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Algerian Democratic and Popular Republic
Ministry of Higher Education and Scientific Research
University of Bejaia

جامعة بجاية
Tasdawit n Bgayet
Université de Béjaïa

Faculty of technological

Departement of technological

In accordance with the LMD First Year Program

Mathematics I

Option : *Science and Technology Field*

By

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Academic Year : 2024/2025

Preface :

Preface

This polycopy is intended for first-year students in the Science and Technology field under the LMD system. The manuscript covers the syllabus of the Mathematics I module, which is dedicated to the first semester program. This course includes numerous typical examples and exercises with solutions.

The syllabus of Mathematics I for the first semester consists of six chapters as follows:

Chapter 1: Methods of Mathematical Reasoning

Chapter 2: Sets, Relations, and Functions

Chapter 3: Real Functions of a Real Variable

Chapter 4: Applications to Elementary Functions

Chapter 5: Limited development

Chapter 6: Linear Algebra

Contents

Chapter 1

Methods of mathematical reasoning

1.1 Mathematical Logic

1.1.1 Assertions

Definition 1.1.1. *A logical proposition is any assertion (sentence or expression, ...) to which we can respond unambiguously, without hesitation or additional information, with either true or false, but not both at the same time. Propositions are denoted by uppercase letters: P, Q, R, ...*

Example 1.1.1.

- *P:* $6 + 3 = 9$ *is a true proposition.*
- *Q:* 17 *is an even number is a false proposition.*
- *R:* $x \in \mathbb{Z} : x^2 = 4$ *is not a proposition, as it is impossible to decide whether it is true or false without knowing the value of* x*.*
- $2 + 2 = 4$ *is a true assertion*
- *For all* $z \in \mathbb{C}$ *we have* $\mid z \mid = 1$ *is a false assertion*

Negation of a logical proposition: Every logical proposition has a negation, denoted by $\overline{P}$ or not P , which is true when P is false and false when P is true. P and $\overline{P}$ can never both be true at the same time.

The following truth table summarizes this:

P	$\overline{P}$
1	0
0	1

Where 1 denotes true and 0 denotes false. These are called truth values.

Example 1.1.2. *The negation of the assertion* $x+2=0$ *she is the assertion* $x+2\neq 0$

1.1.2 Logical Connectors

In mathematical reasoning, multiple propositions are often linked together by relations called logical connectors. There are four main connectors: conjunction, disjunction, implication, and equivalence.

Conjunction $\wedge$ (and)

Let P and Q be two logical propositions. The conjunction of propositions P and Q, denoted $P \wedge Q$, is a proposition that is true only if both P and Q are true. This is summarized in the following truth table:

P	1	1	0	0
Q	1	0	1	0
$p \wedge Q$	1	0	0	0

Example 1.1.3.

1. $(3+5=8)\wedge(3\times 6=18)$ *is a true assertion.*
2. $(2+2=4)\wedge(2\times 3=7)$ *is a false assertion.*

Remark: $P \wedge \overline{P}$ is always false.

Disjunction $\vee$ (or)

Let P and Q be two logical propositions. The disjunction of two propositions P and Q, denoted $P \vee Q$, is a logical proposition that is true if at least one of the propositions P or Q is true. This is summarized in the following truth table:

P	1	1	0	0
Q	1	0	1	0
$p \vee Q$	1	1	1	0

Example 1.1.4.

1. $(2+2=4\vee 3\times 2=6)$ *is a true assertion.*
2. $(2=4\vee 4\times 2=7)$ *is a false assertion.*

Remark: $P \vee \overline{P}$ is always true.

Logical Implication $P \Rightarrow Q$

Let P and Q be two logical propositions. The proposition $P \Rightarrow Q$ is called a logical implication, and it is read as "P implies Q." This is summarized in the following truth table:

P	1	1	0	0
Q	1	0	1	0
$p \Rightarrow Q$	1	0	1	1

Remark: From this table, we observe that a true proposition does not imply a false proposition.

Example 1.1.5. $2+2=5 \Rightarrow \sqrt{2}=2$ *is true , Yes, if P is false then the assertion $P \Rightarrow Q$ is always true.*

Logical Equivalence $P \Leftrightarrow Q$

Equivalence is defined by $(P \Leftrightarrow Q)$ is the assertion $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$. We will say (P is equivalent to Q) or (P if and only if Q). This assertion is true when P and Q are true or when P and Q are false.
The truth table is :

P	1	1	0	0
Q	1	0	1	0
$p \Leftrightarrow Q$	1	0	0	1

Remark: Two propositions are equivalent if and only if they have the same truth value.

Example 1.1.6. *For $x; y \in \mathbb{R}$; Equivalence $x \times y = 0 \Leftrightarrow x = 0$ or $y = 0$ is true*

Example 1.1.7.

1. $(3 < 15)$ *is a true proposition.*
2. $(3 > 15)$ *is a false proposition.*
3. $(2 < 5) \wedge (7 = 2 + 4)$ *is a false proposition.*
4. $(13 + 2 = 15) \vee (1 > 3)$ *is a true proposition.*
5. $(3 + 12 = 15) \Rightarrow (4 \times 5 = 11)$ *is a false proposition.*
6. $(2 > 5) \Rightarrow (7 = 2 + 4)$ *is a true proposition.*

Proposition 1.1.1. *Let P, Q, and R be three logical propositions. We have:*

1. $P \Rightarrow P$
2. $P \wedge Q \Rightarrow P \vee Q$
3. $P \vee Q \Rightarrow P \wedge Q$
4. $[P \Rightarrow Q$ *and* $Q \Rightarrow R] \Rightarrow (P \Rightarrow R)$
5. $[(P \vee Q) \wedge R] \Rightarrow [P \wedge (Q \vee R)]$, *and* $[(P \wedge Q) \wedge R] \Rightarrow [P \wedge (Q \wedge R)]$
6. $[(P \vee Q) \wedge R] \Rightarrow [(P \wedge R) \vee (Q \wedge R)]$, *and* $[(P \wedge Q) \vee R] \Rightarrow [(P \vee R) \wedge (Q \vee R)]$
7. $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$

Example 1.1.8. *Prove that:* $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$

We use the truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$
1	1	1	1	1
1	0	0	1	1
0	1	1	0	1
0	0	1	1	1

Note that the last column is always 1; this shows that $P \Rightarrow Q$ *is equivalent to* $Q \Rightarrow P$.

1.1.3 Quantifiers

1) The quantifier $\forall$: "for all "

The assertion $\forall x \in E; P(x)$ is a true assertion when the assertions $P(x)$ are true for all elements x of the set E. We read : For all x in E, $P(x)$ is true.

Example 1.1.9.

1. $\forall x \in \mathbb{R}$, $x^2 \geq 0$ *is a true assertion.*
2. $\forall x \in \mathbb{R}$, $x^2 \geq 1$ *is a false assertion.*

2) The quantifier $\exists$: "there exists "

The assertion $\exists x \in E; P(x)$ is a true assertion when we can find at least one element x of E for which $P(x)$ is true. We read : there exists x in E such that $P(x)$ (be true)

Example 1.1.10.

1. *$\exists x \in \mathbb{R}$, $x^2 \leq 0$ is true, for example $x = 0$.*

2. *$\exists x \in \mathbb{R}$, $x^2 < 0$ is false.*

1.2 Reasonings

1.2.1 Direct reasoning

We want to show that the assertion $P \Rightarrow Q$ is true. We assume that P is true and we show that then Q is true.

Example 1.2.1. *Let $a; b \in \mathbb{R}$; Show that $a = b \Rightarrow \dfrac{a+b}{2} = b$:*
Let's take $a = b$; then $\dfrac{a}{2} = \dfrac{b}{2}$; so

$$\begin{aligned} \frac{a}{2} + \frac{b}{2} &= \frac{b}{2} + \frac{b}{2} \\ &\Rightarrow \frac{a+b}{2} = b \end{aligned}$$

1.2.2 Reasoning by contrapositive

Reasoning by contraposition is based on the following equivalence : The assertion $(P \Rightarrow Q)$ is equivalent to $(\overline{Q} \Rightarrow \overline{P})$

$$(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$$

Example 1.2.2. *Let $x \in \mathbb{R}$. Show that*

$$(x \neq 2 \text{ and } x_2 \neq -2) \Rightarrow (x^2 \neq 4)$$

Proof. By contraposition this is equivalent to

$$(x^2 = 4) \Rightarrow (x = 2 \text{ or } x = -2)$$

Indeed, let's take $x^2 = 4$, then

$$(x^2 - 2^2 = 0) \Rightarrow (x - 2)(x + 2) = 0$$

so $x = 2$ or $x = -2$. □

Example 1.2.3. *Let $n \in \mathbb{N}$. Show that if n^2 is even then n is even.*

Proof. By contraposition, we assume that n is not even. We want to show that then n^2 is not even. As n is not even, it is odd and therefore there exists $k \in \mathbb{N}$ such that $n = 2k + 1$: Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2) + 1$$

with $k^{'} = 2k^2 + 2$ we have $n^2 = 2k^{'} + 1$, So n^2 is odd.
Conclusion : we have shown that if n is odd $\Rightarrow n^2$ is odd. By contraposition, this is equivalent to : If n^2 is even $\Rightarrow n$ is even. $\square$

1.2.3 Reasoning by Absurd (contradiction)

- To demonstrate that a proposition is true, we assume it is false and arrive at a contradiction.
- Case of an implication: To demonstrate that an implication
$P \Rightarrow Q$ is true, we use the following principle: we assume that P is true and that Q is false, and we look for a contradiction. Thus, if P is true, then Q must also be true, and therefore $P \Rightarrow Q$ is true.

Example 1.2.4. *Let* $a;\ b > 0$. *Show that if*

$$\frac{a}{1+b} = \frac{b}{1+a} \Rightarrow a = b$$

Proof. We reason with the absurd assuming that $\frac{a}{1+b} = \frac{b}{1+a}$ and $a \neq b$: This leads to

$$\begin{aligned} \frac{a}{1+b} = \frac{b}{1+a} \quad &\Leftrightarrow \quad a(1+a) = b(1+b) \\ &\Leftrightarrow \quad a^2 + a = b^2 + b \\ &\Leftrightarrow \quad a^2 - b^2 = -(a-b) \\ &\Leftrightarrow \quad (a-b)(a+b) = -(a-b) \end{aligned}$$

As $a \neq b$ then $a - b \neq 0$ and therefore dividing by $a - b$ we obtain $a + b = -1$ The sum of two positive numbers cannot be negative. We get a contradiction. So we conclude If

$$\frac{a}{1+b} = \frac{b}{1+a} \textit{ then } a = b$$

$\square$

1.2.4 Reasoning by counter example

If we want to show that an assertion of the type $(\forall x \in E; P(x))$ is true then for each x of E we must show that $P(x)$ is true. On the other hand, to show that this assertion is false then it is enough to find $x \in E$ such that $P(x)$ is false

Example 1.2.5. *Show that the following statement is false*

$$\forall x \in \mathbb{R}; x^2 - 1 > 1$$

A counter example is $x = 0 \in \mathbb{R}$*; because* $(0)^2 - 1 < 1$*. so* $(0)^2 - 1 > 1$ *is false.*

1.2.5 Reasoning by Indiction

The principle of indiction allows us to show that an assertion $P(n)$, depending on n, is true for all $n \in \mathbb{N}$. The proof by induction proceeds in three steps :

I) We prove $P(0)$ is true.

II) We assume $\forall n \in \mathbb{N}$, $P(n)$ is true

III) We demonstrate that the assertion $P(n+1)$ is true.

Finally, in the conclusion, we recall that by the principle of indiction $P(n)$ is true for all $n \in \mathbb{N}$

Example 1.2.6. *Show that for all* $n \in \mathbb{N} : 2^n > n$

Proof. Let us note : $P(n)$: $2^n > n$; for all $n \in \mathbb{N}$:
We will demonstrate by indiction that $P(n)$ is true for all $n \in \mathbb{N}$.
i) For $n = 0$ we have $2^0 = 1 > 0$, so $P(0)$ is true.
ii) Let $n \in \mathbb{N}$, suppose $P(n)$ is true $(2^n > n)$
iii) We will show that $P(n+1)$ is true.

$$\begin{aligned} 2^{n+1} &= 2^n + 2^n \\ &\geq n + 2^n \text{ ; } \textit{because by } P(n) \textit{ we know that } 2^n > n; \\ &\geq n + 1 \text{ ; } \textit{because } 2^n \geq 1 : \end{aligned}$$

So $P(n+1)$ is true □

Chapter 2

SETS, RELATIONS AND APPLICATIONS

2.1 Sets

Definition 2.1.1. *A set is a collection of objects that share a common property. Each object is an element of the set.*

Remark 2.1.1. *An element x is distinct from the set $\{x\}$, meaning $x \neq \{x\}$.*

Example 2.1.1. *Let E be the set of divisors of 20, so $E = \{1, 2, 4, 5, 10, 20\}$.*

2.1.1 Element, Inclusion, and Equality

Let E be a set.

a) If x is an element of E, we write $x \in E$. If x is not an element of E, we write $x \notin E$.

b) Set E is included in set F if every element of E is also an element of F. This is written as $E \subset F$ if and only if $\forall x, x \in E \Rightarrow x \in F$. We also say E is a subset of F.

c) Two sets E and F are equal if $E \subset F$ and $F \subset E$, and we write $E = F$ if and only if $\forall x, x \in E \Leftrightarrow x \in F$.

The empty set, denoted $\emptyset$, is the set with no elements, and it is included in every set E.

2.1.2 Union and Intersection

a) The intersection of two sets E and F is the set of elements common to both sets, written as

$$E \cap F = \{x \mid x \in E \text{ and } x \in F\}.$$

If $E \cap F = \emptyset$, then we say E and F are disjoint.

b) The union of two sets E and F is the set of all elements from both sets, written as

$$E \cup F = \{x \mid x \in E \text{ or } x \in F\}.$$

2.1.3 Difference of Two Sets

The difference of two sets E and F is the set of elements in E that are not in F, written as

$$E - F = \{x \mid x \in E \text{ and } x \notin F\}.$$

If $F \subset E$, then $E - F$ is called the complement of F in E, denoted $C_E F$.

2.1.4 Symmetric Difference

The symmetric difference of two sets E and F is defined as

$$E \Delta F = (E - F) \cup (F - E).$$

2.1.5 Power Set

Let E be a set. The power set $P(E)$ is the set of all subsets of E, and is written as

$$P(E) = \{X \mid X \subset E\}.$$

Remark 2.1.2. *a) The empty set and E itself are elements of $P(E)$.*

b) For $E = \{a, b, c\}$, the power set is

$$P(E) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

2.1.6 Partition of a Set

Let E be a set, and let A be a collection of subsets of E. We say that A is a partition of E if:

1. Each element of A is non-empty.
2. The elements of A are pairwise disjoint.
3. The union of the elements of A is equal to E.

Example 2.1.2. *Let $E = \{a, b\}$, then $A = \{\{a\}, \{b\}\}$ is a partition of E.*

2.1.7 Cartesian Product

Definition 2.1.2. *The Cartesian product of two sets E and F is the set of ordered pairs (x, y) such that $x \in E$ and $y \in F$, and is written as*

$$E \times F = \{(x, y) \mid x \in E \text{ and } y \in F\}.$$

Properties

- $E \times F = \emptyset$ if and only if $E = \emptyset$ or $F = \emptyset$.
- $E \times F = F \times E$ if and only if $E = F$, or $E = F = \emptyset$.
- $E \times (F \cup G) = (E \times F) \cup (E \times G)$.
- $(E \cup G) \times F = (E \times F) \cup (G \times F)$.
- $(E \times F) \cap (G \times H) = (E \cap G) \times (F \cap H)$.
- $(E \times F) \cup (G \times H) \neq (E \cup G) \times (F \cup H)$.

Example 2.1.3. *Let $E = F = \{0\}$ and $G = H = \{1\}$. Then:*

$$E \times F = \{(0, 0)\}, \quad G \times H = \{(1, 1)\}, \quad (E \cup G) \times (F \cup H) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

2.2 Binary Relations in a Set

Definition and Properties

Definition 2.2.1. *Let E be a set, and $x, y \in E$. If there is a link connecting x and y, we say they are related by a relation R, and we write xRy or $R(x, y)$.*

Example 2.2.1. *Let $E = \mathbb{R}$, and for all $x, y \in E$, xRy if and only if $|x| - |y| = x - y$.*

Properties of a Binary Relation

A binary relation R can have several important properties:

1. **Reflexivity:** We say R is reflexive on E if, for every $x \in E$, xRx.
2. **Symmetry:** We say R is symmetric on E if, for all $x, y \in E$, $xRy \implies yRx$.
3. **Antisymmetry:** We say R is antisymmetric on E if, for all $x, y \in E$, xRy and $yRx \implies x = y$.
4. **Transitivity:** We say R is transitive on E if, for all $x, y, z \in E$, xRy and $yRz \implies xRz$.

2.2.1 Equivalence Relation

We say R is an **equivalence relation** on E if it is:

- Reflexive,
- Symmetric,
- Transitive.

Equivalence Class: Let R be an equivalence relation on E, and let $a \in E$. The **equivalence class** of a, denoted $\overline{a}$, is the set of all elements $y \in E$ that are related to a by R, i.e.:

$$\overline{a} = \{y \in E, yRa\}$$

Quotient Set: Let R be an equivalence relation on E. The **quotient set** of E by the relation R, denoted E/R, is the set of equivalence classes of all elements of E:

$$E/R = \{\overline{a}, a \in E\}$$

Example 2.2.2. *Let $E = \mathbb{R}$, and the binary relation R defined by xRy if and only if $x^2 - y^2 = x - y$.*

i) Proving that R is an equivalence relation

1. **Reflexivity:** For any $x \in \mathbb{R}$, we have $x^2 - x^2 = x - x = 0$, so xRx.
2. **Symmetry:** If xRy, then $x^2 - y^2 = x - y$, which implies $y^2 - x^2 = y - x$, so yRx.
3. **Transitivity:** If xRy and yRz, then $x^2 - y^2 = x - y$ and $y^2 - z^2 = y - z$, which gives:

$$(x^2 - y^2) + (y^2 - z^2) = (x - y) + (y - z)$$

Simplifying this, we get $x^2 - z^2 = x - z$, so xRz.

Thus, R is an equivalence relation.

ii) Equivalence Class of $\overline{a}$
Let $a \in \mathbb{R}$. The equivalence class of a is given by:

$$\overline{a} = \{x \in \mathbb{R}, x^2 - a^2 = x - a\}$$

This equation factors as $(x - a)(x + a - 1) = 0$, so the solutions are $x = a$ or $x = 1 - a$. Therefore, the equivalence class of a is $\{a, 1 - a\}$.

2.2.2 Order Relation

A binary relation R in a set E is an **order relation** if it is:

- Reflexive,
- Antisymmetric,
- Transitive.

Partial Order and Total Order:

- If for all $x, y \in E$, we have xRy or yRx, then R is a **total order**.
- If this is not the case, we say that R is a **partial order**, meaning there exist $x, y \in E$ such that neither xRy nor yRx is true.

Example 2.2.3. *Let $E = \{a, b, c\}$, and $P(E)$ be the power set of E, with the binary relation R defined by ARB if $A \subset B$, where $A, B \in P(E)$.*

i) Proving that R is an order relation

1. **Reflexivity:** For any $A \in P(E)$, it is clear that $A \subset A$, so ARA.

2. **Antisymmetry:** If ARB and BRA, then $A \subset B$ and $B \subset A$, so $A = B$.

3. **Transitivity:** If ARB and BRC, then $A \subset B$ and $B \subset C$, so $A \subset C$, thus ARC.

Thus, R is an order relation.
ii) Is this order total

We have $E = \{a, b, c\}$, so $P(E) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The order is partial, because there exist elements such as $A = \{a\}$ and $B = \{b\}$ where $A \not\subset B$ and $B \not\subset A$.

2.3 Applications, Functions

Definition 2.3.1. *Let E and F be two sets.*

1. *A function from set E to set F is a relation from E to F such that for each element $x \in E$, there corresponds at most one element $y \in F$. x is called the antecedent, E is the domain or the set of antecedents, y is called the image, and F is the codomain or set of images.*

2. *A mapping from E to F is a relation from E to F such that for each element $x \in E$, there corresponds exactly one element $y \in F$.*

3. *Two mappings are equal if their domains are equal, their codomains are equal, and their values are also equal.*

In general, a function or mapping f is denoted as:

$$f : E \to F \quad x \mapsto y = f(x).$$

The set $\Gamma = \{(x, f(x)) \mid x \in E\}$ is called the graph of f.

Example 2.3.1.

$$f : \mathbb{R} \to \mathbb{R} \quad x \mapsto \frac{x}{x-1}, \quad g : \mathbb{R} \setminus \{1\} \to \mathbb{R} \quad x \mapsto \frac{x}{x-1}.$$

In this example, g is a mapping, but f is a function and not a mapping because the element 1 does not have an image in $\mathbb{R}$.

2.3.1 Restriction and Extension of a Mapping

Let E_0 be a subset of E, and $f : E \to F$ a mapping. The mapping $g : E_0 \to F$ such that $\forall x \in E_0, g(x) = f(x)$ is called the restriction of f to E_0, and is written as $g = f/E_0$. We also say that f is the extension of g to E.

2.3.2 Composition of Mappings

Let E, F, and G be three sets, and $f : E \to F$, $g : F \to G$ two mappings. The composition of f and g, denoted $g \circ f$, is defined by:

$$\forall x \in E, (g \circ f)(x) = g(f(x)).$$

2.3.3 Injection, Surjection, and Bijection

Let $f : E \to F$ be a mapping.

1. f is injective if $\forall x, x' \in E, f(x) = f(x') \implies x = x'$, or equivalently, $\forall x, x' \in E, x \neq x' \implies f(x) \neq f(x')$.
2. f is surjective if $\forall y \in F, \exists x \in E : y = f(x)$.
3. f is bijective if f is both injective and surjective.

Properties:

1. f is injective $\iff$ the equation $y = f(x)$ has at most one solution.
2. f is surjective $\iff$ the equation $y = f(x)$ has at least one solution.
3. f is bijective $\iff$ the equation $y = f(x)$ has exactly one solution.

Proposition 2.1: Let $f : E \to F$ and $g : F \to G$ be two mappings. Then:

1. If $g \circ f$ is injective, then f is injective.
2. If $g \circ f$ is surjective, then g is surjective.
3. If $g \circ f$ is bijective, then f is injective and g is surjective.

2.3.4 Inverse Mappings

Definition 2.4: Let $f : E \to F$ be a bijective mapping. Then there exists an inverse mapping $f^{-1} : F \to E$ defined by:

$$y = f(x) \iff x = f^{-1}(y),$$

which is called the inverse mapping of f.

Theorem 2.1: Let $f : E \to F$ be a bijective mapping. Then its inverse mapping f^{-1} satisfies:

$$f \circ f^{-1} = \mathrm{Id}_F \quad \text{and} \quad f^{-1} \circ f = \mathrm{Id}_E.$$

where $\mathrm{Id}_E : E \to E$ is the identity mapping on E, defined by $x \mapsto x$.

2.3.5 Properties of Composition of Mappings

Proposition 2.2: Let $f : E \to F$ and $g : F \to G$ be two mappings. Then:

1. If f and g are injective, then $g \circ f$ is injective.
2. If f and g are surjective, then $g \circ f$ is surjective.
3. If f and g are bijective, then $g \circ f$ is bijective, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2.3.6 Direct and Inverse Images

1. **Direct Image:** Let $f : E \to F$ be a mapping and A a subset of E. The direct image of A under f, denoted $f(A)$, is the subset of F defined by:

$$f(A) = \{y \in F \mid \exists x \in A : y = f(x)\} = \{f(x) \mid x \in A\}.$$

Example 2.8: Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $A = [-2, 1]$. Then:

$$f(A) = \{f(x) \mid x \in A\} = \{x^2 \mid x \in [-2, 1]\} = [0, 4].$$

2. **Inverse Image:** Let $f : E \to F$ be a mapping and B a subset of F. The inverse image of B under f, denoted $f^{-1}(B)$, is the subset of E defined by:

$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}.$$

Example 2.9: Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $B = [0, 4]$. Then:

$$f^{-1}([0, 4]) = \{x \in \mathbb{R} \mid x^2 \in [0, 4]\} = [-2, 2].$$

Proposition 2.3.1. *Let $f : E \to F$ be a function. Let A_1, A_2 be subsets of E. Then:*

(a) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

(b) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(c) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ *(the other inclusion holds if f is injective).*

If B_1, B_2 are two subsets of F, then:

(a') $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(b') $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(c') $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(d') $f^{-1}(F \setminus B_1) = E \setminus f^{-1}(B_1)$.

Moreover, if $A \subset E$ and $B \subset F$, then:

(a") $f(f^{-1}(B)) \subset B$ *and equality holds if f is surjective.*

(b") $A \subset f^{-1}(f(A))$ *and equality holds if f is injective.*

Proof:

(a) Suppose $A_1 \subset A_2$, and we show that $f(A_1) \subset f(A_2)$. Let $y \in f(A_1)$. Then:

$$y \in f(A_1) \Rightarrow \exists x \in A_1 : y = f(x)$$

Since $A_1 \subset A_2$, we have:

$$\exists x \in A_2 : y = f(x)$$

Hence, $y \in f(A_2)$, and therefore $f(A_1) \subset f(A_2)$.

(b) Let $y \in f(A_1 \cup A_2)$. Then:

$$y \in f(A_1 \cup A_2) \Leftrightarrow \exists x \in (A_1 \cup A_2) : y = f(x)$$

This implies:

$$\exists x \in A_1 \text{ or } \exists x \in A_2 : y = f(x)$$

Hence:

$$y \in f(A_1) \text{ or } y \in f(A_2) \Rightarrow y \in f(A_1) \cup f(A_2)$$

Therefore, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(c) Let $y \in f(A_1 \cap A_2)$. Then:

$$y \in f(A_1 \cap A_2) \Leftrightarrow \exists x \in (A_1 \cap A_2) : y = f(x)$$

This implies:

$$\exists x \in A_1 \text{ and } \exists x \in A_2 : y = f(x)$$

Hence:

$$y \in f(A_1) \text{ and } y \in f(A_2) \Rightarrow y \in f(A_1) \cap f(A_2)$$

Thus, $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. Suppose f is injective and show the second inclusion. Let $y \in f(A_1) \cap f(A_2)$. Then:

$$y \in f(A_1) \cap f(A_2) \Rightarrow y \in f(A_1) \text{ and } y \in f(A_2)$$

Hence, there exist $x_1 \in A_1$ and $x_2 \in A_2$ such that:

$$y = f(x_1) = f(x_2)$$

Since f is injective, it follows that $x_1 = x_2 = x$. Thus, $x \in A_1 \cap A_2$, and $y = f(x)$, so $y \in f(A_1 \cap A_2)$.

Exercise 2.3.1. *Consider the function:*

$$f : [-1, 1] \to \mathbb{R}, \quad x \mapsto f(x) = \frac{1}{1 + x^2}$$

1) Calculate $f^{-1}(\{2\})$ and $f^{-1}\left(\frac{1}{2}\right)$.
2) Study the injectivity and surjectivity of f.

Solution:
1) $f^{-1}\left(\frac{1}{2}\right) =$? Let $x \in f^{-1}\left(\frac{1}{2}\right)$. Then:

$$x \in [-1, 1] \text{ and } f(x) = \frac{1}{2}$$

This implies:

$$x \in [-1, 1] \text{ and } \frac{1}{1 + x^2} = \frac{1}{2}$$

Solving for x^2:

$$x^2 - 1 = 0 \Rightarrow x = \pm 1$$

Thus:

$$f^{-1}\left(\frac{1}{2}\right) = \{-1, 1\}$$

$f^{-1}(\{2\}) =?$ Let $x \in f^{-1}(\{2\})$. Then:

$$x \in [-1, 1] \text{ and } f(x) = 2$$

This implies:

$$x \in [-1, 1] \text{ and } \frac{1}{1 + x^2} = 2$$

which leads to a contradiction because $x^2 = -1$, which is impossible. Therefore:

$$f^{-1}(\{2\}) = \emptyset$$

2) Is f injective? From the first question, we have $f(1) = f(-1) = \frac{1}{2}$. Hence, f is not injective.

Is f surjective? Since there is no $x \in [-1, 1]$ such that $f(x) = 2$, f is not surjective.

Chapter 3

Real Functions of a Real Variable

3.1 Generalities

Definition 3.1.1. *A numerical function defined on a domain X is any application f such that for each point x in X, a single element y in $\mathbb{R}$ is associated. This is written as*

$$f : X \to \mathbb{R}, \quad x \mapsto f(x) = y.$$

X is the domain of definition of f. The set of values of f, or the image set of f, is given by

$$f(X) = Im(f) = \{y \in \mathbb{R}, \exists x \in X; f(x) = y\}.$$

Graph of a Function

The graph of a function f is the geometric locus of points $M(x, y)$ where $x \in X$ and $y = f(x)$. This is written as

$$G_f = \{(x, y), x \in X, y = f(x)\}.$$

Operations on Real Functions

Let $f, g : X \to \mathbb{R}$.

Equality and Inequality

1. We say that f is equal to g, and write

$$f = g \iff f(x) = g(x),\ \forall x \in X.$$

2. We say that f is less than or equal to g, and write

$$f \leq g \iff f(x) \leq g(x),\ \forall x \in X.$$

3. We say that f is greater than or equal to g, and write

$$f \geq g \iff f(x) \geq g(x),\ \forall x \in X.$$

Arithmetic Operations

1. Sum:

$$(f+g)(x) = f(x) + g(x),\ \forall x \in X.$$

2. Product:

$$(f \cdot g)(x) = f(x) \cdot g(x),\ \forall x \in X.$$

3. Quotient:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ \forall x \in X,\ g(x) \neq 0.$$

Composition of Functions

Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$, with $f(X) \subset Y$. The composition of f and g is defined and written as

$$(g \circ f)(x) = g(f(x)),\ \forall x \in X.$$

Example 3.1.1. *Let $f(x) = \cos x$ and $g(x) = x^2$, where $x \in \mathbb{R}$. Then*

$$(g \circ f)(x) = g(f(x)) = (f(x))^2 = \cos^2 x,$$

$$(f \circ g)(x) = f(g(x)) = \cos(g(x)) = \cos(x^2).$$

Clearly, $f \circ g \neq g \circ f$.

3.1.1 General Properties of Functions

Even and Odd Functions

Definition 3.1.2. *The function f defined on the symmetric set X is said to be*

1. *even if for all $x \in X$, $f(x) = f(-x)$,*

2. *odd if for all $x \in X$, $f(-x) = -f(x)$.*

Periodic Functions

A function $f : X \to \mathbb{R}$ is said to be periodic if there exists $\alpha \in \mathbb{R}^+$ such that

1. $x + \alpha \in X$,

2. $f(x + \alpha) = f(x),\ \forall x \in X$.

It is clear that $f(x+k\alpha)=f(x),\ k\in\mathbb{N}^*$.

Definition 3.1.3. *The period of f is the smallest positive number T such that $f(x+T)=f(x),\ \forall x\in X$.*

Monotonic Functions

Let $f:X\to\mathbb{R}$, f is said to be:

1. *increasing* if for all $x_1,x_2\in X$, $x_1\leq x_2 \implies f(x_1)\leq f(x_2)$,
2. *strictly increasing* if for all $x_1,x_2\in X$, $x_1< x_2 \implies f(x_1)< f(x_2)$,
3. *decreasing* if for all $x_1,x_2\in X$, $x_1\leq x_2 \implies f(x_1)\geq f(x_2)$,
4. *strictly decreasing* if for all $x_1,x_2\in X$, $x_1< x_2 \implies f(x_1)> f(x_2)$.

Bounded Functions

The function f is said to be:

1. *upper bounded* on X if there exists $M\in\mathbb{R}$ such that $f(x)\leq M,\ \forall x\in X$,
2. *lower bounded* on X if there exists $m\in\mathbb{R}$ such that $f(x)\geq m,\ \forall x\in X$,
3. *bounded* on X if it is both upper and lower bounded simultaneously.

Definition 3.1.4. *The least upper bound (supremum) and greatest lower bound (infimum) of f on X are defined as the smallest upper bound and the largest lower bound of f respectively, and are written as*

$$\sup_{x\in X} f(x)=M \iff \forall\epsilon>0,\ \exists x_0\in X,\ M-\epsilon<f(x_0),$$

$$\inf_{x\in X} f(x)=m \iff \forall\epsilon>0,\ \exists x_0\in X,\ f(x_0)<m+\epsilon.$$

Theorem 3.1.1. *Every upper bounded (resp. lower bounded) function has a least upper bound (resp. greatest lower bound).*

Maximum and Minimum of a Function

Definition 3.1.5. *We say that a function has a maximum (resp. minimum) at point $x_0\in X$ if for all $x\in X$, $f(x)\leq f(x_0)$ (resp. $f(x)\geq f(x_0)$).*

3.1.2 Inverse Functions

Let $f : X \to Y$, f is said to be invertible if there exists $g : Y \to X$ such that

$$(g \circ f)(x) = x \quad \text{and} \quad (f \circ g)(y) = y.$$

The function g is called the inverse of f and is denoted by f^{-1}, and it is written as

$$y = f(x) \iff x = f^{-1}(y).$$

Properties

Let $f : X \to Y$ be invertible (bijective), then:

1. The inverse of f^{-1} is f, i.e. $(f^{-1})^{-1} = f$.
2. If f is odd (resp. even), then f^{-1} is odd (resp. even).
3. If f is strictly monotone, then f^{-1} is also strictly monotone.

Graph of an Inverse Function

Let G_f be the graph of an invertible function and $G_{f^{-1}} = \{(y, f^{-1}(y)) : y \in Y\}$ be the graph of f^{-1}. Then:

$$(x, y) \in G_{f^{-1}} \iff x = f^{-1}(y), y \in Y \iff y = f(x), x \in X \iff (x, y) \in G_f.$$

The graphs G_f and $G_{f^{-1}}$ are symmetric with respect to the first bisector $y = x$.

3.2 Limit of a Function

Definition 3.2.1. *Let f be a function defined in a neighborhood of x_0, except possibly at x_0. The number l is called the limit of f as x approaches x_0, and we write $l = \lim_{x \to x_0} f(x)$ if*

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon).$$

Example 3.2.1. *Prove that $\lim_{x \to 0} \sin x = 0$. In fact, we have $|\sin x - 0| = |\sin x| \leq |x|$. Therefore,*

$$\forall \epsilon > 0, \exists \delta = \epsilon > 0, \forall x, (0 < |x| < \delta \Rightarrow |\sin x - 0| < \epsilon).$$

Definition 3.2.2. *The number l is said to be the limit of f as x approaches x_0 if and only if for every sequence (x_n) in $V_0(x_0)$ (a punctured neighborhood of x_0) converging to x_0, the sequence $y_n = f(x_n)$ converges to l, and we write*

$$\forall (x_n) \in V_0(x_0), \left(\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = l \right).$$

Remark 3.2.1. *From the previous definition, if two sequences (u_n), (v_n) converge to x_0 such that*

$$\lim_{n\to\infty} f(u_n) \neq \lim_{n\to\infty} f(v_n),$$

then the limit of f does not exist at x_0.

3.2.1 Extension of the Concept of Limit

Right and Left Limits

Let $V \subset \mathbb{R}$ be a set containing the interval (a, x_0) (or (x_0, b)).

Definition 3.2.3. *The number $l \in \mathbb{R}$ is said to be the right-hand limit (resp. left-hand limit) of f at x_0 if:*

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall x \in V, (x - x_0 < \delta \Rightarrow |f(x) - l| < \epsilon)$$

(resp.

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall x \in V, (x_0 - x < \delta \Rightarrow |f(x) - l| < \epsilon)).$$

The right-hand limit (resp. left-hand limit) is denoted by

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^+} f(x) = f(x_0 + 0)$$

(resp.

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0 - 0)).$$

Theorem 3.2.1.

$$\lim_{x\to x_0} f(x) = l \iff \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x).$$

Limit at Infinity

$$\lim_{x\to+\infty} f(x) = l \iff \forall \epsilon > 0, \exists A > 0, \forall x \in V(-\infty), (x < -A \Rightarrow |f(x) - l| < \epsilon),$$

and

$$\lim_{x \to -\infty} f(x) = l \iff \forall \epsilon > 0, \exists A > 0, \forall x \in V(+\infty), (x > A \Rightarrow |f(x) - l| < \epsilon).$$

Infinite Limit

$$\lim_{x \to x_0} f(x) = +\infty \iff \forall A > 0, \exists \delta > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow f(x) > A),$$

$$\lim_{x \to x_0} f(x) = -\infty \iff \forall A > 0, \exists \delta > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow f(x) < -A),$$

$$\lim_{x \to +\infty} f(x) = +\infty \iff \forall A > 0, \exists B > 0, \forall x \in V(+\infty), (x > B \Rightarrow f(x) > A).$$

Uniqueness of the Limit

Theorem 3.2.2. *If a function f has a limit at x_0, then this limit is unique.*

Proof. Suppose that f has two limits l_1 and l_2 at x_0. Then we have:

$$\lim_{x \to x_0} f(x) = l_1 \iff \forall \epsilon > 0, \exists \delta_1 = \delta_1(\epsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - l_1| < 2\epsilon)$$

and

$$\lim_{x \to x_0} f(x) = l_2 \iff \forall \epsilon > 0, \exists \delta_2 = \delta_2(\epsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - l_2| < 2\epsilon).$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for $0 < |x - x_0| < \delta$, we have:

$$|l_1 - l_2| = |(l_1 - f(x)) + (f(x) - l_2)| \leq |l_1 - f(x)| + |f(x) - l_2| < \epsilon + \epsilon = \epsilon.$$

Thus, $l_1 = l_2$. □

Theorem 3.2.3. ***Local Properties*** *If $\lim_{x \to x_0} f(x) = l$, then there exists a neighborhood of the point x_0 in which f is bounded, i.e.*

$$\exists V(x_0) : \forall x \in V(x_0), |f(x)| \leq M.$$

Limit Transition in Inequalities:

Theorem 3.2.4. *Let f and g be two functions defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$ with $l_1 < l_2$. Then, there exists a neighborhood of the point x_0 in which $f(x) \leq g(x)$.*

Proof. We have

$$\lim_{x\to x_0} f(x) = l_1 \iff \forall\epsilon > 0, \exists\delta_1 = \delta_1(\epsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - l_1| < \epsilon)$$

and

$$\lim_{x\to x_0} g(x) = l_2 \iff \forall\epsilon > 0, \exists\delta_2 = \delta_2(\epsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - l_2| < \epsilon).$$

We define $\delta = \min(\delta_1, \delta_2)$, and let $l_1 < l < l_2$. By choosing $\epsilon = l - l_1 > 0$ in the first equivalence (resp. $\epsilon = l_2 - l > 0$ in the second), we obtain, for $0 < |x - x_0| < \delta$,

$$f(x) < l \quad \text{and} \quad g(x) > l.$$

Thus, the result follows. □

Corollary 3.2.1. *Let f be a function defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = l$ and $f(x) > a$, then*

$$l \geq a.$$

Theorem 3.2.5. *Let f, g, and h be three functions defined in a neighborhood of x_0 such that*

$$f(x) \leq g(x) \leq h(x), \quad \forall x \in V_0(x_0).$$

If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l$, then $\lim_{x\to x_0} g(x) = l$.

Proof. We have

$$\lim_{x\to x_0} f(x) = l \iff \forall (x_n) \in V_0(x_0), \left(\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = l\right)$$

and

$$\lim_{x\to x_0} h(x) = l \iff \forall (x_n) \in V_0(x_0), \left(\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} h(x_n) = l\right).$$

We also have

$$f(x_n) \leq g(x_n) \leq h(x_n) \quad \forall x_n \in V_0(x_0),$$

and the result follows from the theorem of the three sequences. □

Example 3.2.2. *Study the limit of $f(x) = x\sin\left(\frac{1}{x}\right)$ at 0. We have $\forall x \in \mathbb{R}^*$,*

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x \leq x\sin\left(\frac{1}{x}\right) \leq x \quad \textit{for} \quad x > 0.$$

Using the previous theorem, we obtain

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$$

Operations on Limits: Let f and g be two functions defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$, then we have:

1. $\lim_{x\to x_0}(f \pm g)(x) = l_1 \pm l_2$,
2. $\lim_{x\to x_0}(f \cdot g)(x) = l_1 \cdot l_2$,
3. $\lim_{x\to x_0}(\lambda f)(x) = \lambda l_1$,
4. $\lim_{x\to x_0}\left(\frac{f}{g}\right)(x) = \frac{l_1}{l_2}, \quad \text{if} \quad \lim_{x\to x_0} g(x) \neq 0$,
5. $\lim_{x\to x_0} |f(x)| = |l_1|$.

Proof. Let us prove (4) as an example. We have

$$\lim_{x\to x_0} f(x) = l_1 \iff \forall (x_n) \in V_0(x_0), \left(\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = l_1\right)$$

and

$$\lim_{x\to x_0} g(x) = l_2 \iff \forall (x_n) \in V_0(x_0), \left(\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} g(x_n) = l_2\right).$$

By the theorem of the limit of the ratio of two sequences, we find

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)} = \frac{l_1}{l_2},$$

and the result follows. □

Example 3.2.3.

$$\lim_{x\to+\infty} \sqrt{x^2+3x-4} + x = \lim_{x\to+\infty} \sqrt{x^2+3x-4} + \lim_{x\to+\infty} x = (+\infty) + (+\infty) = +\infty$$

3.3 Continuity of a Function

3.3.1 Definitions and Properties

Definition 3.3.1. *A function $f : X \to \mathbb{R}$ is continuous at the point $x_0 \in X$ if $\lim_{x\to x_0} f(x) = f(x_0)$, i.e.,*

$$\forall\epsilon > 0, \exists\delta = \delta(\epsilon) > 0, \forall x \in V(x_0), (0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)\,.$$

Remark 3.3.1. *If f is not defined at x_0, it cannot be continuous at x_0.*
f is said to be continuous on X if it is continuous at every point of X.

Example 3.3.1. $f(x) = |x|$, $x_0 \in \mathbb{R}$

$$|f(x) - f(x_0)| = ||x| - |x_0|| \leq |x - x_0|.$$

Therefore, it suffices to choose $\delta = \varepsilon$ in the previous definition.

Definition 3.3.2. *(Continuity from the left and right): A function defined at x_0 and to the right (resp. to the left) of x_0 is continuous from the right (resp. from the left) at x_0 if*

$$\lim_{x\to x_0^+} f(x) = f(x_0) \quad \textit{(resp.} \lim_{x\to x_0^-} f(x) = f(x_0)).$$

Thus, f is continuous at $x_0 \iff \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0)$.

Example 3.3.2. *Let the function h be defined by*

$$h(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

We have $h(0) = 0$.

$$\lim_{x\to 0^+} h(x) = 1 \neq h(0), \quad \textit{so } h \textit{ is not continuous from the right at } 0.$$

$$\lim_{x\to 0^-} h(x) = 0 = h(0), \quad \textit{so } h \textit{ is continuous from the left at } 0.$$

Therefore, h is not continuous at 0.

3.3.2 Operations on Continuous Functions

Proposition 3.3.1. *Let f and g be two functions continuous at x_0, then*

1. *For all $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is continuous at x_0.*

2. *$f \cdot g$ is continuous at x_0.*

3. *If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.*

4. *$|f|$ is continuous at x_0.*

Continuity of Composite Functions

Proposition 3.3.2. *Let $f : I \to I'$ and $g : I' \to \mathbb{R}$ be continuous at x_0 and $f(x_0)$, respectively. Then the composite function $g \circ f : I \to \mathbb{R}$ is continuous at x_0.*

Proof. Let $(x_n) \subset I$ be a sequence converging to $x_0 \in I$. We need to show that $(g \circ f)(x_n)$ converges to $(g \circ f)(x_0)$. We have

$$x_n \to x_0 \implies f(x_n) \to f(x_0) \quad \text{(since } f \text{ is continuous at } x_0\text{)},$$

$$\implies g(f(x_n)) \to g(f(x_0)) \quad \text{(since } g \text{ is continuous at } f(x_0)\text{)}.$$

Thus, $(g \circ f)(x_n) \to (g \circ f)(x_0)$. □

Remark 3.3.2. *We say that f is discontinuous at x_0 if*

1. *f is not defined at x_0,*

2. *The limit exists but is different from $f(x_0)$,*

3. *The limit does not exist.*

Extension by Continuity If f is not defined at x_0 (i.e., f is defined on $I - \{x_0\}$) and $\lim_{x \to x_0} f(x) = l$, where $l \in \mathbb{R}$, then we define the extension by continuity of f at x_0 by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0, \\ l & \text{if } x = x_0. \end{cases}$$

Example 3.3.3. *Let $f(x) = e^{\frac{1}{-x^2}}$, $x \in \mathbb{R}^*$. We have $\lim_{x \to 0} f(x) = 0$, and thus*

$$\tilde{f}(x) = \begin{cases} e^{\frac{1}{-x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This is the extension by continuity of f at 0.

3.3.3 Fundamental Theorems on Continuous Functions

Theorem 3.3.1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then:*

1. *f is bounded.*
2. *f attains its bounds, i.e.,*

$$\exists \alpha \in [a, b] : \sup_{x \in [a,b]} f(x) = f(\alpha) = M$$

and

$$\exists \beta \in [a, b] : \inf_{x \in [a,b]} f(x) = f(\beta) = m.$$

Theorem 3.3.2. *(Intermediate Value Theorem)*
Let $f : [a, b] \to \mathbb{R}$ be a function such that:

1. *f is continuous on $[a, b]$,*
2. *$f(a) \cdot f(b) < 0$.*

Then

$$\exists x_0 \in]a, b[: f(x_0) = 0.$$

Furthermore, if f is strictly monotonic, then x_0 is unique.

Example 3.3.4. *1. Show that $\ln x - \frac{1}{x} = 0$ has a unique solution on $[1, 2]$.*
Let $F(x) = \ln x - \frac{1}{x}$, F is continuous on $[1, 2]$.

$$F(1) = \ln 1 - 1 = -1, \quad F(2) = \ln 2 - \frac{1}{2} = 0.19.$$

By the Intermediate Value Theorem, there exists $x_0 \in]1, 2[$ such that

$$F(x_0) = 0, \quad i.e., \quad \ln x_0 - \frac{1}{x_0} = 0.$$

Uniqueness: $F'(x) = \frac{1}{x} + \frac{1}{x^2}$, so F is strictly increasing in $]1, 2[$. Thus, the solution is unique.

2. $f(x) = E(x) - \frac{1}{2}, x \in [0, 1]$. The Intermediate Value Theorem does not apply to f because it is not continuous at 1.

3.4 Differentiable Functions

Definition 3.4.1. *Let I be an interval of $\mathbb{R}$, $x_0 \in I$, and $f : I \to \mathbb{R}$ be a function. We say that f is differentiable at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called the derivative of f at x_0 and is denoted $f'(x_0)$.

Another form of the derivative at a point:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example 3.4.1. *Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = x^3.$$

Find the derivative of f at a point $x_0 \in \mathbb{R}$. We have $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} = \lim_{x \to x_0} \frac{(x-x_0)(x^2+x_0x+x_0^2)}{x-x_0} = \lim_{x \to x_0}(x^2 + x_0 x + x_0^2) = 3x_0^2$.

Right-hand Derivative and Left-hand Derivative

Definition 3.4.2. *The right-hand derivative of f at x_0 is defined as*

$$f'_r(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

Similarly, the left-hand derivative of f at x_0 is defined as

$$f'_l(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Thus, f is differentiable at x_0 if and only if

$$f'_r(x_0) = f'_l(x_0) = f'(x_0).$$

Example 3.4.2. *Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 0, \\ 1 - 2x & \text{if } x < 0. \end{cases}$$

We have $f(0) = 1$, is f differentiable at 0? We calculate

$$f'_r(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x + 1 - 1}{x} = \lim_{x \to 0^+} 1 = 1.$$

and

$$f_l'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{1-2x-1}{x} = \lim_{x\to 0^-} -2 = -2.$$

Thus, f is not differentiable at 0 because $f_r'(0) \neq f_l'(0)$.

Definition 3.4.3. *f is differentiable on I if it is differentiable at every point of I, and the map*

$$f' : I \to \mathbb{R}, \quad x \mapsto f'(x)$$

is called the derivative function of f.

3.4.1 Fundamental Theorems on Derivable Functions

Equation of the tangent:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

Here, $f'(x_0)$ represents the slope of the tangent line to the curve C at the point $M(x_0, f(x_0))$.

Derivability and Continuity

If f is derivable at x_0, then f is continuous at x_0. The converse is generally false.

Example 3.4.3. $f(x) = |x|$, $x \in \mathbb{R}$. *f is continuous at* 0, *but it is not derivable at* 0, *because*

$$f_r'(0) = 1 \neq -1 = f_l'(0).$$

Operations on Derivable Functions

Proposition 3.4.1. *Let f and g be two derivable functions at $x_0 \in \mathbb{R}$, then $(\alpha f), \alpha \in \mathbb{R}, f+g, f\cdot g$ are derivable at x_0, and $\frac{f}{g}$ is derivable at x_0 if $g(x_0) \neq 0$. Moreover,*

1. $(\alpha f)'(x_0) = \alpha f'(x_0)$,
2. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$,
3. $(f\cdot g)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$,
4. $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{g(x_0)^2}$.

Derivative of a Composite Function

Proposition 3.4.2. *Let $f : I \to I'$ and $g : I' \to \mathbb{R}$ be two derivable functions at x_0 and $f(x_0)$ respectively. Then, $g \circ f : I \to \mathbb{R}$ is derivable at x_0, and*

$$(g \circ f)'(x_0) = f'(x_0) \cdot g'(f(x_0)).$$

Proof. We have

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}.$$

As $y = f(x)$, the limit becomes:

$$= \lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0)) \cdot f'(x_0).$$

□

Example 3.4.4. *Let the functions f and g be defined by*

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) = x^2, \quad g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto g(x) = \cos x.$$

Then

$$g \circ f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto (g \circ f)(x) = \cos(x^2),$$

and

$$(g \circ f)'(x) = f'(x) \cdot g'(f(x)) = 2x \cdot \sin(x^2).$$

Derivative of a Reciprocal Function

Proposition 3.4.3. *If f is derivable at x_0, then f^{-1} is derivable at $f(x_0)$, and we have*

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Proof. We have

$$(f^{-1})'(f(x_0)) = \lim_{y \to f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \lim_{y \to f(x_0)} \frac{1}{f(y) - f(x_0)} \cdot \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{1}{f'(x_0)}.$$

□

Example 3.4.5. *The function*

$$f : \mathbb{R} \to (0, +\infty), \quad x \mapsto f(x) = e^x,$$

is bijective and therefore has an inverse function

$$f^{-1} :]0, +\infty[\to \mathbb{R}, \quad x \mapsto f^{-1}(x) = \ln(x),$$

with the property

$$y = e^x \iff \ln y = x.$$

We have

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{y}.$$

3.4.2 Fundamental Theorems on Derivable Functions

Theorem 3.4.1. *(Rolle's Theorem) Let $f : [a, b] \to \mathbb{R}$ be a function such that:*

1. *f is continuous on $[a, b]$,*
2. *f is differentiable on $]a, b[$,*
3. *$f(a) = f(b)$.*

Then, there exists $c \in]a, b[$ such that $f'(c) = 0$.

Rolle's theorem affirms that there is a point c where the tangent is parallel to the x-axis.

Theorem 3.4.2. *(Lagrange's Theorem or Mean Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a function such that:*

1. *f is continuous on $[a, b]$,*
2. *f is differentiable on $]a, b[$.*

Then, there exists $c \in]a, b[$ such that:

$$f(b) - f(a) = (b - a)f'(c).$$

Proof. Consider the function $g : [a, b] \to \mathbb{R}$ defined by:

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a),$$

for all $x \in [a, b]$. The function g is:

1. continuous on $[a, b]$ (and differentiable on $]a, b[$) because it is the sum and product of continuous (and differentiable) functions on $[a, b]$ (and $]a, b[$),

2. $g(a) = 0$, $g(b) = 0$.

By Rolle's Theorem, there exists $c \in]a, b[$ such that $g'(c) = 0$. We have:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in]a, b[$ such that:

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \Longrightarrow \quad f(b) - f(a) = (b - a)f'(c).$$

Lagrange's theorem tells us that there exists a point $c \in]a, b[$ where the tangent to the curve is parallel to the line joining the points $(a, f(a))$ and $(b, f(b))$. □

Example 3.4.6. *Use the Mean Value Theorem to show that for all $x > 0$, we have $\sin x \leq x$.*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(t) = t - \sin t \quad \textit{for all} \quad t \in \mathbb{R}.$$

The function f is:

1. *continuous on $[0, x]$ and differentiable on $]0, x[$ for all $x > 0$,*

2. *the sum of continuous (and differentiable) functions on $[0, x]$ (and $]0, x[$).*

By the Mean Value Theorem, there exists $c \in]0, x[$ such that:

$$f(x) - f(0) = (x - 0)f'(c),$$

which implies:

$$x - \sin x = x(1 - \cos c).$$

Since $x > 0$ and $\cos c \leq 1$, we obtain that for all $x > 0$, $\sin x \leq x$.

Theorem 3.4.3. *(Cauchy's Theorem) Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that:*

1. *f and g are continuous on $[a, b]$,*

2. *f and g are differentiable on $]a, b[$.*

Then, there exists $c \in]a,b[$ *such that:*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Higher Order Derivatives Let $f : I \to \mathbb{R}$ be a differentiable function on I. Then f', the first derivative of f, is called the first order derivative of f. If f' is differentiable on I, its derivative is called the second order derivative of f, denoted by f'' or $f^{(2)}$.

In general, the n-th order derivative of f is defined as:

$$f^{(n)} = \left(f^{(n-1)}\right)' \quad \forall n \geq 1, f^{(0)} = f.$$

We say that f is of class C^1 on I if f is differentiable on I and f' is continuous on I.
We say that f is of class C^n on I (and write $f \in C^n(I)$) if f is n-times differentiable on I and $f^{(n)}$ is continuous on I.
f is said to be of class C^∞ on I if it is of class C^n for all $n \in \mathbb{N}$.

The Leibniz Formula for the n-th Derivative of a Product

Theorem 3.4.4. *Let* $f, g : [a,b] \to \mathbb{R}$ *be* n *times differentiable. Then* $f \cdot g$ *is* n *times differentiable, and we have for all* $x \in [a,b]$*:*

$$(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x),$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Example 3.4.7. *Calculate* $x^3 \cos(4x)^{(4)}$*. We have:*

$$x^3 \cos(4x)^{(4)} = \sum_{k=0}^{4} \binom{4}{k} \left(x^3\right)^{(4-k)} (\cos(4x))^{(k)}.$$

3.4.3 Application of Derivatives

Monotonicity Criterion

Proposition 3.4.4. *Let* f *be a function from* I *to* $\mathbb{R}$*, differentiable on* I*. Then:*

1. $f' \geq 0$ *on* $I \iff f$ *is increasing on* I*,*
2. $f' \leq 0$ *on* $I \iff f$ *is decreasing on* I*.*

Proof. $\Rightarrow$ Assume that f is increasing on I and show that $f' \geq 0$ on I. Let $x_0 \in I$. For all $h \in \mathbb{R}^+$, we have:

$$\frac{f(x_0+h)-f(x_0)}{h} \geq 0 \quad \Rightarrow \quad \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \geq 0 \quad \Rightarrow \quad f'(x_0) \geq 0.$$

□

L'Hopital's Rule

Theorem 3.4.5. *(L'Hopital's Rule, First Case) Let $f, g : I \to \mathbb{R}$ be two functions continuous on I, differentiable on $I \setminus \{x_0\}$, and satisfying the following conditions:*

1. $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$,
2. $g'(x) \neq 0$ *for all* $x \in I \setminus \{x_0\}$,

then

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l \quad \textit{implies} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = l.$$

Example 3.4.8.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

The reverse is generally false.

Example 3.4.9. *Let $f(x) = x^2 \cos(\frac{1}{x})$, and $g(x) = x$. We have:*

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} x \cos(\frac{1}{x}) = 0,$$

while

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2x\cos(\frac{1}{x}) + \sin(\frac{1}{x})}{1}$$

does not exist because $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

Remark 3.4.1. *L'Hopital's rule is valid when $x \to +\infty$. Indeed, let $x = \frac{1}{t}$, then*

$$\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t\to 0} \frac{f'(1/t)}{g'(1/t)} = \lim_{x\to +\infty} \frac{f'(x)}{g'(x)}.$$

If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = 0$ and $f'(x), g'(x)$ satisfy the conditions of the theorem, L'Hopital's rule can be applied again.

Theorem 3.4.6. *(L'Hopital's Rule, Second Case) Let $f, g : I \to \mathbb{R}$ be two functions continuous on I, differentiable on $I \setminus \{x_0\}$, and satisfying the following conditions:*

1. $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = +\infty$,

2. $g'(x) \neq 0$ *for all* $x \in I \setminus \{x_0\}$,

then

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = l \quad \textit{implies} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = l.$$

The previous remark is true in this case as well.

Example 3.4.10. $\lim_{x\to+\infty} \frac{x^n}{e^x} = \lim_{x\to+\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to+\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to+\infty} \frac{n(n-1)...2\times 1}{e^x} = 0.$

Chapter 4

Applications to Elementary Functions

4.1 Inverse Functions of Trigonometric Functions

Arcsine Function

The function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by $f(x) = \sin x$ is continuous and strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (since $f'(x) = \cos x > 0$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$). It has an inverse function $f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ called the arcsine function, which is continuous and strictly increasing on $[-1, 1]$. It is denoted as arcsin. Thus,

$$y = \arcsin(x) \quad \text{if and only if} \quad x = \sin(y) \quad \text{where} \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

and

$$(\arcsin)'(x) = \frac{1}{\sqrt{1 - x^2}}, \quad |x| < 1.$$

Arccosine Function

The function $f : [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos x$ is continuous and strictly decreasing on $[0, \pi]$ (since $f'(x) = -\sin x < 0$ for all $x \in (0, \pi)$). It has an inverse function $f^{-1} : [-1, 1] \to [0, \pi]$ called the arccosine function, which is continuous and strictly decreasing on $[-1, 1]$. It is denoted as arccos. Thus,

$$y = \arccos(x) \quad \text{if and only if} \quad x = \cos(y) \quad \text{where} \quad y \in [0, \pi],$$

and

$$(\arccos)'(x) = \frac{-1}{\sqrt{1 - x^2}}, \quad |x| < 1.$$

Arctangent Function

The function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ defined by $f(x) = \tan x$ is continuous and strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (since $f'(x) = \frac{1}{\cos^2 x} > 0$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$). It has an inverse function $f^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ called the arctangent function, which is continuous and strictly increasing on $\mathbb{R}$. It is denoted as arctan. Thus,

$$y = \arctan(x) \quad \text{if and only if} \quad x = \tan(y) \quad \text{where} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

and

$$(\arctan)'(x) = \frac{1}{1+x^2}, \quad \forall x \in \mathbb{R}.$$

Arccotangent Function

The function $f : (0, \pi) \to \mathbb{R}$ defined by $f(x) = \cot x$ is continuous and strictly decreasing on $(0, \pi)$ (since $f'(x) = -\frac{1}{\sin^2 x} < 0$ for all $x \in (0, \pi)$). It has an inverse function $f^{-1} : \mathbb{R} \to (0, \pi)$ called the arccotangent function, which is continuous and strictly decreasing on $\mathbb{R}$. It is denoted as arccot. Thus,

$$y = \text{arccot}(x) \quad \text{if and only if} \quad x = \cot(y), \quad y \in (0, \pi),$$

and

$$(\text{arccot})'(x) = \frac{-1}{1+x^2}, \quad \forall x \in \mathbb{R}.$$

Properties

For all $x \in D_f$, the following identities hold:

- $\sin(\arcsin x) = x$,
- $\cos(\arccos x) = x$,
- $\tan(\arctan x) = x$,
- $\cot(\text{arccot} x) = x$.

However, we have:

- $\arcsin(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,
- $\arccos(\cos x) = x$ if $x \in [0, \pi]$,
- ...

1. $\arcsin x + \arccos x = \frac{\pi}{2}$

2. $\arctan x + \ arccot x = \frac{\pi}{2}$

3. $\arcsin(-x) = -\arcsin x$

4. $\arccos(-x) = \pi - \arccos x$

5. $\ arctan(-x) = -\arctan x$

6. $\ arccot(-x) = \pi - \ arccot x$

7. $\cos(\arcsin x) = \sin(\arccos x) = \sqrt{1 - x^2}$

8. $\cos(\arctan x) = \sin(\ arccot x) = \sqrt{1 - \frac{1}{1+x^2}}$

9. $\sin(\arctan x) = \cos(\ arccot x) = \sqrt{\frac{x^2}{1+x^2}}$

10. $\tan(\arccos x) = \cot(\arcsin x) = \frac{\sqrt{1-x^2}}{x}$

11. $\arccos\left(\frac{1}{x}\right) + \arcsin\left(\frac{1}{x}\right) = \frac{\pi}{2}$

4.2 Hyperbolic Functions and Their Inverses

Hyperbolic Sine Function

The hyperbolic sine function is defined as:

$$\sinh : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sinh x = \frac{e^x - e^{-x}}{2}.$$

Properties:

- $\sinh \in C^\infty(\mathbb{R}, \mathbb{R})$
- $\lim_{x\to\pm\infty} \sinh x = \pm\infty$

Hyperbolic Cosine Function

The hyperbolic cosine function is defined as:

$$\cosh : \mathbb{R} \to [1, +\infty), \quad x \mapsto \cosh x = \frac{e^x + e^{-x}}{2}.$$

Properties:

- cosh is an even function, $(\cosh)'(x) = \sinh x$, for all $x \in \mathbb{R}$.
- $\cosh \in C^\infty(\mathbb{R}, [1, +\infty))$

- $\lim_{x\to-\infty} \cosh x = +\infty$, $\lim_{x\to+\infty} \cosh x = +\infty$
- cosh is strictly decreasing on $\mathbb{R}^-$ and strictly increasing on $\mathbb{R}^+$

Hyperbolic Tangent Function
The hyperbolic tangent function is defined as:

$$\tanh : \mathbb{R} \to (-1, 1), \quad x \mapsto \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Properties:

- tanh is an odd function, $\tanh \in C^\infty(\mathbb{R}, (-1, 1))$
- $(\tanh)'(x) = \frac{1-\tanh^2 x}{\cosh^2 x}$, for all $x \in \mathbb{R}$
- $\lim_{x\to-\infty} \tanh x = -1$, $\lim_{x\to+\infty} \tanh x = +1$

Hyperbolic Cotangent Function
The hyperbolic cotangent function is defined as:

$$\coth : \mathbb{R}^* \to (-\infty, -1) \cup (1, +\infty), \quad x \mapsto \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Properties:

- coth is an odd function, strictly decreasing, and bijective from $\mathbb{R}^*$ to $(-\infty, -1) \cup (1, +\infty)$
- $\coth \in C^\infty(\mathbb{R}^*, (-\infty, -1) \cup (1, +\infty))$
- $(\coth)'(x) = \frac{1-\coth^2 x}{\sinh^2 x}$, for all $x \in \mathbb{R}^*$
- $\lim_{x\to-\infty} \coth x = -1$, $\lim_{x\to+\infty} \coth x = +1$
- $\lim_{x\to-1^-} \coth x = -\infty$, $\lim_{x\to1^+} \coth x = +\infty$

Properties of Hyperbolic Functions

- $\cosh x + \sinh x = e^x$
- $\cosh x - \sinh x = e^{-x}$
- $\cosh^2 x - \sinh^2 x = 1$
- $\cosh(x + y) = \cosh x \cdot \cosh y + \sinh x \cdot \sinh y$

- $\sinh(x+y) = \sinh x \cdot \cosh y + \sinh y \cdot \cosh x$
- $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 - \tanh x \cdot \tanh y}$
- $1 - \tanh^2 x = \frac{1}{\cosh^2 x}$
- $\cosh(2x) = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

4.2.1 Inverse Hyperbolic Function

Inverse Hyperbolic Sine Function The function sinh is defined, continuous, and strictly increasing on $\mathbb{R}$. It admits a reciprocal function $f^{-1} : \mathbb{R} \to \mathbb{R}$ called the inverse hyperbolic sine function, denoted as arg sinh. Thus:

$$y = \text{arg sinh}(x) \quad \Leftrightarrow \quad x = \sinh y \quad \text{where} \quad y \in \mathbb{R}.$$

Properties:

1. The function arg sinh : $\mathbb{R} \to \mathbb{R}$ is defined, continuous, and strictly increasing on $\mathbb{R}$
2. $(\text{arg sinh})'(x) = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$, for all $x \in \mathbb{R}$
3. arg sinh $\in C^\infty(\mathbb{R}, \mathbb{R})$
4. $\lim_{x\to-\infty} \text{arg sinh} x = -\infty$, $\lim_{x\to+\infty} \text{arg sinh} x = +\infty$

An alternative form of the inverse hyperbolic sine function is:

$$\cosh^2 y - \sinh^2 y =$$

Another form of the function $\operatorname{arg\,sinh}$
We have:

$$\cosh^2 y - \sinh^2 y = 1 \quad \Rightarrow \quad \cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1+x^2}.$$

Thus, we get:

$$\cosh y + \sinh y = e^y = x + \sqrt{1+x^2},$$

which implies:

$$\operatorname{arg\,sinh} x = y = \ln e^y = \ln\left(x + \sqrt{1+x^2}\right).$$

Hyperbolic Cosine Argument Function

The function $f : [0, +\infty[\to [1, +\infty[$ defined by $f(x) = \cosh x$ is continuous and strictly increasing on $\mathbb{R}_+$. It admits an inverse function $f^{-1} : [1, +\infty[\to [0, +\infty[$ which is continuous and strictly increasing, called the inverse hyperbolic cosine function. We denote it as $\arg\cosh$. Thus:

$$y = \arg\cosh x \quad \Leftrightarrow \quad x = \cosh y, \quad y \in [0, +\infty[.$$

Properties

The function $\arg\sinh : \mathbb{R} \to \mathbb{R}$ is defined, continuous, and strictly increasing on $\mathbb{R}$, and we have:

$$\arg\cosh x = 1 \quad \text{for} \quad x \geq 1.$$

Also:

$$\sinh y = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}, \quad \forall x \geq 1.$$

Furthermore:

$$\lim_{x \to +\infty} \arg\cosh x = +\infty.$$

Another form of the $\arg\cosh$ function:

$$\cosh^2 y - \sinh^2 y = 1 \quad \Rightarrow \quad \sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}.$$

Thus:

$$\cosh y + \sinh y = e^y = x + \sqrt{x^2 - 1},$$

which implies:

$$\arg\cosh x = y = \ln e^y = \ln\left(x + \sqrt{x^2 - 1}\right), \quad \forall x \geq 1.$$

Hyperbolic Tangent Argument Function

The function $f : \mathbb{R} \to]-1, +1[$ defined by $f(x) = \tanh x$ is continuous and strictly increasing on $\mathbb{R}$. It admits an inverse function $f^{-1} :]-1, +1[\to \mathbb{R}$ which is continuous and strictly increasing, called the inverse hyperbolic tangent function. We denote it as $\arg\tanh$. Thus:

$$y = \arg\tanh x \quad \Leftrightarrow \quad x = \tanh y.$$

Properties

For $|x| < 1$, we have:

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

Thus:

$$e^{2y} = \frac{1+x}{1-x},$$

which implies:

$$\arg\tanh x = y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Also:

$$\lim_{x\to -1^+} \arg\tanh x = -\infty, \quad \lim_{x\to +1^-} \arg\tanh x = +\infty.$$

The function $\arg\tan \in C^\infty(]-1,+1[,\mathbb{R})$.

Hyperbolic Cotangent Argument Function

The function $\coth : \mathbb{R}^* \to]-\infty,-1[\cup]1,+\infty[$ defined by:

$$x \mapsto \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

is bijective, and thus admits an inverse function called the inverse hyperbolic cotangent function. We denote it as $\arg\coth$. Thus:

$$y = \arg\coth x \quad \Leftrightarrow \quad x = \coth y, \quad y \in \mathbb{R}^*.$$

Properties

We can show that:

$$\arg\coth x = y = \frac{1}{2}\ln\left(\frac{1+x}{x-1}\right), \quad |x| > 1.$$

Chapter 5

Limited development

5.1 Taylor Formulas

A function f that is continuous on $[a, b]$ and differentiable at $x_0 \in]a, b[$ can be expressed in the vicinity of x_0 as:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + (x - x_0)\epsilon(x)$$

where $\lim_{x \to x_0} \epsilon(x) = 0$. This implies that f can be approximated by a polynomial of degree 1:

$$x \mapsto P(x) = f(x_0) + (x - x_0)f'(x_0).$$

The error $R(x) = (x - x_0)\epsilon(x)$ tends to 0 as x approaches x_0. The Taylor formula generalizes this result to functions that are differentiable n times and can be approximated (in the vicinity of x_0) by polynomials of degree n. More precisely:

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!}f'(x_0) + \frac{(x - x_0)^2}{2!}f^{(2)}(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + R_n(x_0, x).$$

Here, $P_n(x)$ is a polynomial of degree n in $(x - x_0)$ that approximates f with an error R_n. $R_n(x_0, x)$ is known as the remainder of order n. Various forms of $R_n(x_0, x)$ exist; the most common form is given below.

5.1.1 Taylor Formula with Lagrange Remainder

Theorem 5.1.1. *Let $f : [a, b] \to \mathbb{R}$ be a function of class C^n on $[a, b]$ (i.e., $f \in C^n([a, b])$), and suppose $f^{(n)}$ is differentiable on $]a, b[$. For $x_0 \in [a, b]$ and $x \in [a, b]$ with $x \neq x_0$, there exists $c \in]a, b[$ such that:*

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!}f'(x_0) + \frac{(x - x_0)^2}{2!}f^{(2)}(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + \frac{(x - x_0)^{n+1}}{(n+1)!}f^{(n+1)}(c).$$

This is the Taylor formula of order n with Lagrange remainder $\frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$.

5.1.2 Maclaurin Taylor Formula

When $x_0 = 0$ in the Taylor-Lagrange formula, we set $c = \theta x$, where $0 < \theta < 1$, and $c \in]0, x[$, yielding:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f^{(2)}(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x).$$

Remark The Maclaurin formula is often used to calculate approximate values.

Example 5.1.1. *Using the Maclaurin formula of order 2 for the function $x \mapsto e^x$, show that $8/3 < e < 3$.*

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} e^{\theta x}.$$

For $x = 1$, $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} e^{\theta}$, i.e., $e = \frac{5}{2} + \frac{1}{6} e^{\theta}$. Using the fact that $0 < \theta < 1$, we get

$$\frac{8}{3} < \frac{5}{2} + \frac{1}{6} e < 3,$$

which implies $e < 3$.

Finally, $\frac{8}{3} < e < 3$.

5.1.3 Taylor Formula with Young Remainder

We shall restrict the assumptions by assuming only that $f^{(n)}(x_0)$ exists.

Theorem 5.1.2. *Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. Suppose $f^{(n)}(x_0)$ exists (finite). Then, for all $x \in V(x_0)$:*

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + o((x - x_0)^n),$$

where $o((x - x_0)^n) = (x - x_0)^n \epsilon(x)$ with $\lim_{x \to x_0} \epsilon(x) = 0$.

5.2 Limited Development near Zero

We have seen that in a neighborhood of x_0, the function $f(x)$ can be approximated by a polynomial P_n of degree n such that:

$$f(x) - P_n(x) = o((x - x_0)^n),$$

provided that $f^{(n)}(x)$ exists. Now, we will see that such a polynomial may exist even if $f^{(n)}$ does not exist, or even if f is not continuous at x_0.

Definition 5.2.1. *Let f be a function defined in a neighborhood of zero. We say that f admits a Limited Development of order n near zero if there exists an open interval I centered at 0 and constants $a_0, a_1, \ldots, a_n$ such that for all $x \in I$, $x \neq 0$,*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x),$$

with $\lim_{x\to 0} \varepsilon(x) = 0$, written as

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n = P_n(x) + o(x^n).$$

1. $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ is called the regular part of the Limited Development.
2. $o(x^n) = x^n \varepsilon(x)$ (where $\lim_{x\to 0} \varepsilon(x) = 0$) represents the remainder.

Example 5.2.1. *For $f(x) = 1 + \frac{5}{2}x + 3x^2 + x^3 \sin \frac{1}{x}$, $x \in \mathbb{R}^*$, this is a second-order Limited Development around zero, denoted as $DL_2(0)$.*

1. If f admits a Limited Development of order n around zero, then $\lim_{x\to 0} f(x)$ exists. In fact, $f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x)$ implies that:

$$\lim_{x\to 0} f(x) = a_0.$$

This does not imply that f is continuous at 0 since $f(0)$ may not exist.
Example: For $f(x) = \frac{1}{x}$, $x \neq 0$, the function does not admit a Limited Development near 0 because $\lim_{x\to 0} f(x) = \infty$.

2. If f admits a Limited Development of order n around zero and $a_0 = f(0)$, then f is differentiable at 0. In fact, for all $x \neq 0$:

$$f(x) = f(0) + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x),$$

with $\lim_{x\to 0} \varepsilon(x) = 0$. Therefore,

$$\frac{f(x) - f(0)}{x} = a_1 + a_2 x + \cdots + a_{n-1} x^{n-1} + x^{n-1} \varepsilon(x),$$

which implies $\lim_{x\to 0} \frac{f(x)-f(0)}{x} = a_1 = f'(0)$.

5.2.1 Properties of Limited development

Proposition 5.2.1. *(Uniqueness) If f admits a Limited Development of order n around zero, then this Development is unique.*

Theorem 5.2.1. *If $f^{(n)}(0)$ exists, then the Limited Development of f is given by:*

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f^{(2)}(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + x^n \varepsilon(x), \quad \textit{where } \lim_{x\to 0} \varepsilon(x) = 0.$$

Corollary 5.2.1. *If $f^{(n)}(0)$ exists and f admits a Limited Development of order n near zero, then:*

$$a_0 = f(0), \quad a_1 = \frac{f'(0)}{1!}, \quad a_2 = \frac{f^{(2)}(0)}{2!}, \dots, a_n = \frac{f^{(n)}(0)}{n!}.$$

Example 5.2.2. *(Limited Development by Successive Division) For $f(x) = \frac{1}{1-x}$:*

$$f(x) = 1 + x + x^2 + \cdots + x^n \varepsilon(x),$$

where $\varepsilon(x) = \frac{x}{1-x} \to 0$ as $x \to 0$. Identifying terms, we find $f^{(k)}(0)/k! = 1$ for all $0 \leq k \leq n$.

Remark 5.2.1. *The existence of a Limited Development does not imply the existence of derivatives. Consider the function:*

$$g(x) = 1 + x + x^2 + \cdots + x^n + x^{n+1} \sin \frac{1}{x}.$$

It is clear that g admits a Limited Development of order n around zero, but it is not differentiable at 0, as it is not defined at 0.

5.3 Algebraic Operations on Limited development

Theorem 5.3.1. *If f and g admit Limited development of order n around 0, then $f + g$ and fg also admit Limited development of order n. Moreover, $\frac{f}{g}$ admits a Limited Development of order n if $\lim_{x \to 0} g(x) \neq 0$.*

Example 5.3.1.

1. *Let $h(x) = \ln(1 + x) + \cos x$. We have*

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^4 \varepsilon_1(x),$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^4 \varepsilon_2(x).$$

Thus,

$$h(x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^4 \varepsilon_1(x) \right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^4 \varepsilon_2(x) \right),$$

$$= 1 + x - x^2 + \frac{x^3}{3} - \frac{5}{24} x^4 + x^4 \varepsilon(x),$$

which is a Limited Development of order 4 around 0.

2. *Let* $f(x) = \sin x \cdot \cos x$. *We have*

$$\sin x = x - \frac{x^3}{3} + x^3\varepsilon_1(x),$$

$$\cos x = 1 - \frac{x^2}{2!} + x^3\varepsilon_2(x).$$

Thus,

$$f(x) = x - \frac{2}{3}x^3 + x^3\varepsilon(x),$$

which is a Limited Development of order 3 around 0.

5.3.1 Limited Development of a Composite Function

Theorem 5.3.2. *If f and g admit Limited development of order n around 0, and if $g(0) = 0$, then $f \circ g$ admits a Limited Development of order n around 0.*

Remark 5.3.1. *The regular part of $f \circ g$ can be obtained by substituting the regular part of g into the regular part of f, keeping only powers less than or equal to n.*

Example 5.3.2. *Let* $h(x) = e^{\sin x}$. *Set* $f(u) = e^u$ *and* $g(x) = \sin x$. *We have* $g(0) = \sin 0 = 0$,

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + o(u^3),$$

$$\sin x = x - \frac{x^3}{6} + o(x^3),$$

so that

$$(f \circ g)(x) = e^{\sin x} = 1 + \left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^3)\right)^3,$$

$$= 1 + x + \frac{x^2}{2} + o(x^3).$$

5.3.2 Differentiation of Limited development

We have seen that the existence of a Limited Development does not require the existence of a derivative. Thus, we cannot necessarily determine the Limited Development of the derivative.

Theorem 5.3.3. *Let f be a differentiable function around 0 admitting a Limited Development of order n around 0:*

$$f(x) = P(x) + x^n\varepsilon(x) \quad \text{with} \quad \lim_{x \to 0} \varepsilon(x) = 0.$$

If the derivative f' admits a Limited Development of order $n-1$ around 0, then

$$f'(x) = Q(x) + x^{n-1}\eta(x) \quad \textit{with} \quad \lim_{x\to 0}\eta(x) = 0,$$

where $Q(x) = P'(x)$.

Example 5.3.3. *Let $f(x) = \frac{1}{1-x}$. We know that*

$$f(x) = 1 + x + x^2 + \cdots + x^n + x^n\varepsilon(x) \quad \textit{with} \quad \lim_{x\to 0}\varepsilon(x) = 0.$$

Therefore,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = f'(x) = 1+2x+\cdots+nx^{n-1}+x^{n-1}\eta(x) \quad \textit{with} \quad \lim_{x\to 0}\eta(x) = 0.$$

5.3.3 Integration of Limited development

Theorem 5.3.4. *Let f be a numerical differentiable function on the interval $I =]-\alpha,\alpha[$, $\alpha > 0$, with derivative f'. If f' admits a Limited Development of order n around 0,*

$$f'(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\varepsilon(x) \quad \textit{with} \quad \lim_{x\to 0}\varepsilon(x) = 0,$$

then f admits a Limited Development of order $n+1$ around 0,

$$f(x) = f(0) + a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_n}{n+1}x^{n+1} + x^{n+1}\eta(x) \quad \textit{with} \quad \lim_{x\to 0}\eta(x) = 0,$$

where $x^{n+1}\eta(x) = \int_0^x t^n\varepsilon(t)\,dt$.

Example 5.3.4. *We have*

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + x^n\varepsilon(x) \quad \textit{with} \quad \lim_{x\to 0}\varepsilon(x) = 0,$$

and thus

$$\ln(1+x) = \ln 1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n}{n+1}x^{n+1} + x^{n+1}\eta(x) \quad \textit{with} \quad \lim_{x\to 0}\eta(x) = 0.$$

5.4 D.L. Near x_0 and Infinity

Definition 5.4.1. *We say that a function f defined near x_0 admits a Limited Development (D.L.) of order n at $V(x_0)$ if the function*

$$F : x \mapsto F(x) = f(x_0 + x)$$

admits a Limited Development of order n at $V(0)$.

We have

$$F(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x),$$

and hence

$$f(x_0 + x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x),$$

that is,

$$f(y) = a_0 + a_1(y - x_0) + \cdots + a_n(y - x_0)^n + (y - x_0)^n \varepsilon((y - x_0)),$$

or equivalently,

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon((x - x_0)).$$

Finally, we shift the neighborhood of x_0 to the neighborhood of 0 by the variable change $z = x - x_0$.

Similarly, the D.L. at infinity is done by the variable change $y = \frac{1}{x}$.

Definition 5.4.2. *We say that a numerical function admits a Limited Development of order n at $V(+\infty)$ if there exists a polynomial P of degree less than or equal to n such that at $V(+\infty)$ we have*

$$f(x) = P\left(\frac{1}{x}\right) + o\left(\frac{1}{x^n}\right).$$

Example 5.4.1. *The Limited Development of $x \mapsto e^x$ at $V(1)$. Let $u = x - 1$, so near 0 we have*

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + o(u^n).$$

Thus,

$$e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \cdots + \frac{(x-1)^n}{n!} + o((x-1)^n).$$

Finally,

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots + \frac{(x-1)^n}{n!} + o((x-1)^n)\right].$$

2) The Limited Development of $x \mapsto e^x$ at $V(+\infty)$. Near infinity, we set $y = \frac{1}{x}$. We have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n),$$

and therefore,

$$e^{\frac{1}{y}} = 1 + \frac{1}{y} + \frac{1}{2y^2} + \frac{1}{3!y^3} + \cdots + \frac{1}{n!y^n} + o\left(\frac{1}{y^n}\right).$$

5.5 Generalized Taylor Series

If f defined at $V(0)$ does not admit a Limited Development at $V(0)$, but $x^\alpha f(x)$, with $\alpha > 0$, admits a Limited Development, we can then write at $V(0)$ for $x \neq 0$

$$x^\alpha f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x), \quad \text{with } \lim_{x\to 0} \varepsilon(x) = 0,$$

from which

$$f(x) = \frac{1}{x^\alpha}\left[a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x)\right]$$

is a generalized Taylor series of f at $V(0)$.

Example 5.5.1. *Let f be defined by*

$$f(x) = \frac{1}{x - x^2}.$$

f does not admit a Limited Development near 0 since $\lim_{x\to 0} f(x) = +\infty$, but we have

$$xf(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n \varepsilon(x).$$

Thus,

$$f(x) = \frac{1}{x} + 1 + x + \cdots + x^{n-1} + o(x^{n-1})$$

is the generalized Taylor series of f.

5.6 Applications of Limited development

Limited development are very useful in finding limits of functions and studying indeterminate forms.

Example 5.6.1.

1. *Find the limit as $x \to 0$ of the function*

$$f(x) = \frac{\sin x - x\cos x}{x(1-\cos x)}.$$

We have

$$\sin x = x - \frac{x^3}{6} + o(x^3),$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2),$$

and therefore,

$$f(x) = \frac{\frac{x^3}{3} + o(x^3)}{\frac{x^3}{2} + o(x^3)}.$$

Finally,

$$\lim_{x \to 0} f(x) = \frac{2}{3}.$$

Exercise 5.6.1. *Calculate the following limits:*

1.
$$\lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2},$$

2.
$$\lim_{x \to 0} \frac{\ln(1+x) - \sin x}{x},$$

3.
$$\lim_{x \to 0} \frac{\cos x - \sqrt{1 - x^2}}{x^4}.$$

Solution 5.6.1.

1. *We have*
$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + o(x^4),$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4).$$
Thus,
$$e^{x^2} - \cos x = \frac{3}{2} x^2 + o(x^2),$$
and therefore,
$$\lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2} = \frac{3}{2}.$$

2. *We have*
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3),$$
$$\sin x = x - \frac{x^3}{3!} + o(x^3).$$
Thus,
$$\ln(1+x) - \sin x = -\frac{x^2}{2} + o(x^2),$$
and therefore,
$$\lim_{x \to 0} \frac{\ln(1+x) - \sin x}{x} = 0.$$

3. *We have*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4),$$

$$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} + o(x^4).$$

Thus,

$$\cos x - \sqrt{1-x^2} = \frac{x^4}{6} + o(x^4),$$

and therefore,

$$\lim_{x\to 0} \frac{\cos x - \sqrt{1-x^2}}{x^4} = \frac{1}{6}.$$

Chapter 6

Linear Algebra

6.1 Laws of internal composition

Definition 6.1.1. *Let G a set. An internal composition on G is an application of $G \times G$ in G. If we denote it*

$$G \times G \to G$$

$$(a, b) \to a * b$$

we are talking about the law $*$ and say that $a * b$ is the compound of a and b for the law $*$.

Example 6.1.1. *On $G = \mathbb{Z}$, the addition defined by*

$$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

$$(a, b) \to a + b$$

the multiplication

$$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

$$(a, b) \to a \times b$$

and the subtraction

$$\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

$$(a, b) \to a - b$$

are internal composition laws.
On $G = \mathbb{R}^2$ the addition

$$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$$

$$((x_1, y_1), (x_2, y_2)) \to (x_1 + x_2, y_1 + y_2)$$

is internal law.

Example 6.1.2. *In $\mathbb{R}^*$ we define the law δ by:*

$$x\delta y = x + y + \ln|xy|$$

then the law δ is internal on $\mathbb{R}^*$, indeed, soit $x; y \in \mathbb{R}^*$, let's show that $x\delta y \in \mathbb{R}^*$, as

$$\begin{aligned}(x\delta y = 0) &\iff (x + y + \ln|xy| = 0)\\ &\iff (\ln|xy| = -(x+y))\\ &\iff (|xy| = e^{-(x+y)})\\ &\iff (x \neq 0 \textit{ and } y \neq 0)\end{aligned}$$

so $x\delta y \in \mathbb{R}^*$ is an internal law.

Definition 6.1.2. *Let $*$ an internal law on a set G.*

1) The law $*$ is commutative if

$$\forall x, y \in G, \quad x * y = y * x$$

2) The law $*$ is associative if

$$\forall x; y; z \in G, \quad (x * y) * z = x * (y * z)$$

3) The law $*$ admits on G a neutral element, noted e, if

$$\exists e \in G, \quad \forall x \in G, \quad x * e = e * x = x$$

If the law $*$ is commutative, [it suffices to demonstrate that]

$$\forall x \in G, \quad x * e = x$$

Example 6.1.3. *In $\mathbb{R} - \left\{\frac{1}{2}\right\}$ The internal law $*$ is defined by:*

$$x * y = x + y - 2xy$$

the law $*$ is internal on $\mathbb{R} - \left\{\frac{1}{2}\right\}$, indeed, let $x, y \in \mathbb{R} - \left\{\frac{1}{2}\right\}$, show that $x * y \in \mathbb{R} - \left\{\frac{1}{2}\right\}$ as

$$x * y = \frac{1}{2} \Leftrightarrow x + y - 2xy = \frac{1}{2}$$

$$\Leftrightarrow x(1-2y) - \frac{1}{2}(1-2y) = 0$$

$$\Leftrightarrow (1-2y)\left(x - \frac{1}{2}\right) = 0$$

$$\Leftrightarrow \left(y-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)=0$$
$$\Leftrightarrow y=\frac{1}{2}, \quad or\, x=\frac{1}{2}$$

so $x*y\in\mathbb{R}-\left\{\frac{1}{2}\right\}$ *and then* $*$ *is an internal law. Let* $x,\ y;\ z\in\mathbb{R}-\left\{\frac{1}{2}\right\}$*, we have*

$$x*y=x+y-2xy=y+x-2yx=y*x$$

then the law $*$ *is commutative.*

$$\begin{aligned}(x*y)*z &= (x+y-2xy)*z=(x+y-2xy)+z-2\,(x+y-2xy)\,z\\ &= x+y+z-2xy-2xz-2yz+4xyz\\ &= x+(y+z-2yz)-2x\,(y+z-2yz)\\ &= x+(y*z)-2x\,(y*z)=x*(y*z)\end{aligned}$$

then the law $*$ *is associative. Let* $e\in\mathbb{R}-\left\{\frac{1}{2}\right\}$*, such that* $x*e=e*x=x$*, then*

$$x+e-2xe=e+x-2ex=x\Leftrightarrow e\,(1-2x)=0\Leftrightarrow e=0$$

then a law accepts as the neutral element the element $e=0$*.*

Definition 6.1.3. *Let* $*$ *an internal law on a set* G*, having a neutral element* e *and let* $x\in G$*. It is said that* x *accepts a symmetrical element* x' *by the law* $*$*, si*

$$x*x'=x'*x=e$$

Example 6.1.4. *On* $\mathbb{R}-\left\{\frac{1}{2}\right\}$*, we define the internal law* $*$ *by:*

$$x*y=x+y-2xy$$

the law $*$ *admits neutral element. Let* $x\in\mathbb{R}-\left\{\frac{1}{2}\right\}$*, such that* $x*x'=x'*x=e$*, then*

$$x+x'-2xx'=0\Rightarrow x'\,(1-2x)=-x\Leftrightarrow x'=\frac{x}{2x-1},$$

then, the symmetrical element of x *is*

$$x'=\frac{x}{2x-1},\ for\ all\ x\in\mathbb{R}-\left\{\frac{1}{2}\right\}$$

show that $x'\in\mathbb{R}-\left\{\frac{1}{2}\right\}$*. Indeed, let* $x,y\in\mathbb{R}-\left\{\frac{1}{2}\right\}$*, show that* $x*y\in\mathbb{R}-\left\{\frac{1}{2}\right\}$*, we have*

$$x' = \frac{1}{2} \Leftrightarrow 2x - 1 = 2x \Leftrightarrow -1 = 0$$

which is absurd, hence $x' \in \mathbb{R} - \left\{\frac{1}{2}\right\}$.

Definition 6.1.4. *Let* G *a set provided with two internal composition laws, denoted* Δ *and* $*$. *They say* $*$ *is distributive in relation to* Δ *if*

$$\forall x, y, z \in G, \quad x * (y \Delta z) = (x * y) \, \Delta \, (x * z)$$

6.2 Group Structure

Definition 6.2.1. *Let* G *provided with a law of internal composition* $*$. *It is said that* $(G, *)$ *is a group if the law* $*$ *satisfies the following three conditions :*

1) $*$ *is associative.*

2) $*$ *admits a neutral element*

3) Each element of G *allows a symmetrical for* $*$.

If, moreover, the law is commutative, it is said that the group is commutative or abelian.

Example 6.2.1. *1)* $(\mathbb{Z}, +)$ *is a commutative group.*

2) $(\mathbb{R}, \times)$ *is not a group because 0 does not allow symmetrical elements.*

3) $(\mathbb{R}^*, \times)$ *is a commutative group.*

Definition 6.2.2. *Let* $(G, *)$ *a group. A part* $H \subset G$ *(non-empty) is a subgroup of* G *if the restriction of the operation* $*$ *to* H *gives it the group structure.*

Proposition 6.2.1. *Let* H *is a non-empty part of the group* G. *Then,* H *is subgroup de* G *if and only if,*

1) For all $x, y \in H$, *we have* $x * y \in H$,

2) For all $x \in H$, *we have* $x' \in H$, *with* x' *is the symmetrical of* x.

Example 6.2.2. $(\mathbb{R}_+^*, \times)$ *is a subgroup of* $(\mathbb{R}_+^*, \times)$. *Indeed :*

1) Si $x, y \in \mathbb{R}_+^*$, *then* $x \times y \in \mathbb{R}_+^*$,

2) Si $x \in \mathbb{R}_+^*$ *then* $x' = \frac{1}{x}$ *symmetrical element of* x *and* $x' = \frac{1}{x} \in \mathbb{R}_+^*$.

Example 6.2.3. *We pose* $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$, $\{2\mathbb{Z}, +\}$ *is a subgroup of* $\mathbb{Z}$. *Indeed :*

1) If $x, y \in 2\mathbb{Z}$, *it exists* $x_1 \in \mathbb{Z}$ *such that* $x = 2x_1$ *and* $y = 2y_1$, *then*

$$x + y = 2x_1 + 2y_1 = 2\,(x_1 + y_1) \in 2\mathbb{Z}.$$

2) If $x \in 2\mathbb{Z}$, *it exists* $x_1 \in \mathbb{Z}$ *such that* $x = 2x_1$ *then*

$$-x = -2x_1 = 2\,(-x_1) \in 2\mathbb{Z}.$$

6.3 Vector Space

6.3.1 Definitions and elementary properties

Let $\mathbb{k}$ be a commutative body (usually it's $\mathbb{R}$ or $\mathbb{C}$) and let E be a non-empty assembly provided with an internal operation denoted by $(+)$

$$(+) : E \times E \to E$$

$$(x, y) \to (x + y)$$

and an external operation noted (.)

$$(\cdot) : \mathbb{k} \times E \to E$$

$$(\lambda, y) \to (\lambda.y)$$

Definition 6.3.1. *A vector space on the body $\mathbb{k}$ or a $\mathbb{k}-$vector space is a triplet $(E, +, .)$ such that :*

1) $(E, +)$ is a commutative group.

2) $\forall \lambda \in \mathbb{k}, \forall x, y \in E,\ \lambda.(x + y) = \lambda.x + \lambda.y.$

3) $\forall \lambda, \mu \in \mathbb{k}, \forall x \in E,\ (\lambda + \mu).x = \lambda.x + \mu.x.$

4) $\forall \lambda, \mu \in \mathbb{k}, \forall x \in E,\ (\lambda.\mu).x = \lambda\,(\mu.x)$

5) $\forall x \in E, 1_{\mathbb{k}}.x = x$

The elements of the vector space are called vectors and those of $\mathbb{k}$ are called scalars.

Example 6.3.1. *1) $(\mathbb{R}, +, .)$ is a $\mathbb{R}-$vector space,*

2) $(\mathbb{C}, +, .)$ is a $\mathbb{C}-$vector space,

3) If we consider $\mathbb{R}^n$ provided with the following two operations

$$(+) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\ ^n$$

$$((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) \to (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\ ^n$$

$$(\lambda\,(y_1, y_2, ..., y_n)) \to (\lambda y_1, \lambda y_2, ..., \lambda y_n)$$

It can easily shown that $(\mathbb{R}^n, +, \cdot)$ is a $\mathbb{R}-$vector space.

Proposition 6.3.1. *If E is a $\Bbbk-$vector space, then we have the following properties :*

1) $\forall x \in E,\ 0_{\Bbbk}.x = 0_E$.

2) $\forall x \in E,\ (-1_{\Bbbk}).x = x$

3) $\forall \lambda \in \Bbbk,\ \lambda.0_E = 0_E$

4) $\forall \lambda \in \Bbbk,\ \forall x, y \in E, \lambda (x - y) = \lambda.x - \lambda.y$

5) $\forall \lambda \in \Bbbk,\ \forall x \in E,\ \lambda.x = 0_E \Leftrightarrow \lambda = 0_{\Bbbk}$ *or* $x = 0_E$.

Definition 6.3.2. *Let $(E, +, \cdot)$ be $\Bbbk-$vector space and let F be non-empty sub set of F. it is said that F is vector subspace if $(F, +, \cdot)$ is also a $\Bbbk-$vector space.*

Remark 6.3.1. *1) When $(F, +, \cdot)$ is $\Bbbk-$vector space of $(E, +, \cdot)$ then $0_E \in F$.*

2) If $0_E \notin F$, then $(F, +, \cdot)$ can't be a $\Bbbk-$vector space of $(E, +, \cdot)$.

Theorem 6.3.1. *Let $(E, +, \cdot)$ be $\Bbbk-$vector space and $F \subset E$, F non empty we have the following equivalence :*

1) F is a vector subspace of E.

2) F is stable by addition and by multiplication, i.e :

$$\forall \lambda \in \Bbbk,\ \ \forall x, y \in F,\ \ \lambda.x \in F \text{ and } x + y \in F$$

3) $\forall \lambda, \mu \in \Bbbk, \forall x, y \in F, \lambda x + \mu y \in F$, *so*

$$F \text{ is a vector subspace } \Leftrightarrow \begin{cases} F \neq \varnothing \\ \forall \lambda, \mu \in \Bbbk, \forall x, y \in F, \lambda.x + \mu.y \in F \end{cases}$$

Example 6.3.2. *We pose $F = \{(x, y) \in \mathbb{R}^2 : x - y = 0\} \subset \mathbb{R}^2$, then F is a vector subspace, indeed,*

1) $0_{\mathbb{R}^2} = (0, 0) \in F$, *because* $0 - 0 = 0$

2) $\forall \lambda, \mu \in \mathbb{R}, \forall (x, y), (x', y') \in F$, *then* $x - y = 0$, *and* $x' - y' = 0$, *so*

$$\lambda (x - y) + \mu (x' - y') = (\lambda x + \mu x') - (\lambda y + \mu y') = 0,$$

i.e, $\lambda (x, y) + \mu (x', y') \in F$, *so F is vector subspace of* $\mathbb{R}^2$.

Remark 6.3.2. *The intersection of a non-empty family of vector subspace is a vector subspace.*

But, Reuniting two vector subspace is not necessarily a vector subspace.

Example 6.3.3. *Let $F_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $F_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ two vector subspace in $\mathbb{R}^2$, $F_1 \cup F_2$ is not a vector subspace, because*

$$u_1 = (0, 1) \in F_1, u_2 = (1, 0) \in F_2 \text{ and } u_1 + u_2 = (1, 1) \notin F_1 \cup F_2$$

6.3.2 Sum of two vector subspaces

Definition 6.3.3. *Let E_1, E_2 two vector subspaces of $\mathbb{k}-$vector space E, it said Sum of two vector subspaces, E_1 and E_2, which we note $E_1 + E_2$ the following set :*

$$E_1 + E_2 = \{x \in E : \exists x_1 \in E_1, \exists x_2 \in E_2 \text{ such that } x = x_1 + x_2\}$$

Example 6.3.4. *Let's $E_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $E_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ two vector subspaces in $\mathbb{R}^2$, if $(x, y) \in \mathbb{R}^2$, then*

$$(x, y) = (0, y) + (x, 0)$$

So $(x, y) \in E_1 + E_2$, then $E_1 + E_2 = \mathbb{R}^2$.

Proposition 6.3.2. *The sum of two vector subspaces E_1 and E_2 (of the same $\mathbb{k}-$vector space) is a vector subspace of E container $E_1 \cup E_2$, i,e,*

$$E_1 \cup E_2 \subset E_1 + E_2$$

6.3.3 Direct sum of two vector subspaces

Definition 6.3.4. *Let E_1, E_2 two vector subspaces of the same $\mathbb{k}-$vector space E. It will be said that the sum $E_1 \oplus E_2$ of two vector subspaces, is direct if $E_1 \cap E_2 = \{0\}$. We write $E_1 \oplus E_2$.*

Proposition 6.3.3. *Let E_1, E_2 two vector subspaces of the same $\mathbb{k}-$vector space E. The sum $E_1 + E_2$ is direct if $\forall x \in E_1 + E_2$, there is a single vector $x_1 \in E_1$, a single vector $x_2 \in E_2$ such that $x = x_1 + x_2$*

Example 6.3.5. *Let's $F_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ and $F_2 = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$ two vector subspaces in $\mathbb{R}^3$.*

1) Let $(x, y, z) \in \mathbb{R}^3$, then

$$(x, y, z) = (0, y, z) + (x, 0, 0)$$

then $(x, y, z) \in F_1 + F_2$, hence $F_1 + F_2 = \mathbb{R}^3$.

2) Let $(x, y, z) \in F_1 \cap F_2$, then $(x, y, z) \in F_1$ and $(x, y, z) \in F_2$, it means that $x = 0$ and $y = z = 0$, then $(x, y, z) = 0_{\mathbb{R}^3}$, i.e, $F_1 \cap F_2 = \{0\}$.

Finally, we conclude that $\mathbb{R}^3 = F_1 \oplus F_2$.

6.3.4 Generating families, free families and bases

Hereinafter, the vector space $(E, +, \cdot)$ will be designated by E.

Definition 6.3.5. *Let E be a vector space and $e_1, e_2, ..., e_n$ elements of E.*

1) They say that $\{e_1, e_2, ..., e_n\}$ are free or linearly independent, if for all $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{k}$:

$$\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0_E \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0_{\mathbb{k}}$$

if they are not, they are said to be related.

2) They say that $\{e_1, e_2, ..., e_n\}$ is a generates family E, or that E is generated by $\{e_1, e_2, ..., e_n\}$ if

$$\forall x \in E, \exists \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{k}, \quad x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$$

3) If $\{e_1, e_2, ..., e_n\}$ is a free and generates family of E, so $\{e_1, e_2, ..., e_n\}$ is called a base of E.

Example 6.3.6. *On $\mathbb{R}^2$, we pose $u_1 = (1, 0)$,$u_2 = (1, -1)$, then $\{u_1, u_2\}$ is a base of $\mathbb{R}^2$. Indeed*

i) $\{u_1, u_2\}$ is free. $\forall \alpha_1, \alpha_2 \in \mathbb{R}$,

$$\begin{aligned} (\alpha_1 u_1 + \alpha_2 u_2 = 0) \;&\Rightarrow\; \alpha_1 (1, 0) + \alpha_2 (1, -1) = (0, , 0) \\ &\Rightarrow\; (\alpha_1 + \alpha_2, -\alpha_2) = (0, , 0) \\ &\Rightarrow\; \alpha_1 = \alpha_2 = 0 \end{aligned}$$

ii) $\{u_1, u_2\}$ is generating. $\forall (x, y) \in \mathbb{R}^2$,

$$(x, y) = \alpha_1 u_1 + \alpha_2 u_2 = (\alpha_1 + \alpha_2, -\alpha_2) \Rightarrow \alpha_2 = -y \in \mathbb{R} \text{ and } \alpha_1 = x + y \in \mathbb{R},$$

then it exists $\alpha_1, \alpha_2 \in \mathbb{R}$

Proposition 6.3.4. *If $\{e_1, e_2, ..., e_n\}$ and $\{u_1, u_2, ..., u_m\}$ are two bases of the vector space E, then $n = m$.*

Remark 6.3.3. *If a vector space E admits a base then all the bases of E have the same number of elements (or same cardinal), this number does not depend on the base but it depends only on the space E.*

Definition 6.3.6. *Let E be a $\mathbb{k}-$vectoriel space of base $B = \{e_1, e_2, ..., e_n\}$, The dimension of E, denoted $\dim E$, is the number defined by $\dim(E) = Card(B)$ where $Card(B)$ is the cardinal of B.*

Example 6.3.7. *We pose $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$, then $\{e_1, e_2, e_3\}$ is a base of $\mathbb{R}^3$, so*

$$\dim\left(\mathbb{R}^3\right) = Card\left(\{e_1, e_2, e_3\}\right) = 3$$

Example 6.3.8. *On $\mathbb{R}_2[x]$, the family $\{1, x, x^2\}$ is a base of $\mathbb{R}_2[x]$*

$$\dim \mathbb{R}_2[x] = Card\left\{1, x, x^2\right\} = 3$$

6.4 Linear application

Definition 6.4.1. *Let's E and F two $\mathbb{k}-$vector spaces. An application f of E on F is linear application if satisfies the following two conditions :*

$$\forall x, y \in E, \quad f(x+y) = f(x) + f(y)$$
$$\forall x \in E, \forall \lambda \in \mathbb{k}, \qquad f(\lambda x) = \lambda f(x)$$

where in an equivalent manner

$$\forall x, y \in E, \quad \forall \lambda \in \mathbb{k}, \quad f(\lambda x + y) = \lambda f(x) + f(y)$$

Remark 6.4.1. *The set's of the linear application of E on F denoted $\mathcal{L}(E, F)$.*

Example 6.4.1. *The application f defined by*

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \to f(x, y, z) = (2x + y, y - z)$$

is a linear application. Indeed, let's (x, y, z) ,$(x', y', z') \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$, we have

$$f[(x, y, z) + (x', y', z')] = f(x + x', y + y', z + z')$$
$$= (2(x + x') + (y + y'), (y + y') - (z + z'))$$
$$= (2x + 2x' + y + y', y + y' - z - z')$$
$$= ((2x + y) + (2x' + y'), (y - z) + (y' - z'))$$

$$= (2x + y, y - z) + (2x' + y', y' - z')$$
$$= f(x, y, z) + f(x', y', z')$$

and

$$f[\lambda (x, y, z)] = f(\lambda x, \lambda y, \lambda z) = (2\lambda x + \lambda y, \lambda y - \lambda z) = (\lambda (2x + y), \lambda (y - z))$$
$$= \lambda (2x + y, y - z)$$
$$= \lambda f(x, y, z)$$

Definition 6.4.2. *Let's E and F are two $\mathbb{K}-$vector spaces, and let $f \in \mathcal{L}(E, F)$. They say that*

1) f is an isomorphism of E on F, if f is bijective.
2) f is an endomorphism, if $(E, +, \cdot) = (F, +, \cdot)$.
3) f is an automorphism, if f is endorphism and isomorphism.

Example 6.4.2. *The application f defined by*

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \to f(x) = -2x$$

is an automorphisme, Indeed, let $x, y, \lambda \in \mathbb{R}$, we have

$$f(\lambda x + y) = -2(\lambda x + y) = \lambda (-2x) + (-2y) = \lambda f(x) + f(y)$$

and the application f is bijective, where

$$f^{-1} : \mathbb{R} \to \mathbb{R}$$
$$x \to f^{-1}(x) = \frac{-1}{2}x$$

Remark 6.4.2. *The null application, denoted $0_{\mathcal{L}(E,F)}$ is given by :*

$$f : E \to F, \;\; x \to f(x) = 0_F$$

the identity application, noted id_E is given by :

$$id_E : E \to F, \;\; x \to id_E(x) = x$$

Proposition 6.4.1. *Let f is a linear application of E on F, we have*

1) $f(0_E) = 0_F$
2) $\forall x \in E : f(-x) = -f(x)$

6.4.1 Kernel, image, and rank of a linear application

Definition 6.4.3. *Let f be a linear application of E on F.*

1) The set $f(E)$ is called the image of the linear application f and is denoted $\operatorname{Im} f$ *i.e*

$$\operatorname{Im} f = \{f(x) : x \in E\}$$

2) The set $f^{-1}(\{0\})$ is calledis the kernel of the linear application and is denoted $\ker f$ *i.e*

$$\ker f = \{x \in E, \ \ f(x) = 0_F\}$$

Example 6.4.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a linear application defined by*

$$(x, y) \longrightarrow f(x, y) = x - y$$

The kernel of the linear application f:

$$ker(f) = \{(x, y) \in \mathbb{R}^2 : x - y = 0\} = \{(x, y) \in \mathbb{R}^2 : x = y\} = \{x(1, 1) : x \in \mathbb{R}\}$$

Thus $ker(f)$ is a vector subspace generated by $e = (1, 1)$ with dimension 1.

The image of f:

$$Im(f) = \{f(x, y) : (x, y) \in \mathbb{R}^2\} = \{x - y : (x, y) \in \mathbb{R}^2\} = \mathbb{R}$$

Proposition 6.4.2. *Let f be a linear application from E to F.*

1) $Im(f)$ is a vector subspace of F.

2) $ker(f)$ is a vector subspace of E.

Definition 6.4.4. *If $\dim Im(f) = n < +\infty$, we call n the rank of f, denoted $rg(f)$.*

Proposition 6.4.3. *For a linear application $f : E \to F$:*

$$f \text{ is surjective} \Leftrightarrow Im(f) = F$$

$$f \text{ is injective} \Leftrightarrow ker(f) = \{0_E\}$$

6.4.2 Linear applications in finite dimensions

Proposition 6.4.4. *Let E, F be $\mathbb{K}$-vector spaces and $f, g : E \to F$ linear applications. If E has finite dimension n with basis $\{e_1, ..., e_n\}$, then:*

$$\forall k \in \{1, ..., n\}, f(e_k) = g(e_k) \Leftrightarrow \forall x \in E, f(x) = g(x)$$

Example 6.4.4. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *with* $f(1,0) = -1$ *and* $f(0,1) = 4$*. Then:*

$$f(x, y) = xf(1, 0) + yf(0, 1) = -x + 4y$$

Proposition 6.4.5 (Rank-Nullity Theorem)**.** *For finite-dimensional* E *and linear* $f : E \to F$*:*

$$\dim E = \dim \mathit{ker}(f) + \dim \mathit{Im}(f)$$

Example 6.4.5. *For* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = -x + 5y$*:*

$$\mathit{ker}(f) = \{(5y, y) : y \in \mathbb{R}\} \Rightarrow \dim \mathit{ker}(f) = 1$$

$$\dim \mathit{Im}(f) = \dim \mathbb{R}^2 - \dim \ \mathit{ker}(f) = 2 - 1 = 1$$

Proposition 6.4.6. *For* $\dim E = \dim F = n$ *and linear* $f : E \to F$*, the following are equivalent:*

- f *is isomorphism*
- f *is surjective*
- $\dim \mathit{Im}(f) = n$
- f *is injective*
- $\mathit{ker}(f) = \{0\}$

Example 6.4.6. *For* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $f(x, y) = (2x - y, x)$*:*

$$\mathit{ker}(f) = \{(x, y) \ : \ f(x, y) = 0 \Rightarrow 2x - y = x = 0\}$$

$$= \{(0, 0)\}$$

as $\dim f = 2$ *and* $\mathit{ker}(f) = 0_{\mathbb{R}^2}$ *; then* f *is an isomorphism*

Conclusion

I hope that this material will help first-year students in assimilating mathematics, and more particularly Analysis and Algebra I, which constitute the foundation of mathematics at the university.

Finally, errors may be found. Please communicate them to me by email at the following addresses: rachid.lakehal93@gmail.com or rachid.lakehal@univ-bejaia.dz.

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