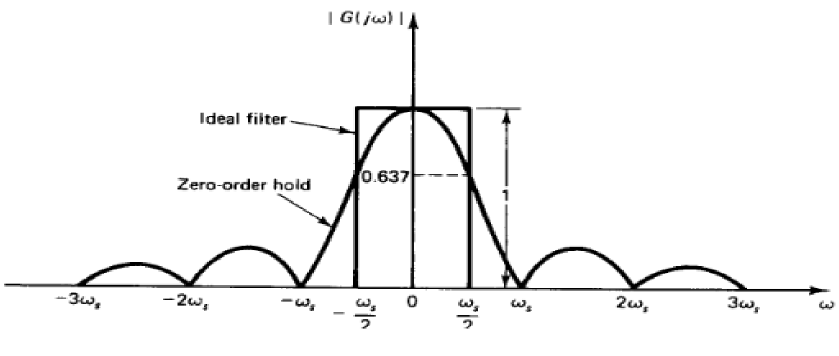


Figure 2.11: comparison of the ideal filter and the zero-order hold

2.3.4 Folding

As shown in figure 2.12, this phenomenon of the overlap in the frequency of spectra is known as *folding*. The frequency $\frac{1}{2}\omega_s$ is called the folding frequency or *Nyquist frequency* ω_N.

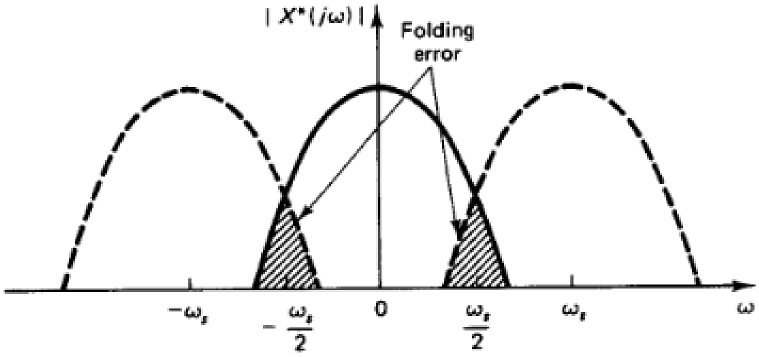


Figure 2.12: Diagram of folding region

CHAPTER THREE

REPRESENTATION OF DISCRETE-TIME SYSTEMS

3.1 Introduction

Continuous systems are designed and analyzed with the use of Laplace transforms. On the other hand, discrete-time systems are represented using a similar technique called z-transform. The basic ideas of reasoning are the same for both scenarios: After determining the impulse response of the system, the response of any other input signal can be extracted by simple arithmetic operations. The behavior and the stability of the system can be predicted from the zeros and poles of the transfer function. As Laplace transform converts the differential equations into algebraic terms with respect to s, z-transform converts the difference equations into algebraic terms with respect to z. Both transformations are matching a complex quantity to the points of a region of the complex plane. It should be noted that the z-plane (i.e., the domain of z-transform) is organized in a polar form, while the s-plane (i.e., the domain of Laplace transform) is in a Cartesian form.

3.2 The History of the Z-Transform

The history of the z-transform goes back to the work of the French mathematician De Moivre, who in 1730 introduced the characteristic function to represent the probability mass function of a discrete random variable. The characteristic function is identical to the z-transform. Also, the z-transform is a special case of the Laurent's series, used to represent complex functions. In the 1950s the Russian engineer and mathematician Yakov Tsypkin (1919–1997) proposed the discrete Laplace transform, which he applied to the study of pulsed systems.

Then Professor John Ragazzini and his students Eliahu Jury and Lofti Zadeh at Columbia University developed the z-transform. Ragazzini (1912–1988) was chairman of the Department of Electrical Engineering at Columbia University. Three of his students are well recognized in electrical engineering for their accomplishments: Jury for the z-transform, nonlinear systems, and the inners stability theory; Zadeh for the z-transform and fuzzy set theory; and Rudolf Kalman for the Kalman filtering. Jury was born in Iraq, and received his doctor of engineering science degree from Columbia University in 1953. He was professor of electrical engineering at the University of California, Berkeley, and at the University of Miami. Among his publications, Professor Jury's "Theory and Application of the z-transform," is a seminal work on the theory and application of the z-transform[*].

[*] L. F. Chaparro "Signals and systems using Matlab" , Elsevier Inc, 2011.

3.3 From Laplace Transform to z-Transform

The z-transform greatly facilitates the study and design of linear time varying discrete-time systems, because it transforms the difference equation that describes the system into an algebraic equation. In Figure 3.1, the procedure followed by using z-transform, where there are three steps to resolve the difference equation (D.E.) and the direct solution of the given D.E. via higher mathematics, which is much more laborious, are given. To show that z- and Laplace transforms are two parallel techniques, the Laplace transform, which is already known, will be used, and capitalizing on it, the mathematical expression of z-transform will be developed.

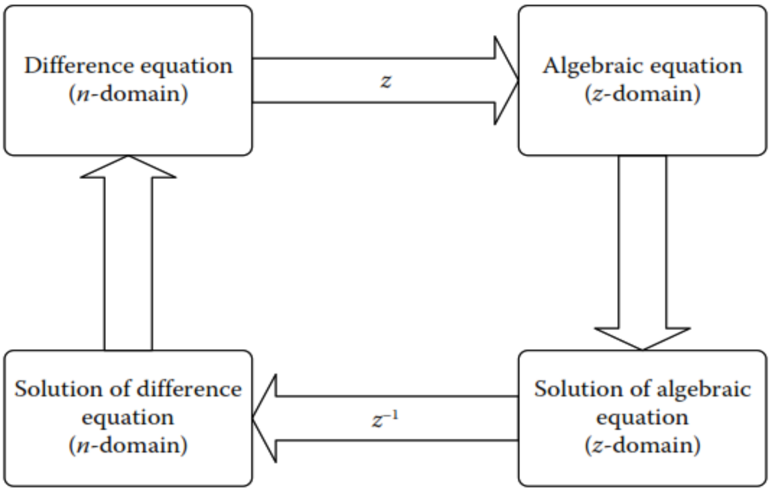


Figure 3.1: Solution of D.E. using z-transform

3.4 Definition of z-Transform

The z-transform is an important tool in the analysis and design of discrete-time systems. It simplifies the solution of discrete-time problems by converting LTI difference equations to algebraic equations and convolution to multiplication. Thus, it plays a role similar to that served by Laplace transforms in continuous-time problems. Because we are primarily interested in application to digital control systems, this brief introduction to the z-transform is restricted to causal signals (i.e., signals with zero values for negative time) and the one-sided z-transform.

Definition 3.1

Given the causal sequence $\{x(k)\}$ for $k \in \mathbb{N}$ and z is a complex variable $z = re^{j\theta}$. We define z-transform of $\{x(k)\}$ as the sum of the following series;

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \qquad |z| > R_0 \tag{3.1}$$

Notation: $X(z) = \mathcal{Z}\big(x(k)\big)$

The variable z^{-1} in equation (3.1) can be regarded as a time delay operator. The z-transform of a given sequence can be easily obtained as in the following example.

Example 3.1: obtain the z-transform of the sequence $\{x(k)\}_{k=0}^{\infty} = \{1, 3, 2, 0, 4, 0, 0, 0, \dots\}$.

Solution

applying the equation (3.1) gives $X(z) = 1 + 3z^{-1} + 2z^{-2} + 4z^{-4}$

3.5 z-Transform of standard discrete-time signals

Having defined the z-transform, we now obtain the z-transforms of commonly used discrete-time signals such as the sampled step, exponential, and the discrete-time impulse. The following identities are used repeatedly to derive several important results:

$$\begin{cases} \displaystyle\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} & a \neq 0 \\ \displaystyle\sum_{k=0}^{\infty} a^k = \frac{1}{1 - a} & |a| < 1 \end{cases} \tag{3.2}$$

Example 3.2: consider the time impulse (Figure 3.2)

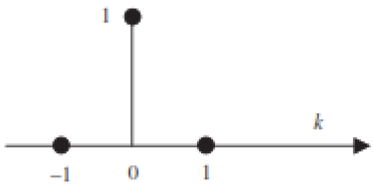


Figure 3.2: Discrete time impulse

$$x(k) = \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \qquad X(z) = 1$$

Example 3.3: consider the sampled unit step (Figure 3.3)

Consider the sequence $\{x(k)\}_{k=0}^{\infty} = \{1, 1, 1, 1, 1, 1, 1, 1, \dots\}$ of figure 3.3

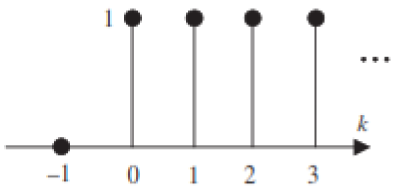


Figure 3.3: Sampled unit step

$$X(z) = 1 + z^{-1} + z^{-2} + \cdots + z^{-k} + \cdots = \sum_{k=0}^{\infty} z^{-k}$$

Using the identity (3.2) by replacing $a = z^{-1}$ gives the following expression of z-transform

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Example 3.4: consider the exponential (Figure 3.4)

$$x(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

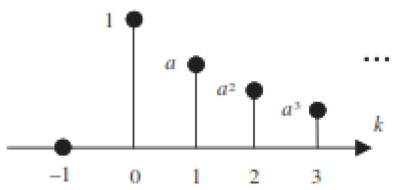


Figure 3.4: Sampled exponential

$$X(z) = 1 + az^{-1} + a^2z^{-2} + \cdots + a^kz^{-k} + \cdots = \sum_{k=0}^{\infty} a^kz^{-k} \qquad 0 < a < 1$$

Using (3.2), we obtain

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \qquad |z| > |a|$$

For the region of convergence, see the figure 3.5

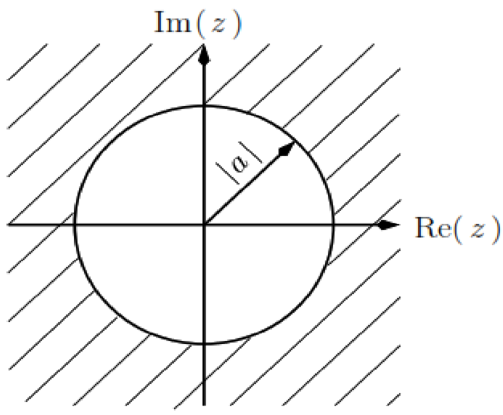


Figure 3.5: Region of convergence of z-transform of $a^ku(k)$

3.6 Properties of z-transform

The z-transform can be derived from the Laplace transform as shown in chapter two. Hence, it shares several useful properties with the Laplace transform, which can be stated without proof. These properties can also be easily proved directly, and the proofs are left as an exercise for the reader. Proofs are provided for properties that do not obviously follow from the Laplace transform.

3.6.1 Linearity

In the same way of as in the Laplace transform.

$$\mathcal{Z}\big(\alpha_1x_1(k) + \alpha_2x_2(k)\big) = \alpha_1X_1(z) + \alpha_2X_2(z), \qquad \alpha_1, \alpha_2 \in \mathbb{R} \tag{3.3}$$

Example 3.5: Find the z-transform of the causal sequence.

$$x(k) = 2u(k) + 4\delta(k)$$

Solution: using the property of linearity, the z-transform of the sequence

$$X(z) = \mathcal{Z}\big(2u(k) + 4\delta(k)\big) = 2\mathcal{Z}\big(u(k)\big) + 4\mathcal{Z}\big(\delta(k)\big)$$

$$= \frac{2z}{z-1} + 4 = \frac{6z-4}{z-1}$$

3.6.2 Time delay

This equation follows from the time delay property of the Laplace transform

$$\mathcal{Z}\big(x(k-n)\big) = z^{-n}X(z) \tag{3.4}$$

Proof

The z-transform of $y(k) = x(k-n)$ is given by

$$Y(z) = \textstyle\sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} x(k-n)z^{-k}$$

With a variable change $m = k - n$ we get

$$Y(z) = z^{-n}\sum_{m=-n}^{\infty} x(m)z^{-m} = z^{-n}\sum_{m=0}^{\infty} x(m)z^{-m} = z^{-n}X(z)$$

Example 3.6: find the z-transform of the causal sequence

$$x(k) = \begin{cases} 4 & k = 2,3,\ldots \\ 0, & otherwise \end{cases}$$

Solution: The given sequence is a sampled step starting at $k = 2$ rather than $k = 0$ (i.e., it is delayed by two sampling periods). Using the delay property, we have

$$X(z) = \mathcal{Z}\big(4u(k-2)\big) = 4z^{-2}\mathcal{Z}\big(u(k)\big) = \frac{4z^{-2}z}{z-1} = \frac{4}{z(z-1)}$$

3.6.3 Time advance

$$\mathcal{Z}\{x(k+1)\} = zX(z) - zx(0)$$

$$\mathcal{Z}\{x(k+n)\} = z^{n}X(z) - z^{n}x(0) - z^{n-1}x(1) - \cdots - zx(n-1) \tag{3.5}$$

Proof: Only the first part of the theorem is proved here. The second part can be easily proved by induction. We begin by applying the z-transform to a discrete-time function advanced by one sampling interval. This gives

$$\mathcal{Z}\{x(k+1)\} = \sum_{k=0}^{\infty} x(k+1)z^{-k} = z\sum_{k=0}^{\infty} x(k+1)z^{-(k+1)}$$

Now add and subtract the initial condition $x(0)$ to obtain

$$\mathcal{Z}\{x(k+1)\} = z\left(\left(x(0) + \sum_{k=0}^{\infty} x(k+1)z^{-(k+1)}\right) - x(0)\right)$$

Next, change the index of summation to $m = k + 1$ and rewrite the z-transform as

$$\mathcal{Z}\{x(k+1)\} = z\left(\sum_{m=0}^{\infty} x(m)z^{-m} - x(0)\right) = zX(z) - zx(0)$$

Example 3.7: Using the time advance property, find the z-transform of the causal sequence

$$\{x(k)\} = \{4, 8, 16, \ldots\}$$

Solution: The sequence can be written as

$x(k) = 2^{k+2} = y(k+2), \qquad k = 0, 1, 2, \ldots$

Where $y(k) = 2^k,\ k = 0, 1, 2, \ldots$

Using the time advance property, we write the transform

$$X(z) = z^2 Y(z) - z^2 y(0) - zy(1) = \frac{z^2 z}{z-2} - z^2 - 2z = \frac{4z}{z-2}$$

Clearly, the solution can be obtained directly by rewriting the sequence as

$$\{x(k)\} = 4\{1, 2, 4, \ldots\}$$

and using the linearity of the z-transform.

3.6.4 Multiplication by exponential

$$\mathcal{Z}\{a^{-k} x(k)\} = X(az) \tag{3.6}$$

Proof:

$$\mathcal{Z}\{a^{-k} x(k)\} = \sum_{k=0}^{\infty} a^{-k} x(k) z^{-k} = z \sum_{k=0}^{\infty} x(k)(az)^{-k} = X(az)$$

Example 3.8: Find the z-transform of the sampled ramp sequence

$$x(k) = e^{-\alpha kT}, k = 0, 1, 2, 3, \ldots$$

<u>**Solution**</u>

Observe that $x(k)$ can be rewritten as

$$x(k) = (e^{\alpha T})^{-k}, k = 0, 1, 2, 3, \ldots$$

Then apply the multiplication by exponential property to obtain

$$\mathcal{Z}\{(e^{\alpha T})^{-k}\} = \sum_{k=0}^{\infty} (e^{\alpha T})^{-k} z^{-k} = z \sum_{k=0}^{\infty} (e^{\alpha T} z)^{-k} = \frac{z}{z - e^{-\alpha T}}$$

3.6.5 Complex differentiation

$$\mathcal{Z}\{k^m x(k)\} = \left(-z \frac{d}{dz}\right)^m X(z) \tag{3.7}$$

Example 3.9: Find the z-transform of the sampled ramp sequence

$$x(k) = k, k = 0,1, 2, \ldots$$

<u>**Solution**</u>

Recall that the z-transform of a sampled step is

$$U(z) = \frac{z}{z-1}$$

And observe that $x(k)$ can be rewritten as

$$x(k) = ku(t), k = 0,1, 2, \ldots$$

Then apply the complex differentiation property to obtain

$$X(z) = -\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = -\frac{z(z-1-z)}{(z-1)^2} = \frac{z}{(z-1)^2}$$

3.6.6 The Initial value theorem

Suppose $x(k)$ is a causal sequence and $X(z)$ its z-transform:

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Making z tends to infinity then, we obtain the value at the instant 0 :

$$x(0) = \lim_{k\to 0} x(k) = \lim_{z\to\infty} X(z) \tag{3.8}$$

3.6.7 The final value theorem

The final value theorem allows us to calculate the limit of a sequence as k tends to infinity, if one exists, from the z-transform of the sequence. If one is only interested in the final value of the sequence, this constitutes a significant shortcut. The main pitfall of the theorem is that there are important cases where the limit does not exist. The two main cases are as follows:

1) an unbounded sequence

2) an oscillatory sequence

The reader is cautioned against blindly using the final value theorem, because this can yield misleading results.

Theorem 3.2: If a sequence approaches a constant limit as k tends to infinity, then the limit is given by

$$x(\infty) = \lim_{k\to\infty} x(k) = \lim_{z\to 1}\frac{z-1}{z}X(z) = \lim_{z\to 1}(z-1)X(z) \tag{3.9}$$

Proof

Either $x(k)$ a causal sequence and $X(z)$ its z-transform. We calculate $\mathcal{Z}[x(k+1)-x(k)]$ in two differe∞nt ways.

Using in the one hand the property of advance (sub-section 3.6.3), we easily established that:

$$\mathcal{Z}[x(k+1)-x(k)] = \mathcal{Z}[x(k+1)] - \mathcal{Z}[x(k)] = (z-1)X(z) - zx(0)$$

Applying in the other hand the definition of z-transform, we obtain:

$$\mathcal{Z}[x(k+1)-x(k)] = \lim_{N\to\infty}\sum_{k=0}^{N-1}[x(k+1)-x(k)]\,z^{-k}$$

$$= \lim_{N\to\infty}\left\{[x(1)-x(0)] + [x(2)-x(1)]z^{-1} + \cdots + [x(N+1)-x(N)]z^{-(N-1)}\right\}$$

$$= \lim_{N\to\infty}\left\{-x(0) + \frac{z-1}{z}x(1) + \frac{z-1}{z^2}x(2) + \cdots + \frac{z-1}{z^{N-1}}x(N-1) + z^{-N+1}x(N)\right\}$$

By equating with the previous relation, we obtain

$$\lim_{N\to\infty}\left\{-x(0)+\frac{z-1}{z}x(1)+\frac{z-1}{z^2}x(2)+\cdots+\frac{z-1}{z^{N-1}}x(N-1)+z^{-N+1}x(N)\right\}=(z-1)X(z)-zx(0)$$

By making z tend to 1

$$\lim_{z\to 1}(z-1)X(z)-x(0)=\lim_{N\to\infty}\{-x(0)+x(N)\}$$

Consequently, when the limit exists, we can write:

$$\lim_{k\to\infty}x(k)=\lim_{z\to 1}(z-1)X(z)$$

The theorem 3.2 is proven

Example 3.10: obtain the final value for the sequence whose z-transform is

$$X(z)=\frac{z^2(z-a)}{(z-1)(z-b)(z-c)}$$

What can you conclude concerning the constants b and c if it is known that the limit exists?

Solution:

3.6.8 The transform of the convolution product

Given two discrete signals $x(k)$ and $y(k)$, causals, the z-transforms respectively are $X(z)$ and $Y(z)$. Prove that the z-transform of their convolution product is the algebraic product of the transforms:

$$\mathcal{Z}\big(x(k)*y(k)\big)=X(z)Y(z) \tag{3.10}$$

this relation is very important because, as will be seen in next chapter, the behavior of all systems in discrete time may be described by the convolution operation.

Proof

$$\mathcal{Z}\big(x(k)*y(k)\big)=\mathcal{Z}\left[\sum_{n=0}^{\infty}x(n)y(k-n)\right]=$$

$$=\sum_{k=0}^{\infty}\left[\sum_{n=0}^{\infty}x(n)y(k-n)\right]z^{-k}=\sum_{n=0}^{\infty}x(n)\sum_{k=0}^{\infty}y(k-n)z^{-k}$$

Using the property of delay and the hypothesis of causality of $y(k)$ it becomes:

$$\mathcal{Z}\big(x(k)*y(k)\big)=\sum_{n=0}^{\infty}x(n)z^{-n}Y(z)=\left(\sum_{n=0}^{\infty}x(n)z^{-n}\right)Y(z)=X(z)Y(z)$$

3.7 The inverse z-transform

Knowing the z-transform of a discrete time signal $x(k)$, we seek to recover the original time form of this signal. In general case this inversion necessitates the knowledge of the region of convergence. However, in the particular case of causal signals, it will be omitted because it is implicit. There are many methods to compute the original signal

$x(k)$ of a z-transform. We discuss three principal methods, the integral formula, long division and the partial fraction decomposition.

3.7.1 integral inversion formula

if

$$X(z) = \sum_{k=0}^{\infty} x(k)z^{-k}, |z| > R_0 \qquad (3.11)$$

We multiply the two members of this expression by z^{n-1}, we obtain:

$$z^{n-1}X(z) = z^{n-1}\sum_{k=0}^{\infty} x(k)z^{-k}$$

Let Γ a closed contour, surrounding the origine and belongs to a domain $|z| > R_0$. We can then write:

$$\oint z^{n-1}X(z)dz = \oint z^{n-1}\sum_{k=0}^{\infty} x(k)z^{-k}\, dz$$

By exchanging the symbols of integration and the sum in the right part, we obtain the following integral relation:

$$\oint z^{n-1}X(z)dz = \sum_{k=0}^{\infty} x(k)\oint z^{n-k-1}\, dz \qquad (3.12)$$

<u>**Theorem 3.1 : Cauchy theorem**</u>

If Γ is a closed contour, surrounding the origine, then:

$$\oint \frac{dz}{z} = \begin{cases} 2\pi j & if\ n = 1 \\ 0 \quad if & n \neq 1 \end{cases}$$

By applying this theorem to integral $\oint z^{n-k-1}\, dz$, we obtain:

$$\oint \frac{dz}{z^{1+k-n}} = \begin{cases} 2\pi j & if\ n = k \\ 0 \quad if & n \neq k \end{cases}$$

We replace this equation in (3.9), it becomes:

$$\oint z^{n-1}X(z)dz = \sum_{k=0}^{\infty} x(k)\oint z^{n-k-1}\, dz = 2\pi j x(n)$$

$$X(z), |z| > R_0 \xrightarrow{Z^{-1}} x(k) = \frac{1}{2\pi j}\oint z^{k-1}X(z)dz \quad \Gamma \in \text{domain } |z| > R_0 \qquad (3.13)$$

If the integral $\oint z^{k-1}X(z)dz$ is computed by the residue method, this formula becomes

$$x(k) = \sum_{\substack{z_i = poles\ of \\ z^{k-1}X(z)}} Res\left(z^{k-1}X(z)\right)_{z=z_i} \qquad (3.14)$$

Example 3.11: we consider the following function

$$X(z) = \frac{z}{(z-a)(z-b)} \qquad |z| > \max(|a|, |b|)$$

Applying (3.11) with $k > 0$, to compute the original signal $x(k)$.

Solution:

3.7.2 Method of long division

This approach is based on the time sequence of its z-transform directly. We first use long division to obtain as many terms as desired of the z-transform expansion; then we use the coefficients of the expansion to write the time sequence. The following two steps give the inverse z-transform of a function $X(z)$:

1) Using long division, expand $X(z)$ as a series to obtain

$$X_l(z) = x_0 + x_1 z^{-1} + \cdots + x_i z^i = \sum_{k=0}^{i} x_k z^{-k}$$

2) Write the inverse transform as the sequence

$$\{x_0\,, x_1, \ldots, x_i + \cdots\}$$

The number of terms i obtained by long division is selected to yield a sufficient number of points in the time sequence.

Example 3.12: obtain the inverse z-transform of the function

$$X(z) = \frac{z+1}{z^2 + 0.2z + 0.1}$$

Solution

3.7.3 Partial fraction expansion

This method is the same as that used in inverting Laplace transforms. However, because most z-functions have the term z in their numerator, it is often convenient to expand $X(z)/z$ rather than $X(z)$. As in Laplace transforms, partial fraction expansion allows us to write the function as the sum of simpler elements that are the z-transforms of known discrete-time functions. The time functions are available in z-transform tables provided in Appendix 1.

The procedure for inverse z-transformation is

1) Find the partial fraction expansion of $X(z)/z$ or $X(z)$.

2) Obtain the inverse transform $x(k)$ using the z-transform tables.

We consider three types of z-domain functions $X(z)$: elements with simple (no repeated) real poles, elements with complex conjugate and real poles, and elements with repeated poles. We discuss examples that demonstrate partial fraction expansion and inverse z-transformation in each case.

Case 1: simple real poles

The most convenient method to obtain the partial fraction expansion of a function with simple real poles is the method of residues. The residue of a complex function $X(z)$ at a simple pole z_i is given by

$$X(z) = \sum_{i=1}^{n} \frac{\alpha_i}{z - z_i}, \quad \alpha_i = (z - z_i)X(z)|z \longrightarrow z_i \tag{3.12}$$

Example 3.13: Obtain the inverse z-transform of the function

$$X(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

Solution

Case 2: Complex conjugate and simple real poles

For a function $X(z)$ with real and complex poles, the partial fraction expansion includes terms with real roots and others with complex roots. Assuming that $X(z)$ has real coefficients, then its complex roots occur in complex conjugate pairs and can be combined to yield a function with real coefficients and a quadratic denominator. To inverse-transform such a function, use the following z-transforms (see Appendix I):

$$\mathcal{Z}(e^{-\alpha k}\sin(k\omega_d)) = \frac{e^{-\alpha}\sin(\omega_d)\, z}{z^2 + 2e^{-\alpha}\cos(\omega_d)\, z + e^{-2\alpha}} \tag{3.13}$$

$$\mathcal{Z}(e^{-\alpha k}\cos(k\omega_d)) = \frac{z(z - e^{-\alpha}\cos(\omega_d))}{z^2 + 2e^{-\alpha}\cos(\omega_d)\, z + e^{-2\alpha}} \tag{3.14}$$

Example 3.13: the inverse of z-transform of the function

$$X(z) = \frac{z^3 + 2z + 1}{(z-1)(z^2 + z + 0.5)}$$

Solution

Case 3: repeated roots

For a function $X(z)$ with a repeated root of multiplicity r, partial fraction coefficients are associated with the repeated root. The partial fraction expansion is of the form

$$X(z) = \frac{N(z)}{(z - z_1)^r \prod_{j=r+1}^{n}(z - z_j)} = \sum_{i=1}^{r} \frac{\alpha_{1i}}{(z - z_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{\alpha_j}{z - z_j} \tag{3.15}$$

The coefficients for the repeated roots can be found as

$$\alpha_{1i} = \frac{1}{(i-1)!}\frac{d^{i-1}}{dz^{i-1}}(z-z_i)^r X(z)\Bigg|z = z_1 \tag{3.16}$$

The coefficients of the simple or complex conjugate roots can be obtained as before.

Example 3.14: Obtain the inverse z-transform of the function

$$X(z) = \frac{1}{z^2(z-0.5)}$$

Solution:

3.8 Difference equations

Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values. The nonlinear difference equation

$$\begin{aligned} y(k+n) = f(&y(k+n-1), y(k+n-2), \dots, y(k+1), y(k) \\ &u(k+n), u(k+n-1), \dots, u(k+1), u(k)) \end{aligned} \tag{3.17}$$

with forcing function $u(k)$ is said to be of order n because the difference between the highest and lowest time arguments of $y(.)$ and $u(.)$ is n. The equations we deal with in this course are almost exclusively linear and are of the form

$$\begin{aligned} &y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) \\ &\qquad = b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k) \end{aligned} \tag{3.18}$$

We further assume that the coefficients a_i, b_i $i = 1, 2, \ldots$ are constant. The difference equation is then referred to as linear time invariant, or LTI. If the forcing function $u(k)$ is equal to zero, the equation is said to be *homogeneous*.

Difference equations can be solved using classical methods analogous to those available for differential equations. Alternatively, z-transforms provide a convenient approach for solving LTI equations.

CHAPTER FOUR
ANALYSIS OF DISCRETE-TIME SYSTEMS

4.1 Introduction

After completing this chapter, the main objective is to be able to do the following:

1- Determine the input output stability of a z-transform function.

2- Determine the asymptotic stability of a z-transfer function.

3- Determine the internal stability of a digital feedback control system.

4- Determine the stability of a z-polynomial using the Routhe Hurwitz criterion.

5- Determine the stability of a z-polynomial using the Jury criterion.

6- Define the stable range of a parameter for a z-polynomial.

7- Establish the closed-loop stability of a digital system using the Nyquist criterion.

8- Determine the gain margin and phase margin of a digital system.

Stability is a basic requirement for discrete and continuous time control systems. Discrete control is based on samples and is updated at every sampling period, and there is a possibility that the system will become unstable between updates. This obviously makes stability analysis different in the discrete case. We examine the stability of linear time invariant discrete-systems based on transfer function using different definitions and tests. In particular, we consider input-output stability and internal stability. We afford several tests for stability, such as the Routhe-Hurwitz criterion, the Jury criterion, and the Nyquist criterion. We also define the gain margin and phase margin for discrete-systems.

4.2 Definition of stability

Definition 4.1: Bounded-Inputs and Bounded-Outputs stability (BIBO stability)

A system is said to be bounded-input and bounded-output (BIBO) stable if its response to any bounded-input remains bounded the input and the output satisfy the following.

$$|u(k)| < \beta_u,\ k = 0, 1, 2, \ldots \tag{4.1}$$

$$0 < \beta_u < \infty$$

$$|y(k)| < \beta_y,\ k = 0, 1, 2, \ldots \tag{4.2}$$

$$0 < \beta_y < \infty$$

Theorem 4.1: A system is stable in the sense of BIBO if, and only if its impulse response is absolutely summable, that's to say:

$$\sum_{k=0}^{\infty} |g(k)| < \infty \tag{4.3}$$

Proof: this proof will be performed in two steps. In the first time we prove that this condition is sufficient.

$$y(k) = u(k) * g(k) = \sum_{n=0}^{\infty} u(n)g(k-n)$$

Sufficient: $\sum_{k=0}^{\infty}|g(k)| < \infty \Longrightarrow BIBO\ stable$

Suppose that the input $u(k)$ is bounded, either $|u(k)| < U_0$, then is possible to limit the output signal, for the system time invariant can be written as

$$y(k) = \sum_{n=-\infty}^{\infty} u(n)g(k-n) \Longrightarrow |y(k)| = \left|\sum_{n=-\infty}^{\infty} u(n)g(k-n)\right| \leq \sum_{n=-\infty}^{\infty} |u(n)g(k-n)|$$

Using the hypothesis of bounded input and change the variable $k - n = m$, we can also write.

$$|y(k)| \leq U_0 \sum_{m=-\infty}^{\infty} |g(m)| \leq U_0 G_0$$

We remark that, for a causal system the impulse response is zero for $m < 0$. The previous relation shows that, if the input is bounded and the impulse response is summable, then the output is bounded.

Necessity: $\sum_{k=0}^{\infty}|g(k)| = \infty \Longrightarrow BIBO\ instable$

In order to prove the necessity, it is sufficient to find an input signal such that, if the condition of the impulse response is summable is not respected, then the output diverge.

$$y(0) = \sum_{n=-\infty}^{\infty} u(n)g(-n)$$

Precise that, in the case of causal system this relation can be seen as the behavior at instant $k = 0$, take the sign input signal;

$$u(n) = sign\big(g(-n)\big) = \begin{cases} +1\ if\ g(-n) > 0 \\ -1\ if\ g(-n) < 0 \end{cases}$$

The output at instant $k = 0$ is then written:

$$y(0) = \sum_{n=-\infty}^{\infty} u(n)g(-n)$$

$$= \sum_{n=-\infty}^{\infty} g(-n)sign\big(g(-n)\big) = \sum_{n=-\infty}^{\infty} g(-n)sign\big(g(-n)\big) = \sum_{n=-\infty}^{\infty} |g(-n)|$$

We conclude that, if the impulse response is not absolutely summable, then it is always exhibiting a bounded input, such that the output diverges, which demonstrates the necessity condition.

Definition 4.2: **Asymptotic stability**

A system is said to be asymptotically stable if its response to any initial conditions decays to zero asymptotically in the steady state, that is, the response due to the initial conditions satisfies

$$\lim_{k\to\infty} y(k) = 0 \tag{4.4}$$

If the response due to the initial conditions remains bounded but does not decay to zero, the system is said to be marginally stable.

4.3 Stability using poles of transfer function

Theorem 4.2: The discrete causal and time invariant system is stable in the sense of BIBO if all the poles of its transfer function are inside the unit circle.

Proof: the transfer function given by

$$G(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{\prod_{i=1}^{n}(z-p_i)^{m_i}} \tag{4.5}$$

The impulse response of this system is obtained for the input $u(k) = \delta(k)$, then

$$Y(z) = G(z) = \frac{N(z)}{\prod_{i=1}^{n}(z-p_i)^{m_i}} = \sum_{i=1}^{n_i}\sum_{j=1}^{m_i}\frac{a_{i,j}z}{(z-p_i)^j}$$

$$= \sum_{i=1}^{n_i}\left[\frac{a_{i,1}z}{z-p_i} + \frac{a_{i,2}z}{(z-p_i)^2} + \cdots + \frac{a_{i,m_i}z}{(z-p_i)^{m_i}}\right]$$

$$g(k) = a_{i,1}p_i^k + a_{i,2}kp_i^{k-1} + \cdots + \frac{a_{i,m_i}}{(m_i-1)!}\left[\prod_{\substack{j=0\\ m_i>0}}^{m_i-2}(k-j)\right]p_i^{k-m_i+1} \tag{4.6}$$

4.4 Algebraic methods

4.4.1 Routh-Hurwitz criterion

The Routh-Hurwitz criterion determines conditions for left half plane (LHP) polynomial roots and cannot be directly used to investigate the stability of discrete-time systems. The bilinear transformation

$$z = \frac{1+w}{1-w} \Leftrightarrow w = \frac{z-1}{z+1} \tag{4.7}$$

transforms the inside of the unit circle to the LHP.

To verify this property, consider the three cases shown in Figure 4.1. They represent the mapping of a point in the LHP, a point in the right half plane (RHP), and a point on the $j\omega$-axis. The angle of ω after bilinear transformation is

$$\angle w = \angle(z-1) - \angle(z+1) \tag{4.8}$$

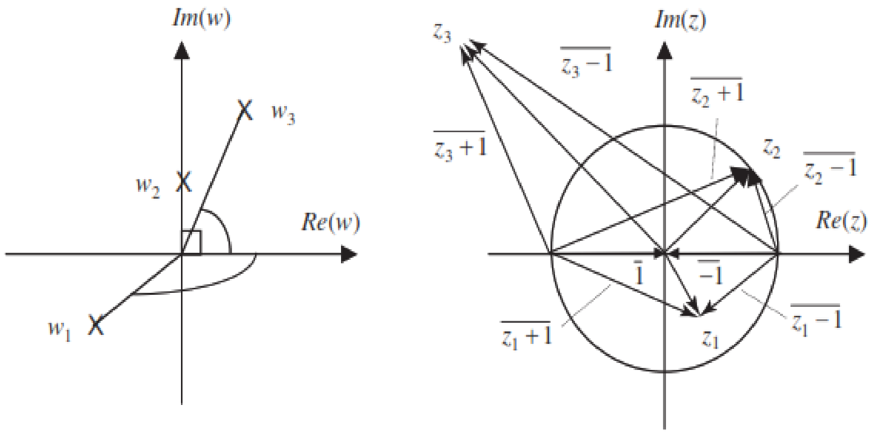


Figure 4.1: Angles associated with bilinear transformation

$$D(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$
$$\rightarrow a_n \left(\frac{1+w}{1-w}\right)^n + a_{n-1}\left(\frac{1+w}{1-w}\right)^{n-1} + \cdots + a_1 \left(\frac{1+w}{1-w}\right) + a_0 \tag{4.9}$$

Proof of this property

The w-transform matches the inside of unit circle of the z-plane to half left plane of w, as shown by the relation below, obtained by setting $z = \rho e^{j\theta}$:

$$w = \frac{\rho e^{j\theta} - 1}{\rho e^{j\theta} + 1} = \frac{\rho\big(\cos(\theta) + jsin(\theta)\big) - 1}{\rho\big(\cos(\theta) + jsin(\theta)\big) + 1} = \frac{(\rho^2 - 1) + j2\rho \sin(\theta)}{(\rho\cos(\theta) + 1)^2 + (\rho\sin(\theta))^2}$$

Examine then the correspondence between planes z and w induced by this transformation.

Unit circle: $\rho = 1 \Rightarrow w = j\acute{\omega}$, unit circle corresponds to purely imaginary axis.

Outside the unit circle: $\rho > 1 \Rightarrow Re(w) > 0$, outside the unit circle corresponds to half right plane.

Inside the unit circle: $\rho < 1 \Rightarrow Re(w) < 0$, inside the unit circle corresponds to the left half plane.

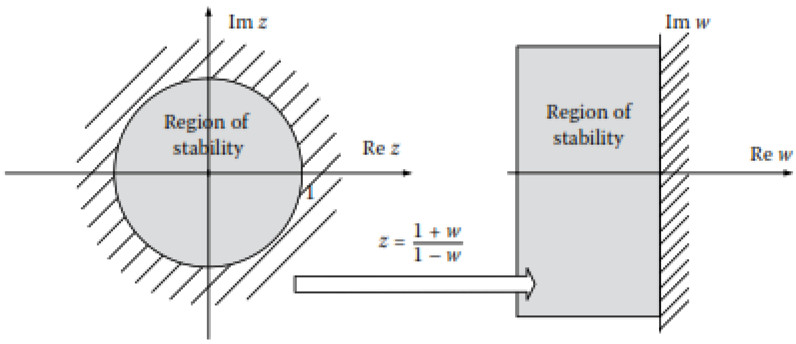


Figure 4.2: z- and w-domains via the Mobius bilinear transform.

This property which makes w-transform interest. It allows to use all the techniques and methods of continuous-time systems and in particular the design of controllers. The return of function in w to the original in the z domain is done by the inverse transform. By introducing the term of normalization, the pulses $\frac{2}{T}$, we obtain a variant of w-transform:

$$w = \frac{2}{T}\frac{z-1}{z+1} \Leftrightarrow z = \frac{\frac{2}{T}+w}{\frac{2}{T}-w} \tag{4.10}$$

This variant possesses the same properties, the interest resides in the correspondence between the pulsation ω in the plane z when $z = e^{j\omega T}$ and the equivalent pulsation $\acute{\omega}$ in the plane w. This relation will be examined in the frequency response part.

4.4.1.1 application to the study of stability

The characteristic equation $T(z) = 0$ becomes $T(w) = 0$, either this transform:

$numerator\big(T(w)\big) = \alpha_n w^n + \alpha_{n-1} w^{n-1} + \cdots + \alpha_1 w + \alpha_0 = 0$

For the reason of the correspondence of planes, the condition of stability being then

$$Re\big(w_j\big) < 0, \forall j, j = 1, 2, \dots, n \tag{4.11}$$

We can use all the criteria applied to continuous time systems, among others the Routh criterium. Let's remember that, this criterium express the necessary and sufficient conditions for a polynomial has all its zeros with real parts strictly negatives.

<u>Example 4.1</u>:

Find stability conditions for

1- the first order polynomial $a_1 z + a_0, a_1 > 0$

2- the second order polynomial $a_2 z^2 + a_1 z + a_0, a_2 > 0$

<u>Solution:</u>

1- The stability of the first-order polynomial can be easily determined by solving for its root. Hence, the stability condition is

$$\left|\frac{a_0}{a_1}\right| < 1$$

2- The roots of the second-order polynomial are in general given by

$$z_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

Thus, it is not easy to determine the stability of the second-order polynomial by solving for its roots. For a monic polynomial (coefficient of z^2 is unity), the constant term is equal to the product of the poles. Hence, for pole magnitudes less than unity, we obtain the necessary stability condition

$$\left|\frac{a_0}{a_2}\right| < 1 \tag{4.12}$$

Or equivalently

This condition is also sufficient in the case of complex conjugate poles where the two poles are of equal magnitude. The condition is only necessary for real poles because the product of a number greater than unity and a number less than unity can be less than unity. For example, for poles at 0.01 and 10, the product of the two poles has magnitude 0.1, which satisfies (4.12), but the system is clearly unstable.

Substituting the bilinear transformation in the second-order polynomial gives

$$a_2\left(\frac{1+w}{1-w}\right)^2 + a_1\left(\frac{1+w}{1-w}\right) + a_0$$

Which reduces to

$$(a_2 - a_1 + a_0)w^2 + 2(a_2 - a_0)w + (a_2 + a_1 + a_0)$$

By the Routhe-Hurwitz criterion, it can be shown that the poles of the second-order w polynomial remain in the LHP if and only if its coefficients are all positive. Hence, the stability conditions are given by

$$\begin{aligned} &a_2 - a_1 + a_0 > 0 \\ &a_2 - a_0 > 0 \\ &a_2 + a_1 + a_0 > 0 \end{aligned} \tag{4.13}$$

Adding the first and third conditions gives

$$a_2 + a_0 > 0 \Rightarrow a_2 > -a_0$$

This condition, obtained earlier in (4.12), is therefore satisfied if the three conditions of (4.13) are satisfied. The reader can verify through numerical examples that if real roots satisfying conditions (4.13) are substituted in the roots given below, we obtain roots between -1 and 1 Without loss of generality, the coefficient a_2 can be assumed to be unity, and the stable parameter range can be depicted in the a_0 versus $a_1$parameter plane as shown in Figure. 4.3.

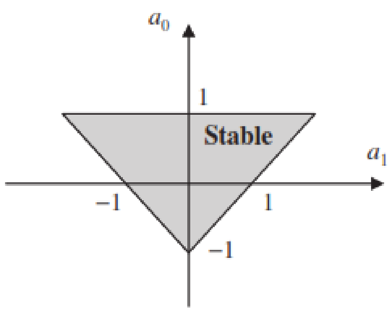


Figure 4.3: Stable parameter range for a second-order z-polynomial.

4.4.1.2 The study of the frequency response

The frequency response being obtained when $z = e^{j\omega T}$, the w −transform becomes:

$$w = \frac{2}{T}\frac{e^{j\omega T} - 1}{e^{j\omega T} + 1} = \frac{2}{T}\frac{e^{\frac{j\omega T}{2}} - e^{-\frac{j\omega T}{2}}}{e^{\frac{j\omega T}{2}} + e^{-\frac{j\omega T}{2}}} = \frac{2}{T}jtg\left(\frac{\omega T}{2}\right) = j\omega\frac{tg\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} = j\acute{\omega} \tag{4.14}$$

We see that the real pulse ω which varies between 0 and π/T corresponds to the w domain, the $\acute{\omega}$-pulse varies from 0 to $+\infty$. These two pulsations are related by this relationship

$$\omega\frac{tg\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}} = \acute{\omega} \tag{4.15}$$

For small ωT we ca use the approached relation $\acute{\omega} \approx \omega$, that is to say that the pulsation in the domain w is directly the real pulsation.

As the frequency response in the domain w is computed by setting $w = j\acute{\omega}$. The function obtained is in relation of $\acute{\omega}$, it results a more easily drawn and interpretation. For example, in the case of Bode diagrams all the rules are the same as continuous time systems.

Example 4.2: Either the continuous time system given by the transfer function

$$G(s) = \frac{1}{s+1}$$

Controlled in discrete with sampling time $T = 0.4\ s$ and whose output is sampled at same time.

Solution: The associated discrete function using ZOH approximation is found as follows

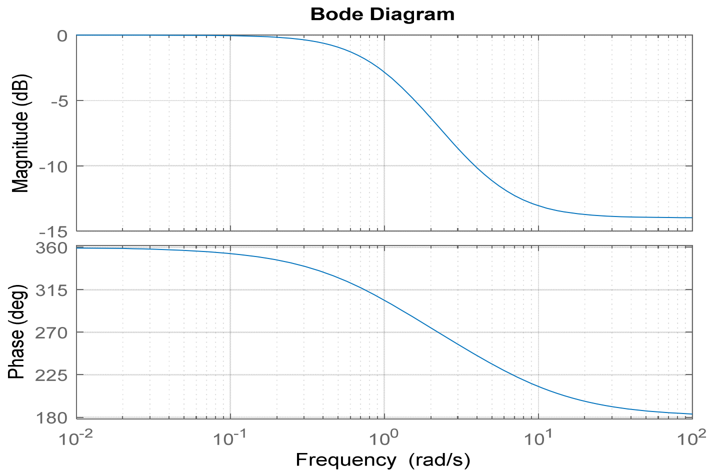
$$F(z) = (1 - z^{-1})\mathrm{Z}\left(\frac{G(s)}{s}\right) = \frac{0.3297}{z - 0.6703}$$

$$z = \frac{\frac{2}{T} + w}{\frac{2}{T} - w} = \frac{5 + w}{5 - w}$$

We obtain after calculation

$$F(w) = \frac{1 - \frac{w}{5}}{1 + w}$$

By setting $w = j\acute{\omega}$, we obtain the frequency response represented the Bode gain and the phase diagrams.



4.4.2 Jury test

It is possible to investigate the stability of z-domain polynomials directly using the Jury test for real coefficients or the Schure-Cohn test for complex coefficients. These tests involve determinant evaluations as in the Routh-Hurwitz test for s-domain polynomials but are more time-consuming. The Jury test is given next.

For the polynomial $D(z)$ with a real coefficients of the form

$$D(z) = a_{0,0} + a_{0,1}z + a_{0,2}z^2 + \cdots + a_{0,n}z^n \quad with \quad a_{0,n} > 0 \tag{4.16}$$

We construct the following $(n-1) \times (n+1)$ matrix

$$M = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots & \cdots & a_{0,n-1} & a_{0,n} \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,n-1} & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & \vdots & \vdots \\ a_{n-2,0} & a_{n-2,1} & a_{n-2,2} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.17}$$

Where the row $j+1$ elements are defined as follows from the coefficients of the row j:

$$a_{j+1,k} \begin{cases} \begin{bmatrix} a_{j,0} & a_{j,n-j-k} \\ a_{j,n-j} & a_{j,k} \end{bmatrix} & for\ 0 \le k \le n-j-1 \\ 0 & for \quad k > n-j-1 \end{cases} \tag{4.18}$$

The polynomial $D(z)$ has not root of the module greater than unity (outside the unit circle) if the $n+1$ following conditions are satisfied:

$$\begin{cases} \sum_{i=0}^{n} a_{0,i} = D(1) > 0 \\ (-1)^n \sum_{i=0}^{n} (-1)^i a_{0,i} = (-1)^n D(-1) > 0 \\ |a_{0,0}| - a_{0,n} < 0 \\ |a_{j,0}| - |a_{j,n-j}| > 0 \quad for\ j = 1, 2, \dots, n-2 \end{cases} \tag{4.19}$$

The use of the jury criterium necessitate then the construction of the $(n-1) \times (n+1)$ matrix obtained by calculating $n(n-1)(n-2) \ldots 3\ 2 \times 2$ determinants. Its application in the case of high order system can therefore prove to be relatively tedious.

Example 4.3: it is interesting to practice the jury test in the case of second and third order systems

$$D(z) = a_{0,0} + a_{0,1}z + a_{0,2}z^2 \qquad n = 2$$
$$D(z) = a_{0,0} + a_{0,1}z + a_{0,2}z^2 + a_{0,3}z^3 \qquad n = 3$$

Construct the table for the jury test

$D(z) = a_{0,0} + a_{0,1}z + a_{0,2}z^2$ $n = 2$	$D(z) = a_{0,0} + a_{0,1}z + a_{0,2}z^2 + a_{0,3}z^3$ $n = 3$
1) $a_{0,0} + a_{0,1} + a_{0,2} > 0$	1) $a_{0,0} + a_{0,1} + a_{0,2} + a_{0,3} > 0$
2) $a_{0,0} - a_{0,1} + a_{0,2} > 0$	2) $-a_{0,0} + a_{0,1} - a_{0,2} + a_{0,3} > 0$
3) $\lvert a_{0,0}\rvert - a_{0,2} < 0$	3) $\lvert a_{0,0}\rvert - a_{0,3} < 0$
4) Can not be applied because $n - 2 = 0$	4) $\lvert a_{1,0}\rvert - \lvert a_{1,2}\rvert > 0$ either: $\lvert (a_{0,0})^2 - (a_{0,3})^2\rvert - \lvert a_{0,0}a_{0,2} - a_{0,1}a_{0,3}\rvert > 0$

4.5 Frequency response of discrete system

The frequency response of systems covers relatively an important feature in the sense where many analysis methods of robustness and performances of the controllers are based on a frequency analysis. To this, it must be added that the sinusoidal input is always the corresponding input using for the test.

Either a linear invariant causal discrete system, where the impulse response is the sequence $g(k)$ and the associated z-transform $G(z) = \mathcal{Z}\big(g(k)\big)$. A causal sinusoidal input is applied $u(k) = U_0 \cos(2\pi\nu k)\, u_1(k)$, where $u_1(k)$ is the unit step and searching the response to this signal.

$$y(k) = u(k) * g(k) = \sum_{n=0}^{k} U_0 \cos\big(2\pi\nu(k-n)\big)\, g(n) \tag{4.20}$$

Putting $\cos(2\pi\nu k) = Re\big(e^{j2\pi\nu k}\big)$ the previous expression can be written again:

$$y(k) = U_0 Re\left(\sum_{n=0}^{k} g(n) e^{j2\pi\nu(k-n)}\right) = U_0 Re\left(e^{j2\pi\nu k} \sum_{n=0}^{k} g(n) e^{-j2\pi\nu n}\right)$$

This expression is not easy to analyze in the general case, but we use the Fourier transform of the causal discrete time signal $g(k)$, namely:

$$G(\nu) = \sum_{n=0}^{\infty} g(n)e^{-j2\pi\nu n} \tag{4.21}$$

And supposing the stability of the system, we can make appear the behavior in steady state, that's to say when $k \to \infty$:

$$\lim_{k\to\infty} y(k) = U_0 Re\left(e^{j2\pi\nu k}\sum_{n=0}^{k} g(n)e^{-j2\pi\nu n}\right)$$
$$= U_0|G(\nu)|\cos\big(2\pi\nu k + arg\big(G(\nu)\big)\big) \quad (4.22)$$

Finally, we obtain $y(k) = Y_0\cos\big(2\pi\nu k + \varphi(\nu)\big)$, expression allowing to make appear the amplitude of the output and its phase comparatively to the input signal:

$$Y_0 = U_0|G(\nu)| \quad \varphi(\nu) = arg\big(G(\nu)\big)$$

The complex number $G(\nu)$ can be interpreted from the transfer function as being the value of $G(z)$ for $z = e^{j2\pi\nu}$, in other terms, the harmonic response of the system is given by:

$$G(z)_{z=e^{j2\pi\nu}} = \begin{Bmatrix} |G(z)| \\ \arg\,(G(z)) \end{Bmatrix}_{z=e^{j2\pi\nu}} \; ; \; \nu \in \left[0; \frac{1}{2}\right] \tag{4.23}$$

We remark that, the Fourier transform of discrete time signal being even and periodic (period equals 1). If the discrete system comes from the sampling with constant time T of continuous system, we can put $2\pi\nu = \omega T$, where ω represent the pulsation of continuous time signals. The frequency response in module and in phase is then given by the relations:

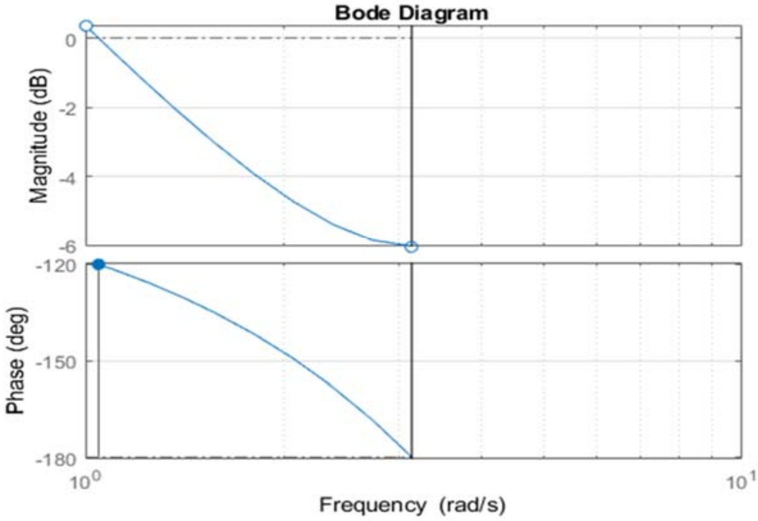
$$G(z)_{z=e^{j\omega T}} = \begin{Bmatrix} |G(z)| \\ \arg\,(G(z)) \end{Bmatrix}_{z=e^{j\omega T}} \; ; \; \omega \in \left[0; \frac{\pi}{T}\right] \tag{4.24}$$

Example 4.4: either the discrete time system given by the transfer function:

$$G(z) = \frac{1}{z-1}$$

The frequency response is obtained putting $z = e^{j2\pi\nu}$

$$G(z)_{z=e^{j2\pi\nu}} = \begin{Bmatrix} |G(z)| \\ \varphi = \arg\,(G(z)) \end{Bmatrix}_{z=e^{j2\pi\nu}} = \begin{cases} \dfrac{1}{\sqrt{2(1-\cos(2\pi\nu))}} \\ -\arg\,(\cos(2\pi\nu - 1) + j\sin(2\pi\nu) \end{cases}$$

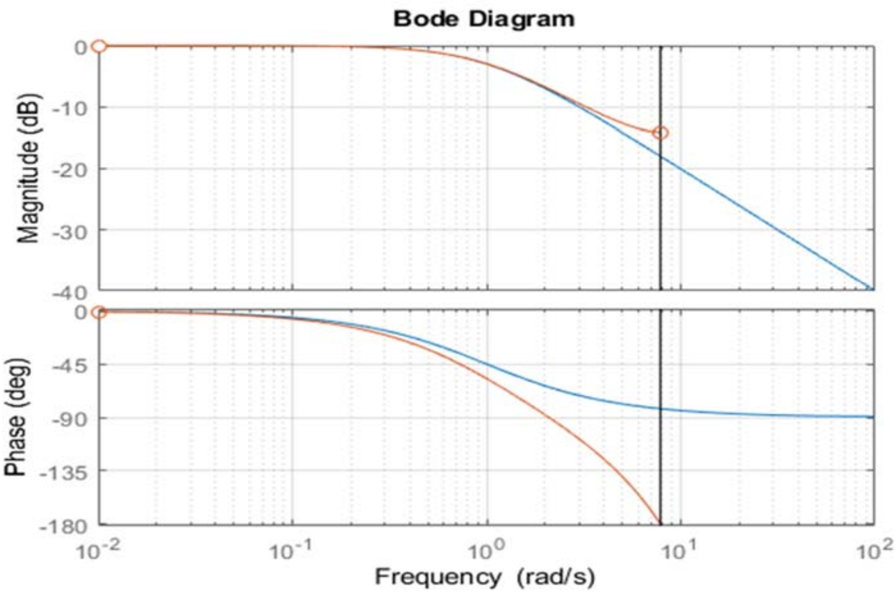


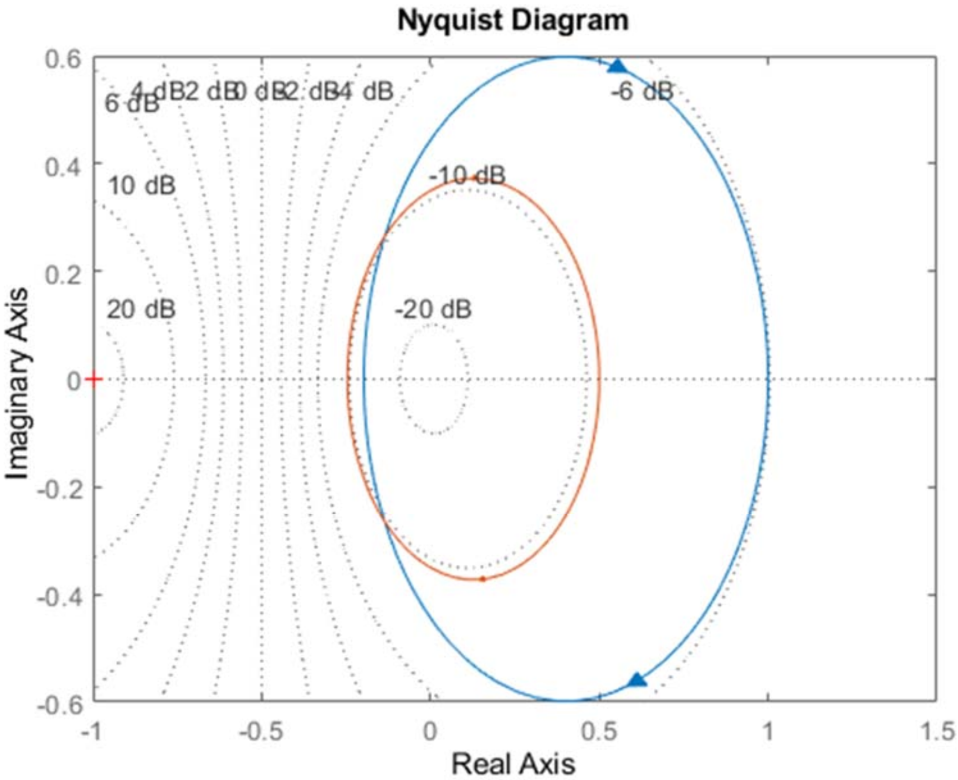
Example 4.5: we consider the continuous time system given by the transfer function with ($T = 0.4\ s$)

$$G(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{1}{s(s+1)}\right) = \frac{0.33}{z - 0.67}$$

The frequency response is obtained putting $z = e^{j\omega T}$

$$G(z)_{z=e^{j2\pi\nu}} = \begin{Bmatrix} |G(z)| \\ \varphi = \arg\left(G(z)\right) \end{Bmatrix}_{z=e^{j\omega T}} = \begin{cases} \dfrac{0.33}{\sqrt{1.45 - 1.34\cos\left(\omega T\right)}} \\ -\arg\left[(\cos\omega T - 0.67) + j\sin(\omega T)\right] \end{cases}$$




Nyquist Diagram
Imaginary Axis
Real Axis
6 dB
4 dB
2 dB
0 dB
-2 dB
-4 dB
-6 dB
10 dB
-10 dB
20 dB
-20 dB
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-1
-0.5
0
0.5
1
1.5

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