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Mathematics II

Option : *Science and Technology Field*

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Preface :

Preface

This polycopy is intended for first-year students in the Science and Technology field under the LMD system. The manuscript covers the syllabus of the Mathematics II module, which is dedicated to the second semester program. This course includes numerous typical examples and exercises with solutions.

The syllabus of Mathematics II for the second semester consists of five chapters as follows:

Chapter 1: primitive and integrals

Chapter 2: Differential equations

Chapter 3: Matrices and Determinants

Chapter 4: Systems of linear equations

Chapter 5: Functions of several variables

Contents

Chapter 1

primitive and integrals

Introduction

This chapter primarily focuses on the technical aspects of integral calculus. Its aim is to present the main techniques for computing primitive and definite integrals. All functions considered here are real-valued functions of a single real variable.

1.1 Indefinite integrals (primitive)

Definition 1.1.1. *Let $f : I \to \mathbb{R}$ be a function defined on an interval $I \subset \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called* primitive *of f on I if it is differentiable and satisfies:*

$$F'(x) = f(x), \quad \forall x \in I.$$

The notation $\int f(x)\,dx$ represents an indefinite integral *of f, and denotes the set of all primitive of f:*

$$\int f(x)\,dx = F(x) + c, \quad c \in \mathbb{R}.$$

Example 1.1.1. *Consider the function $f(x) = 4x^3 - 3x^2 + 2x + 3$. An primitive of f is:*

$$F(x) = x^4 - x^3 + x^2 + 3x.$$

Thus, the general form of all primitive of f is:

$$\int (4x^3 - 3x^2 + 2x + 3)\,dx = x^4 - x^3 + x^2 + 3x + c, \quad c \in \mathbb{R}.$$

Example 1.1.2. *The function $f(x) = \cos(2x)$ has the primitive:*

$$F(x) = \frac{1}{2}\sin(2x),$$

so the general solution is:

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + c, \quad c \in \mathbb{R}.$$

Remark 1.1.1. *The primitive of a function if it exists is not unique.*
If F is an primitive of f, then all other primitive of f are of the form $F(x) + c$, where c is a constant.

Example 1.1.3. *Let $f(x) = 4x + 3$. Then, the following are all primitive of f:*

$$F(x) = 2x^2 + 3x, \quad G(x) = 2x^2 + 3x + 1, \quad H(x) = 2x^2 + 3x + 2, \ldots$$

Proposition 1.1.1. *Let F and G be primitive of functions f and g on an interval I. Then:*

1. $\int (f + g)(x)\,dx = F(x) + G(x)$
2. $\int (\lambda f)(x)\,dx = \lambda F(x) \qquad \lambda \in \mathbb{R},$
3. $\int (fG + Fg)(x)\,dx = (F \cdot G)(x),$
4. $\int \frac{fG - Fg}{G^2}(x)\,dx = \left(\frac{F}{G}\right)(x),$ *provided* $G(x) \neq 0.$

1.1.1 Primitives of usual functions

1. $\int a\,dx = ax + c, \quad a \in \mathbb{R},$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1,$
3. $\int \frac{1}{x}\,dx = \ln|x| + c,$
4. $\int \frac{1}{2\sqrt{x}} = \sqrt{x} + c$
5. $\int \sin(x)\,dx = -\cos(x) + c,$
6. $\int \cos(x)\,dx = \sin(x) + c,$
7. $\int \frac{1}{1+x^2}\,dx = \arctan x + c,$
8. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c, \quad |x| < 1.$
9. $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \arccos x + c, \quad |x| < 1$

In case general,if f be a continuous function on I we have :

1. $\int f' \times f^n \, dx = \frac{f^{n+1}}{n+1} + c, \quad n \neq -1,$
2. $\int \frac{f'}{f} \, dx = \ln|f| + c,$
3. $\int \frac{f'}{2\sqrt{f}} = \sqrt{f} + c$
4. $\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c, \quad a \neq 0,$
5. $\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c, \quad a \neq 0,$
6. $\int \frac{f'}{1+f^2} \, dx = \arctan(f) + c,$
7. $\int \frac{f'}{\sqrt{1-f^2}} \, dx = \arcsin(f) + c, \quad |f| < 1.$
8. $\int \frac{-f'}{\sqrt{1-f^2}} \, dx = \arccos(f) + c, \quad |f| < 1$

1.2 Techniques for calculate primitive

1.2.1 Integration by Parts

The first method for computing primitive is known as **integration by parts**, which is based on the product rule for derivatives.

Proposition 1.2.1. *Let $U(x)$ and $V(x)$ be two functions of class $\mathcal{C}^1$ on $[a,b]$. Then:*

$$\int U(x)V'(x) \, dx = U(x)V(x) - \int U'(x)V(x) \, dx.$$

Proof. We start from the derivative of a product:

$$\frac{d}{dx}[U(x)V(x)] = U'(x)V(x) + U(x)V'(x).$$

Integrating both sides:

$$\int U(x)V'(x) \, dx = U(x)V(x) - \int U'(x)V(x) \, dx.$$

□

Example 1.2.1. *Let's calculate the primitive* $\int xe^x \, dx$.

We set:

$$\begin{cases} U(x) = x, & \Rightarrow U'(x) = 1, \\ V'(x) = e^x, & \Rightarrow V(x) = e^x. \end{cases}$$

Then:

$$\int xe^x \, dx = xe^x - \int 1 \cdot e^x \, dx = xe^x - e^x + c = (x-1)e^x + c.$$

Remark 1.2.1. *This method is especially useful for integrals of the forms:*

$$\int P(x)\sin x\,dx, \quad \int P(x)\cos x\,dx, \quad \int P(x)e^{-x}\,dx, \quad \int P(x)\ln x\,dx.$$

$$\int P(x)\ \arccos(x)dx, \quad \int P(x)\ \arcsin(x)dx, \quad \int P(x)\ \arctan(x)dx,$$

Such that $P(x)$ is polynomial.

1.2.2 Integration by Substitution (Change of Variable)

Another technique for finding primitive is based on the chain rule for derivatives.

Substitution Rule. If a direct computation of $\int f(x)\,dx$ is difficult, one may perform a change of variable by letting $x = g(t)$, where g is differentiable. Then:

$$dx = g'(t)\,dt \quad \Rightarrow \quad \int f(x)\,dx = \int f(g(t))g'(t)\,dt.$$

Remark 1.2.2. *The success of this technique depends on selecting an appropriate substitution that simplifies the integral. The process involves three main steps:*

1. *Choose a suitable substitution $t = g(x)$.*
2. *Express dx in terms of dt using $dx = g'(t)\,dt$.*
3. *Rewrite and compute the new integral, adjusting limits if the integral is definite.*

Example 1.2.2. *Let's calculate the primitive $\int (x-1)^5\,dx$.*

Let $t = x - 1$, so $dt = dx$. Then:

$$\int (x-1)^5\,dx = \int t^5\,dt = \frac{1}{6}t^6 + c = \frac{1}{6}(x-1)^6 + c.$$

Example 1.2.3. *Let's calculate the primitive $\int \sin^3(x)\cos(x)\,dx$.*

Let $t = \cos x$, so $dt = -\sin x\,dx \Rightarrow dx = -\frac{dt}{\sin x}$. Then:

$$\int \sin^3(x)\cos(x)\,dx = -\int \sin^2(x)\cdot t\,dt = -\int (1-t^2)t\,dt = -\int (t-t^3)\,dt.$$

$$= -\left(\frac{1}{2}t^2 - \frac{1}{4}t^4\right) + c = -\left(\frac{1}{2}\cos^2(x) - \frac{1}{4}\cos^4(x)\right) + c.$$

1.3 Calculation of integrals

Integration of Rational Functions

Rational functions (ratios of polynomials) can often be integrated using the method of partial fraction decomposition. This reduces the integral to a combination of simpler types, such as:

$$\int \frac{1}{(x+\alpha)^n}\,dx, \quad \int \frac{ax+b}{x^2+px+q}\,dx,$$

where $a, b, p, q, \alpha \in \mathbb{R}$, and $n \in \mathbb{N}^*$.

(a) Polynomial Integration

When the integrand is a polynomial $P(x)$, integrate term by term:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$
$$\int P(x)\,dx = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + \frac{a_1}{2}x^2 + a_0 x + c.$$

(b) Integrals of the Form $\int \frac{1}{x+\alpha}\,dx$

$$\int \frac{1}{x+\alpha}\,dx = \ln|x+\alpha| + c.$$

(c) Integrals of the Form $\int \frac{1}{(x+\alpha)^n}\,dx$, with $n \neq 1$

$$\int \frac{1}{(x+\alpha)^n}\,dx = 1 \times (x+\alpha)^{-n} = \frac{1}{1-n}(x+\alpha)^{1-n} + c.$$

(d) Integrals of the Form $\int \frac{ax+b}{x^2+px+q}\,dx$

If the denominator can be factored into real roots α, β, decompose:

$$\frac{ax+b}{x^2+px+q} = \frac{ax+b}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta},$$

and integrate each term separately:

$$\int \frac{ax+b}{x^2+px+q}\,dx = A\ln|x-\alpha| + B\ln|x-\beta| + c.$$

Example 1.3.1. *Compute* $\int \frac{1}{x^2-1}\,dx$.

We use the decomposition:

$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)},$$

so:

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c.$$

Integration of Rational Functions with a Double Root in the Denominator

To compute an integral of the form:

$$\int \frac{P(x)}{Q(x)}\,dx$$

where $P(x)$ is a polynomial of degree 1 and $Q(x)$ is a polynomial of degree 2 with a **double root** (i.e., $\Delta = 0$), follow these steps:

1. Factor the denominator:

$$Q(x) = a(x-x_0)^2 \quad \text{where} \quad x_0 = -\frac{b}{2a}$$

2. Use the substitution:

$$t = x - x_0 \quad \Rightarrow \quad x = t + x_0$$

3. Express the integral in terms of t:

$$\int \frac{P(x)}{Q(x)}\,dx = \int \frac{P(t+x_0)}{at^2}\,dt$$

4. Simplify and integrate using elementary rules:

$$\int \left(\frac{1}{t} - \frac{1}{t^2}\right)dt = \ln|t| + \frac{1}{t} + C$$

5. Return to the original variable x.

Example 1.3.2. *Evaluate the integral:*

$$\int \frac{x+1}{x^2+4x+4}\,dx$$

We have : $x^2+4x+4 = (x+2)^2$

Let $t = x + 2 \Rightarrow x = t - 2$, so:

$$\int \frac{x+1}{(x+2)^2}\,dx = \int \frac{(t-2)+1}{t^2}\,dt = \int \frac{t-1}{t^2}\,dt$$

Then

$$\int \left(\frac{1}{t} - \frac{1}{t^2}\right) dt = \ln|t| + \frac{1}{t} + C$$

after replace t by $x + 2$:

$$\boxed{\int \frac{x+1}{(x+2)^2}\,dx = \ln|x+2| + \frac{1}{x+2} + C}$$

If the denominator has no real roots, complete the square:

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right).$$

pose $\alpha = \frac{-p}{2}$ and $\beta = (q - \frac{p^2}{4})$ then $x^2 + px + q = (x-\alpha)^2 + \beta^2$ after use the substitution:

$$x - \alpha = \beta t, \quad \text{then } dx = \beta dt. \textit{ and } (x-\alpha)^2 + \beta^2 = \beta^2(t^2+1)$$

The integral becomes a sum of:
$\int \frac{ax+b}{x^2+px+c} = \int \frac{ax+b}{(x-\alpha)^2+\beta^2}$
$= \dots\dots\dots = \int \frac{Mt+N}{t^2+1}\,dt = M \cdot \frac{1}{2}\ln(t^2+1) + N \cdot \arctan t + c.$

Example 1.3.3. *Compute* $\int \frac{x+4}{x^2+2x+5}\,dx.$

Complete the square: $x^2 + 2x + 5 = (x+1)^2 + 4$, and set $x + 1 = 2t$, so $dx = 2dt$. Then:

$$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{2t+3}{4(t^2+1)} \cdot 2dt = \frac{1}{2}\int \frac{2t}{t^2+1}\,dt + \frac{3}{2}\int \frac{1}{t^2+1}\,dt$$

$$= \frac{1}{2}\ln(t^2+1) + \frac{3}{2}\arctan(t) + c = \frac{1}{2}\ln\left(\frac{x^2+2x+5}{4}\right) + \frac{3}{2}\arctan\left(\frac{x+1}{2}\right) + c.$$

Integration of Expressions Involving Exponentials

To compute integrals of the form $\int P(x)e^{\alpha x}\,dx$, where $P(x)$ is a polynomial and $\alpha \in \mathbb{R}$, one can either:

- Use repeated integration by parts (if the degree of P is small),

- Or assume a solution of the form $Q(x)e^{\alpha x}$, where $\deg(Q) = \deg(P)$, and determine Q using undetermined coefficients.

Example 1.3.4. *Compute the integral:*

$$\int (5x^2 + 3x - 1)e^x \, dx$$

We know that an integral of the form:

$$\int P(x)e^x \, dx$$

where $P(x)$ is a polynomial, can be written as:

$$(ax^2 + bx + c)e^x$$

We aim to find constants $a, b, c \in \mathbb{R}$ such that:

$$\frac{d}{dx}\Big((ax^2 + bx + c)e^x\Big) = (5x^2 + 3x - 1)e^x$$

Using the product rule:

$$\begin{aligned}\frac{d}{dx}\Big((ax^2 + bx + c)e^x\Big) &= \Big(ax^2 + bx + c\Big)' e^x + (ax^2 + bx + c)e^x \\ &= (2ax + b)e^x + (ax^2 + bx + c)e^x \\ &= \Big(ax^2 + (2a + b)x + (b + c)\Big) e^x\end{aligned}$$

We compare with the original integrand:

$$(5x^2 + 3x - 1)e^x$$

So by identifying terms:

$$\begin{cases} a = 5 \\ 2a + b = 3 \\ b + c = -1 \end{cases}$$

Solve the system:
From the first equation: $a = 5$
Substitute into the second: $2(5) + b = 3 \Rightarrow 10 + b = 3 \Rightarrow b = -7$
Substitute into the third: $-7 + c = -1 \Rightarrow c = 6$
Final Result is :

$$\int (5x^2 + 3x - 1)e^x \, dx = (5x^2 - 7x + 6)e^x + C$$

1.3.1 Integration of Trigonometric Functions

Integration of Trigonometric Powers

To integrate expressions like $\int \cos^p x \sin^q x \, dx$, where $p, q \in \mathbb{N}$, use the following rules:

Case 1: p is odd. Extract one cosine and express the rest in terms of sine:

$$\int \cos^{2k+1} x \sin^q x \, dx = \int (1 - \sin^2 x)^k \sin^q x \cos x \, dx.$$

Use the substitution $t = \sin x$.

Case 2: q is odd. Extract one sine and express the rest in terms of cosine:

$$\int \cos^p x \sin^{2k+1} x \, dx = \int \cos^p x (1 - \cos^2 x)^k \sin x \, dx.$$

Use the substitution $t = \cos x$.

Case 3: p and q both even. Use double-angle identities or the substitution $t = \tan \frac{x}{2}$, which converts all sine and cosine terms to rational expressions in t:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt.$$

And we can use the identities:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}, \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Example 1.3.5. ***case 1***

We consider the integral:

$$\int \cos^3(x) \sin^2(x) \, dx$$

Since the power of cosine is odd, we can write:

$$\cos^3(x) = \cos(x)(1 - \sin^2(x))$$

Then the integral becomes:

$$\int \cos^3(x) \sin^2(x) \, dx = \int \cos(x)(1 - \sin^2(x)) \sin^2(x) \, dx$$

Let:

$$t = \sin(x) \quad \Rightarrow \quad dt = \cos(x) dx$$

Then:

$$\int \cos(x)(1-\sin^2(x))\sin^2(x)\,dx = \int (1-t^2)t^2\,dt$$

Simplify:

$$\int (1-t^2)t^2\,dt = \int (t^2-t^4)\,dt = \int t^2\,dt - \int t^4\,dt$$

Compute:

$$\frac{t^3}{3} - \frac{t^5}{5} + C$$

Return to variable x:

$$\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$$

Then

$$\boxed{\int \cos^3(x)\sin^2(x)\,dx = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

case 2 *We consider the integral:*

$$\int \cos^2(x)\sin^3(x)\,dx$$

Rewrite $\sin^3(x) = \sin(x)(1-\cos^2(x))$, *so:*

$$\int \cos^2(x)(1-\cos^2(x))\sin(x)\,dx$$

Let $t = \cos(x), \quad dt = -\sin(x)\,dx$, *we get:*

$$-\int t^2(1-t^2)\,dt = -\int (t^2-t^4)\,dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + C$$

Back-substitute:

$$\int \cos^2(x)\sin^3(x)\,dx = \frac{\cos^5(x)}{5} - \frac{\cos^3(x)}{3} + C$$

Case 3*We consider the integral:*

$$\int \cos^2(x)\sin^2(x)\,dx$$

Use identities:

$$\cos^2(x)\sin^2(x) = \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right) = \frac{1-\cos^2(2x)}{4}$$

$$\cos^2(2x) = \frac{1+\cos(4x)}{2} \Rightarrow \frac{1-\cos^2(2x)}{4} = \frac{1}{4}\left(1 - \frac{1+\cos(4x)}{2}\right) = \frac{1}{4}\cdot\frac{1-\cos(4x)}{2} = \frac{1-\cos(4x)}{8}$$

Then:

$$\int \cos^2(x)\sin^2(x)\,dx = \int \frac{1-\cos(4x)}{8}\,dx = \frac{1}{8}\int (1-\cos(4x))\,dx = \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

$$\boxed{\int \cos^2(x)\sin^2(x)\,dx = \frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

Products of Trigonometric Functions with Different Angles

Use product-to-sum identities to simplify expressions such as:

$$\cos(\alpha x)\cos(\beta x), \quad \sin(\alpha x)\cos(\beta x), \quad \sin(\alpha x)\sin(\beta x).$$

$$\begin{aligned}
\cos(\alpha x)\cos(\beta x) &= \frac{1}{2}\left[\cos((\alpha+\beta)x)+\cos((\alpha-\beta)x)\right],\\
\sin(\alpha x)\cos(\beta x) &= \frac{1}{2}\left[\sin((\alpha+\beta)x)+\sin((\alpha-\beta)x)\right],\\
\sin(\alpha x)\sin(\beta x) &= \frac{1}{2}\left[\cos((\alpha-\beta)x)-\cos((\alpha+\beta)x)\right].
\end{aligned}$$

Example 1.3.6.

$$\int \cos(5x)\cos(x)\,dx = \frac{1}{2}\int \cos(6x)+\cos(4x)\,dx = \frac{1}{12}\sin(6x)+\frac{1}{8}\sin(4x)+c.$$

1.4 The Definite Integral

Definition 1.4.1. *If F is an primitive of a continuous function f on the interval $[a,b]$, then:*

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

This result emphasizes the importance of knowing the primitive of a function when computing definite integrals.

Remark 1.4.1. *There is a fundamental distinction between indefinite and definite integrals:*

- $\int f(x)\,dx$*: an* indefinite integral*, represents a family of functions (the primitive of f).*
- $\int_a^b f(x)\,dx$*: a* definite integral*, represents a real number (the net area under the curve between $x=a$ and $x=b$).*
- *If f is differentiable on $[a,b]$, then:*

  $$\int_a^b f'(x)\,dx = f(b)-f(a).$$

Example 1.4.1. *Compute the definite integral:*

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

Basic Properties of the Definite Integral

Let f and g be integrable functions on $[a, b]$, and let $\lambda \in \mathbb{R}$. Then:

1. $f + g$, fg, λf, and $|f|$ are all integrable on $[a, b]$.
2. Linearity:
$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$
3. Reversal of limits:
$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$
4. Scalar multiplication:
$$\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx.$$
5. Absolute value inequality:
$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$
6. If $f(x) = 0$ for all $x \in [a, b]$, then:
$$\int_a^b f(x)\, dx = 0.$$
7. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then:
$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$
8. If $n \leq f(x) \leq m$ for all $x \in [a, b]$, then:
$$n(b - a) \leq \int_a^b f(x)\, dx \leq m(b - a).$$
9. If $n \leq f(x) \leq m$ and $g(x) \geq 0$ on $[a, b]$, then:
$$n \int_a^b g(x)\, dx \leq \int_a^b f(x)g(x)\, dx \leq m \int_a^b g(x)\, dx.$$

Cauchy–Schwarz Inequality (Integral Form)

Proposition 1.4.1. *Let f and g be integrable on $[a, b]$. Then:*

$$\left(\int_a^b f(x)g(x)\, dx \right)^2 \leq \left(\int_a^b f^2(x)\, dx \right) \left(\int_a^b g^2(x)\, dx \right).$$

1.5 Exercises on Integration Techniques

Exercise 01 Compute the following indefinite integral:

$$\int \frac{x^2+2}{x+1}\,dx$$

Solution. We perform polynomial division:

$$\frac{x^2+2}{x+1} = x - 1 + \frac{3}{x+1}$$

Thus:

$$\int \frac{x^2+2}{x+1}\,dx = \int \left(x - 1 + \frac{3}{x+1} \right) dx = \frac{x^2}{2} - x + 3\ln|x+1| + c$$

Exercise 02 Compute:

$$\int (x \ln x)\,dx$$

Solution. Use integration by parts. Let:

$$\begin{cases} u = \ln x & \Rightarrow du = \frac{1}{x}dx \\ dv = xdx & \Rightarrow v = \frac{x^2}{2} \end{cases}$$

Then:

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Exercise 03 Compute:

$$\int \frac{e^{2x}}{e^x+1}\,dx$$

Solution. Let $t = e^x + 1 \Rightarrow dt = e^x dx$, and note that $e^{2x} = (e^x)^2$.
So:

$$\int \frac{e^{2x}}{e^x+1}dx = \int \frac{(t-1)^2}{t} \cdot \frac{dt}{t-1} = \int \frac{t^2-2t+1}{t}dt = \int (t - 2 + \frac{1}{t})dt$$

$$= \frac{t^2}{2} - 2t + \ln|t| + c = \frac{(e^x+1)^2}{2} - 2(e^x+1) + \ln(e^x+1) + c$$

Exercise 04 Compute:

$$\int \frac{1}{x^2-4}dx$$

Solution. Factor the denominator:

$$x^2 - 4 = (x-2)(x+2)$$

Use partial fractions:

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \Rightarrow 1 = A(x+2) + B(x-2)$$

Solve for A and B:

$$\begin{cases} x = 2 \Rightarrow 1 = A(4) + B(0) \Rightarrow A = \frac{1}{4} \\ x = -2 \Rightarrow 1 = A(0) + B(-4) \Rightarrow B = -\frac{1}{4} \end{cases}$$

Therefore:

$$\int \frac{1}{x^2-4}dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + c$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + c$$

Exercise 05 Evaluate:

$$\int \frac{1}{1+e^x}dx$$

Solution. Let $u = 1 + e^x \Rightarrow du = e^x dx$

But $dx = \frac{du}{e^x} = \frac{du}{u-1}$, so:

$$\int \frac{1}{u} \cdot \frac{1}{u-1}du \quad \text{is messy. Better: try substitution } u = \frac{1}{1+e^x}$$

Or use the trick:

$$\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}}, \quad \text{then let } u = 1 + e^{-x}, \quad du = -e^{-x}dx$$

So:

$$\int \frac{e^{-x}}{1+e^{-x}}dx = -\int \frac{1}{u}du = -\ln|1+e^{-x}| + c$$

Exercise 06 Evaluate:

$$\int \sin^3 x\,dx$$

Solution. Use identity: $\sin^3 x = \sin x(1-\cos^2 x)$

Let $u = \cos x \Rightarrow du = -\sin x dx \Rightarrow -du = \sin x dx$

$$\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)dx = -\int(1-u^2)du = -\left(u - \frac{u^3}{3}\right) + c = -\cos x + \frac{\cos^3 x}{3} + c$$

Exercise 07 Evaluate:

$$\int_1^e \ln x\,dx$$

Solution. Use integration by parts:

$$\begin{cases} u = \ln x & \Rightarrow du = \frac{1}{x} dx \\ dv = dx & \Rightarrow v = x \end{cases}$$

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + c \Rightarrow \int_1^e \ln x \, dx = [x \ln x - x]_1^e$$

$$= (e \cdot 1 - e) - (1 \cdot 0 - 1) = (e - e) - (-1) = 1$$

Chapter 2

Differential equations

Introduction

The laws of physics and mechanics, as well as many chemical, biological, and economic phenomena, can often be reduced to finding functions whose derivatives satisfy specific relationships.

2.1 Generalities

Definition 2.1.1. *A differential equation is any equation involving an unknown function y of a variable x and its derivatives of various orders:*

$$F(x, y, y', \dots, y^{(k)}) = 0$$

where F expresses a relationship between x, y, and its derivatives.

The order of the differential equation is the order of the highest derivative appearing in the equation.

Such an equation is called an ordinary differential equation (ODE), as the unknown function depends on a single variable.

Example 2.1.1.

- $y'' + (y')^3 + 2y = 0$ *is a second-order differential equation.*
- $y' + \dfrac{xy}{x+1} = 0$ *is a first-order differential equation.*
- $x^2y'' + xy' + 2y^4 = 0$ *is a second-order differential equation.*
- $xy''' + 2y + xe^x = 0$ *is a third-order differential equation.*

Definition 2.1.2. *A solution (or integral) of a differential equation is a pair* (I, f) *consisting of an interval* $I \subset \mathbb{R}$ *and a function* f *satisfying:*

- f *is* k*-times differentiable on* I*;*
- *for all* $x \in I$*,* $F(x, f(x), f'(x), \ldots, f^{(k)}(x)) = 0$*.*

The graph of a solution f *is called an integral curve of the differential equation.*

Remark 2.1.1. *Solving (or integrating) a differential equation means finding all of its solutions, when they exist.*

2.2 Differential equations of the first order

2.2.1 Differential equations of separate variables

A 1^{st} differential equation with separate variables is any equation of the form :

$$f(y).y^{'} = g(x)$$

where f and g are two real functions defined and continuous on the intervals I and J of $\mathbb{R}$ respectively.
Solving method: Let F and G be two primitives of f and g respectively, we have

$$\begin{aligned} f(y)y^{'} = g(x) &\Leftrightarrow f(y)\frac{dy}{dx} = g(x) \\ &\Leftrightarrow \int f(y)dy = \int g(x)dx \\ &\Leftrightarrow F(y) = G(x) + c, \end{aligned}$$

where c is a real constant.

Example 2.2.1. *Solve the differential equation* $y^{'} = y\sin(x)$*.*
Assume that y *does not cancel, hence*

$$\begin{aligned} y^{'} = y\sin(x) &\Longleftrightarrow \frac{y^{'}}{y} = \sin(x) \\ &\Longleftrightarrow \int \frac{dy}{y} = \int \sin(x)dx \\ &\Longleftrightarrow \ln|y| = -\cos(x) + c, \qquad c \in \mathbb{R} \\ &\Longleftrightarrow y = \lambda e^{-\cos(x)}, \qquad \lambda \in \mathbb{R}. \end{aligned}$$

2.2.2 Homogeneous differential equations

Let $f : I \longrightarrow \mathbb{R}$ be a continuous function on I of $\mathbb{R}$. A homogeneous differential equation is any equation of the form

$$y^{'} = f(\frac{y}{x}).$$

Solving method:

we use the change of unknown $y = zx \Longrightarrow y^{'} = z^{'}x + z$. As a result

$$\begin{aligned} y^{'} = f(\frac{y}{x}) &\Longleftrightarrow z^{'}x + z = f(z) \\ &\Longleftrightarrow z + x\frac{dz}{dx} = f(z) \\ &\Longleftrightarrow xdz = (f(z) - z)dx \\ &\Longleftrightarrow \int \frac{dx}{x} = \int \frac{dz}{f(z) - z} \\ &\Longleftrightarrow \ln x = \int \frac{dz}{f(z) - z} + c \quad , c \in \mathbb{R} \\ &\Longleftrightarrow x = \lambda e^{\int \frac{dz}{f(z) - z}}, \qquad \lambda \in \mathbb{R}. \end{aligned}$$

Then the solutions of equation defined by $y = zx$ where $x = \lambda e^{\int \frac{dz}{f(z) - z}}$.

Example 2.2.2. *Integrate the following equation*

$$xy^{'} = y - x.$$

For $x \neq 0$, equation is written as $y^{'} = \frac{y}{x} - 1 = f(\frac{y}{x})$.
let's pose the change of variable $y = zx \Longrightarrow y^{'} = z^{'}x + z$, then

$$\begin{aligned} xy^{'} = y - x &\Longrightarrow x(z^{'}x + z) = zx - x \\ &\Longrightarrow z^{'}x + z = z - 1 \\ &\Longrightarrow z^{'}x = -1 \\ &\Longrightarrow \int dz = -\int \frac{dx}{x} \\ &\Longrightarrow z = -\ln|x| + c \qquad , c \in \mathbb{R}. \end{aligned}$$

but $y = zx$, so we conclate $y = x(-\ln|x| + c)$.

2.3 First-order linear differential equations

Definition 2.3.1. *Let* $b(x)$ *and* $f(x)$ *be two continuous defferential functions on an open interval* I *of* $\mathbb{R}$*. A linear differential equation* (LDE) *of the first order is an equation of the type*

$$y^{'} + b(x)y = f(x) \qquad (E)$$

The associated homogeneous equation of (E) *is*

$$y^{'} + b(x)y = 0$$

We also say equation without a second member.

Resolution method:

a) The homogeneous equation is an equation with separate variables.

$$\begin{aligned} y^{'} + b(x)y = 0 &\Longrightarrow \frac{dy}{dx} = -b(x)y \\ &\Longrightarrow \int \frac{dy}{y} = -\int b(x)dx \\ &\Longrightarrow \ln|y| = -\int b(x)dx + c \\ &\Longrightarrow y = e^{(-\int b(x)dx+c)} \\ &\Longrightarrow y = c_1 e^{-\int b(x)dx} \quad where \quad c_1 \in \mathbb{R} \end{aligned}$$

Then,the homogeneous solution is $y_h = c_1 e^{-\int b(x)dx}$

b) The equation with second member:

If y_p is a particular solution of $y^{'} + b(x)y = f(x)$, then the general solution is $y_g = y_h + y_p$.

Remark 2.3.1. *To find* y_p*, we use the method of variation of the constant and put* $y_p = c_1(x)e^{-\int b(x)dx}$.

Example 2.3.1. *Solve the linear differential equation* $y^{'} - y = e^x$.

- *The homogeneous solution* y_h*:*

$$\begin{aligned} y^{'} - y = 0 &\Longrightarrow \frac{dy}{y} = dx \\ &\Longrightarrow \ln|y| = x + c \\ &\Longrightarrow y_h = c_1 e^x, \quad c_1 \in \mathbb{R}. \end{aligned}$$

- *The particular solution* y_p*:*

by the method of variation of the constant, we pose $y_p = c_1(x)e^x \Longrightarrow y_p^{'} = c_1^{'}(x)e^x + c_1(x)e^x$*, then*

$$\begin{aligned} y^{'} - y = e^x &\Longrightarrow c_1^{'}(x)e^x + c_1(x)e^x - c_1(x)e^x = e^x \\ &\Longrightarrow c_1^{'}(x) = 1 \\ &\Longrightarrow c_1(x) = x + c_2, \quad c_2 \in \mathbb{R} \\ &\Longrightarrow y_p = (x + c_2)e^x. \end{aligned}$$

Finally the general solution is $y_g = y_h + y_p = c_1e^x + (x + c_2)e^x = xe^x + c_3e^x$*.*

Theorem 2.3.1. ***Cauchy-Lipschitz)*** *For* $x_0 \in I$ *and* $y_0 \in \mathbb{R}$*, the equation* (E) *admits a single solution of class* C^1 *at* I *such that* $y(x_0) = y_0$*.*

2.3.1 Bernoulli's differential equation

Let f, g be two continuous functions on an interval I and α a real with $\alpha \neq 0$ and $\alpha \neq 1$. Bernoulli's differential equation is an equation of the form

$$y^{'} + f(x)y = g(x)y^{\alpha}.$$

Solving method

If $y \neq 0, \alpha \neq 1$ and $\alpha \neq 0$, divide the two members of equation by y^{α}, then

$$\frac{y^{'}}{y^{\alpha}} + f(x)\frac{1}{y^{\alpha-1}} = g(x) \Longleftrightarrow y^{'}y^{-\alpha} + f(x)y^{1-\alpha} = g(x).$$

We put $z = \frac{1}{y^{\alpha-1}} = y^{1-\alpha} \Longrightarrow z^{'} = (1-\alpha)y^{-\alpha}y^{'}$, then last equation becomes in z

$$\tfrac{z^{'}}{1-\alpha} + f(x)z = g(x)$$

which is a 1^{st} order (LDE). After calculate we deduce the solution ($y = z^{\frac{1}{1-\alpha}}$)

Example 2.3.2. *Integrate the Bernoulli's equation*

$$3y^{'}\cos(x) - y\sin(x) = y^4.$$

We have

$$3y^{'}\cos(x) - y\sin(x) = y^4 \Longleftrightarrow \frac{3y^{'}\cos(x)}{y^4} - \frac{1}{y^3}\sin(x) = 1$$

we pose $z = \frac{1}{y^3} \Longrightarrow z^{'} = -3y^{'}y^{-4}$, *then* $z^{'}\cos(x) + z\sin(x) = -1$ *, is a* (EDL).
Solution z_h*:*

$$\begin{aligned} z^{'}\cos(x) + z\sin(x) = 0 &\Longleftrightarrow z^{'} = (-\tan(x))z \\ &\Longleftrightarrow \int \frac{dz}{z} = \int \frac{-\sin(x)}{\cos(x)}dx \\ &\Longleftrightarrow \ln z = \ln(\cos(x)) + c \\ &\Longleftrightarrow z_h = c_1\cos(x), \quad c_1 \in \mathbb{R}. \end{aligned}$$

Solution z_p : *By the constant variation method, we put*

$$z = c_1(x)\cos(x) \Longrightarrow z^{'} = c_1^{'}(x)\cos(x) - c_1(x)\sin(x)$$

then

$$\Big(c_1^{'}(x)\cos(x) - c_1(x)\sin(x)\Big)\cos(x) + c_1(x)\cos(x)\sin(x) = -1$$

hence

$$c_1^{'}(x)\cos^2(x) = -1 \Longleftrightarrow c_1(x) = \int \frac{-1}{\cos^2(x)}dx = -\tan(x) + c_2,$$

consequently, the solution y_p *is*

$$z_p = c_1(x)\cos(x) = (-\frac{\sin(x)}{\cos(x)} + c_2)\cos(x) = -\sin(x) + c_2\cos(x).$$

hence

$$z_g = z_h + z_p = c_1\cos(x) - \sin(x) + c_2\cos(x) = -\sin(x) + c_3\cos(x).$$

As $z = \frac{1}{y^3} \Longrightarrow y = \frac{1}{\sqrt[3]{z}}$, *then the solution final is*

$$y = \frac{1}{\sqrt[3]{-\sin(x) + c_3\cos(x)}}, \quad c_3 \in \mathbb{R}.$$

2.4 Second-order differential equations

Definition 2.4.1. *A second-order differential equation is any relation of the form* $F(x, y, y^{'}, y^{''}) = 0$ *between the variable* x*; the unknown function* y*; and its derivatives* $y^{'}, y^{''}$.

Example 2.4.1. $y^{''} + 4y = 0$ *and* $2y^{''} + 4y^{'} - 5y = 0$ *are* 2^{nd}*-order differential equation.*

2.4.1 Differential equations without y

Let be the differential equation of the form $F(x, y', y'') = 0$. Let's apply the change of variable $y' = z$, hence

$$F(x, y', y'') = 0 \Longleftrightarrow F(x, z, z') = 0.$$

Example 2.4.2. *Let the 2^{nd} order differential equation $xy'' + 2y' = 0$,*
If we pose $z = y' \Longrightarrow z' = y''$, hence $xz' + 2z = 0$; is a linear differential equation of the 1^{st} order, we will have

$$\int \frac{dz}{z} = -2\int \frac{dx}{x} \Longrightarrow \ln z = \ln \frac{1}{x^2} + c \Longrightarrow z = \frac{c_1}{x^2} \Longrightarrow y' = \frac{c_1}{x^2}$$

We conclude $y = \frac{-c_1}{x} + c_2$. where $c_1, c_2 \in \mathbb{R}$.

2.4.2 Differential equation with constant coefficients

Definition 2.4.2. *Let f a continuous function on an open interval I. A second-order linear differential equation with constant coefficients is an equation of the form*

$$ay'' + by' + cy = f(x) \qquad (E)$$

Where $a, b, c \in \mathbb{R}$ $(a \neq 0)$.
The associated homogeneous equation of (E) is

$$ay'' + by' + cy = 0. \qquad (H.E)$$

it's an equation without a second member.

Proposition 2.4.1. *A general solution y_g of (E) is $y_g = y_h + y_p$, where y_h is a solution of $(H.E)$ and y_p is particular solution of (E).*

Resolution of the homogeneous equation (H.E)

Let's look for solutions of equation $(H.E)$ in the form $y = e^{rx}$ where $r \in \mathbb{R}$. So we have $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Then $(H.E)$ become

$$ar^2 + br + c = 0. \qquad (C.E)$$

Equation (C.E) is called the characteristic equation associated with $(H.E)$, and its roots r_1, r_2 are called the characteristic values. There are three cases depending on the sign of discriminant $\Delta = b^2 - 4ac$.

1. If $\Delta > 0$: the equation (C.E) has two distinct real roots $r_1 \neq r_2$, and the solution of (H.E) is
$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$
2. If $\Delta = 0$: the equation (C.E) has 1 double root $r_1 = r_2 = r$, and the solution of (H.E) is
$$y(x) = (c_1 x + c_2)e^{rx}.$$
3. If $\Delta < 0$: the equation (C.E) has two conjugate complex roots $r_1 = \alpha + \beta i, \quad r_2 = \alpha - \beta i$, and the solution of (H.E) is
$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

   where $c_1, c_2 \in \mathbb{R}$.

Example 2.4.3. $y^{''} - 4y^{'} + 3y = 0$, *the characteristic equation is* $r^2 - 4r + 3 = 0$ $\Delta = 4 \Longrightarrow r_1 = 1, r_2 = 3$. *The solutions is* $y(x) = c_1 e^x + c_2 e^{3x}$ *with* $c_1,\ c_2 \in \mathbb{R}$.

Example 2.4.4. $y^{''} + 2y^{'} + y = 0$, *the characteristic equation is* $r^2 + 2r + 1 = 0$ $\Delta = 0 \Longrightarrow r_1 = r_2 = r = -1$. *The solutions is* $y(x) = (c_1 x + c_2)e^x$ *with* $c_1,\ c_2 \in \mathbb{R}$.

Example 2.4.5. $y^{''} + 2y^{'} + 4y = 0$, *the characteristic equation is* $r^2 + 2r + 4 = 0$ $\Delta = -12 \Longrightarrow r_1 = -1 + \sqrt{3i}, r_2 = -1 - \sqrt{3i}$. *The solutions is* $y(x) = (c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)e^{-x}$ *with* $c_1,\ c_2 \in \mathbb{R}$.

Solving equations with second member Solving equations with second member

Proposition 2.4.2. *The general solution of (E) is the sum of the general solution of (H.E) and a particular solution of (E). So* $y_g = y_h + y_p$.

Theorem 2.4.1. ***Cauchy-Lipschitz)*** *For* $(y_0, y_1) \in \mathbb{R}^2$*, the equation* (E) *admits a single solution* y *at* I *such that* $y(x_0) = y_0$ *and* $y^{'}(x_0) = y_1$.

Finding a particular solution y_p:

I) Constant variation method

If y_1, y_2 are two linearly independent solutions of the homogeneous equation $(E.H)$, we search a particular solution of (E) in the form $y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are two functions verifying
$$\begin{cases} c_1^{'}(x)y_1 + c_2^{'}(x)y_2 = 0 \\ c_1^{'}(x)y_1^{'} + c_2^{'}(x)y_2^{'} = \dfrac{f(x)}{a} \end{cases}$$

Example 2.4.6. *Solve the differential equation:*

$$y^{''} - 3y^{'} + 2y = e^{x} \tag{2.1}$$

1) For $y^{''} - 3y^{'} + 2y = 0$*, the associated characteristic equation is* $r^2 - 3r + 2 = 0$ *where* $r_1 = 1$ *and* $r_2 = 2$ *so* $y_h = c_1 e^{x} + c_2 e^{2x}$
2) A particular solution y_p *is in the form* $y_p = c_1(x)e^{x} + c_2(x)e^{2x}$ *where* $c_1(x)$ *and* $c_2(x)$ *are two functions which verify*

$$\begin{cases} c_1^{'}(x)e^{x} + c_2^{'}(x)e^{2x} = 0 \\ c_1^{'}(x)e^{x} + 2c_2^{'}(x)e^{2x} = e^{x} \end{cases}$$

By multiplying the first line by (2) *and substract from the second lines, we obtain*

$$c_1^{'} = -1 \Longrightarrow c_1(x) = -x + c_3$$

By substracting the second line from the first line, we obtain

$$c_2^{'} = e^{-x} \Longrightarrow c_2(x) = -e^{-x} + c_4$$

Then $y_p = (-x + c_3)e^{x} + (-e^{-x} + c_4)e^{2x}$
Finally $y_g = y_h + y_p$.

II) Method of Undetermined Coefficients: Case of $f(x) = e^{\lambda x}P(x)$

If the non homogeneous term is of the form $e^{\lambda x}P(x)$, where $\lambda \in \mathbb{R}$ and $P(x) \in \mathbb{R}[x]$, then we look for a particular solution of the form:

$$y_p = x^{m} e^{\lambda x} Q(x)$$

where $Q(x)$ is a polynomial of the same degree as $P(x)$, and:

- If λ is not a root of the characteristic equation (C,E), then we set $m = 0$, i.e., $y_p = e^{\lambda x}Q(x)$.
- If λ is a simple root of the characteristic equation (C,E), then $m = 1$, i.e., $y_p = xe^{\lambda x}Q(x)$.
- If λ is a double root of the characteristic equation (C,E), then $m = 2$, i.e., $y_p = x^2 e^{\lambda x}Q(x)$.

Example 2.4.7. *Solve the following differential equation:*

$$y'' - 4y' + 4y = 6x^2 - 2x + 1$$

Then solve the following Cauchy problem:

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

1. Homogeneous Equation

We first solve the associated homogeneous equation:

$$y'' - 4y' + 4y = 0$$

The characteristic equation is:

$$r^2 - 4r + 4 = 0 \quad \Leftrightarrow \quad (r-2)^2 = 0$$

This gives a double root $r = 2$*. Hence, the general solution of the homogeneous equation is:*

$$y_h(x) = (A + Bx)e^{2x}, \quad A, B \in \mathbb{R}$$

2. Particular Solution

We now find a particular solution y_p*. The non homogeneous term is:*

$$f(x) = 6x^2 - 2x + 1 = (6x^2 - 2x + 1)e^{0x}$$

Since $\alpha = 0$ *is not a root of the characteristic equation, we try:*

$$y_p(x) = ax^2 + bx + c$$

Then:

$$y_p'(x) = 2ax + b, \quad y_p''(x) = 2a$$

Substitute into the differential equation:

$$y'' - 4y' + 4y = 2a - 4(2ax + b) + 4(ax^2 + bx + c)$$

$$= 2a - 8ax - 4b + 4ax^2 + 4bx + 4c$$

$$= 4ax^2 + (-8a + 4b)x + (2a - 4b + 4c)$$

We compare with the right-hand side: $6x^2 - 2x + 1$ *Matching coefficients:*

$$\begin{cases} 4a = 6 \Rightarrow a = \dfrac{3}{2} \\ -8a + 4b = -2 \Rightarrow -12 + 4b = -2 \Rightarrow b = \dfrac{5}{2} \\ 2a - 4b + 4c = 1 \Rightarrow 3 - 10 + 4c = 1 \Rightarrow c = 2 \end{cases}$$

Thus, the particular solution is:

$$y_p(x) = \frac{3}{2}x^2 + \frac{5}{2}x + 2$$

The general solution of the differential equation is:

$$y(x) = y_h(x) + y_p(x) = (A + Bx)e^{2x} + \frac{3}{2}x^2 + \frac{5}{2}x + 2$$

3. Solving the Cauchy Problem

Apply the initial conditions:

At $x = 0$ we have

$$y(0) = A + 0 + 0 + 0 + 2 = 1 \Rightarrow A = -1$$

after we have :

$$\begin{aligned} y'(x) &= \Big[(B)e^{2x} + (A + Bx) \cdot 2e^{2x}\Big] + 3x + \frac{5}{2} \\ &= [B + 2A + 2Bx]\, e^{2x} + 3x + \frac{5}{2} \end{aligned}$$

then at $x = 0$:

$$\begin{aligned} y'(0) &= (B + 2A)e^0 + \frac{5}{2} = B + 2A + \frac{5}{2} \\ 0 &= B + 2(-1) + \frac{5}{2} \Rightarrow B = -\frac{1}{2} \end{aligned}$$

Final we have the solution of the Cauchy problem is:

$$\boxed{y(x) = \left(-1 - \frac{1}{2}x\right) e^{2x} + \frac{3}{2}x^2 + \frac{5}{2}x + 2}$$

Example 2.4.8. *Solve the following differential equation:*

$$y'' - 3y' + 2y = e^x(x - 1)$$

Then solve the Cauchy problem:

$$\begin{cases} y(0) = 2 \\ y'(0) = 1 \end{cases}$$

1. Homogeneous Equation *We first solve the associated homogeneous equation:* $y'' - 3y' + 2y = 0$

The characteristic equation is: $r^2 - 3r + 2 = 0 \quad \Rightarrow \quad (r - 1)(r - 2) = 0 \Rightarrow r_1 = 1, \quad r_2 = 2$ *So the general solution of the homogeneous equation is:*

$$y_h(x) = Ae^x + Be^{2x}, \quad A, B \in \mathbb{R}$$

2. Particular Solution *We now look for a particular solution y_p. The non homogeneous term is:* $f(x) = (x-1)e^x = P(x)e^x, \quad$ *where* $P(x) = x - 1$
Since $\alpha = 1$ *is a simple root of the characteristic equation, we take:* $y_p(x) = xe^x(ax+b)$ *Now compute derivatives:*
$y_p' = \frac{d}{dx}\left[x(ax+b)e^x\right] = \left[(2ax+b) + (ax^2+bx)\right]e^x = (ax^2 + (2a+b)x + b)e^x$ *and* $y_p'' = \frac{d}{dx}\left[(ax^2 + (2a+b)x + b)e^x\right] = (2ax + (2a+b) + ax^2 + (2a+b)x + b)e^x = (ax^2 + (4a+b)x + (2a+2b))e^x$

Now substitute into the differential equation: $y_p'' - 3y_p' + 2y_p = f(x)$ *Substitute and collect terms:*

$$\begin{array}{l} \Big[ax^2 + (4a+b)x + (2a+2b)\Big] \\ -3\Big[ax^2 + (2a+b)x + b\Big] \qquad \Rightarrow \Big[0x^2 + (4a+b-6a-3b+2b)x + (2a+2b-3b)\Big]e^x \\ +2\Big[ax^2 + bx\Big] \end{array}$$

Final expression becomes:

$$(-2a)x + (2a-b) = x - 1$$

Now match coefficients:

$$\begin{cases} -2a = 1 \Rightarrow a = -\dfrac{1}{2} \\ 2a - b = -1 \Rightarrow -1 - b = -1 \Rightarrow b = 0 \end{cases}$$

So the particular solution is: $y_p(x) = xe^x(-\dfrac{1}{2}x) = -\dfrac{1}{2}x^2e^x$ *So the general Solution given by :*

$$y(x) = y_h(x) + y_p(x) = Ae^x + Be^{2x} - \frac{1}{2}x^2e^x$$

3. Apply Initial Conditions

$y(0) = A+B = 2 \quad$ *(1)*$y'(x) = Ae^x + 2Be^{2x} - (x^2+x)e^x \Rightarrow y'(0) = A + 2B - 0 = 1 \quad$ *(2)*

Solve system:

$$\begin{cases} A + B = 2 \\ A + 2B = 1 \end{cases} \Rightarrow B = -1, \quad A = 3$$

Then the Final Solution is :

$$\boxed{y(x) = 3e^x - e^{2x} - \frac{1}{2}x^2e^x}$$

If second member sous form $f(x) = e^{\beta x}(P_1(x)\cos\varphi x + P_2(x)\sin\varphi x)$
Let $f(x) = e^{\beta x}(P_1(x)\cos\varphi x + P_2(x)\sin\varphi x)$, where $\beta \in \mathbb{R}$ and $P_1(x), P_2(x) \in \mathbb{R}[x]$. We search for a particular solution in the form:

- $y_p = e^{\beta x}(Q_1(x)\cos\varphi x + Q_2(x)\sin\varphi x)$, if $\beta + i\varphi$ is **not** a root of the characteristic equation.
- $y_p = xe^{\beta x}(Q_1(x)\cos\varphi x + Q_2(x)\sin\varphi x)$, if $\beta + i\varphi$ **is** a root of the characteristic equation.

In both cases, $Q_1(x)$ and $Q_2(x)$ are polynomials of degree $n = \max(\deg P_1, \deg P_2)$.

Example 2.4.9. *Solve the differential equation:* $y'' + 4y = \cos x \qquad (1)$
1. Homogeneous Equation *The associated homogeneous equation is:*

$$y'' + 4y = 0 \qquad (2)$$

The characteristic equation is: $r^2 + 4 = 0 \Rightarrow r = \pm 2i$
Therefore, the general solution of the homogeneous equation is:

$$y_h(x) = A\cos 2x + B\sin 2x, \quad A, B \in \mathbb{R}$$

2. Particular Solution
We look for a particular solution $y_p(x)$*. The forcing term is:* $f(x) = \cos x$

Since i *is not a root of the characteristic equation* $r^2 + 4 = 0$*, we propose:*

$$y_p(x) = h\cos x + k\sin x$$

Compute derivatives:

$$y_p'(x) = -h\sin x + k\cos x$$
$$y_p''(x) = -h\cos x - k\sin x$$

Substitute into the differential equation:

$$y_p'' + 4y_p = (-h\cos x - k\sin x) + 4(h\cos x + k\sin x) = (3h\cos x + 3k\sin x)$$

Set equal to $f(x) = \cos x$*, then identify coefficients:*

$$3h\cos x + 3k\sin x = \cos x \Rightarrow \begin{cases} 3h = 1 \Rightarrow h = \dfrac{1}{3} \\ 3k = 0 \Rightarrow k = 0 \end{cases}$$

So the particular solution is:

$$y_p(x) = \frac{1}{3}\cos x$$

The general solution is:

$$\boxed{y(x) = A\cos 2x + B\sin 2x + \frac{1}{3}\cos x} \quad \textit{with } A, B \in \mathbb{R}$$

Example 2.4.10. *Solve the differential equation:* $y''-2y'+2y = e^x \sin x$ (1)

1. Homogeneous Equation*:*

We first solve the associated homogeneous equation:

$$y'' - 2y' + 2y = 0 \tag{2}$$

The characteristic equation is: $r^2 - 2r + 2 = 0$

Solving:

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

So the general solution of the homogeneous equation is:

$$y_h(x) = e^x(A\cos x + B\sin x), \quad A, B \in \mathbb{R}$$

2. Particular Solution

The non homogeneous term is: $f(x) = e^x \sin x$

Since $e^x \cos x$ *and* $e^x \sin x$ *are already part of the homogeneous solution (due to repeated complex root* $r = 1 \pm i$*), we multiply by* x *to avoid duplication.*

We propose: $y_p(x) = xe^x(h\cos x + k\sin x)$

Compute derivatives:

$$y_p'(x) = e^x(h\cos x + k\sin x) + xe^x[h(-\sin x) + k\cos x] + e^x(h\cos x + k\sin x)$$

$$= e^x(h\cos x + k\sin x) + xe^x(-h\sin x + k\cos x) + e^x(h\cos x + k\sin x)$$

$$= 2e^x(h\cos x + k\sin x) + xe^x(-h\sin x + k\cos x)$$

Now: By similar way calculate $y_p''(x)$ *Rather than expanding everything, we substitute* y_p*,* y_p'*, and* y_p'' *into the left-hand side of equation (1), simplify, and match coefficients with the right-hand side:*

$$y'' - 2y' + 2y = e^x \sin x$$

After simplifying and comparing coefficients, we find:

$$h = 0, \quad k = \frac{1}{2}$$

Thus, the particular solution is:

$$y_p(x) = \frac{1}{2} x e^x \sin x$$

The general solution of the differential equation is:

$$\boxed{y(x) = e^x (A\cos x + B\sin x) + \frac{1}{2} x e^x \sin x} \quad \textit{where } A, B \in \mathbb{R}$$

2.5 Principle of Superposition

Let $y'' + ay' + by = f(x)$ we have :
If $f(x) = f_1(x) + f_2(x)$, then a particular solution y_p is given by:

$$y_p = y_{p1} + y_{p2}$$

where y_{p1} is a particular solution of the equation:

$$y'' + ay' + by = f_1(x)$$

and y_{p2} is a particular solution of the equation:

$$y'' + ay' + by = f_2(x)$$

So $y_G = y_h + y_{p1} + y_{p2}$

Example 2.5.1. *Solve the differential equation:* $y''-5y'+6y = 2e^{3x}+e^{4x}$ (1)

1. ***Associated Homogeneous Equation***

$$y'' - 5y' + 6y = 0 \qquad (2)$$

The characteristic equation is: $r^2 - 5r + 6 = 0$ (3)
Discriminant: $\Delta = (-5)^2 - 4(1)(6) = 1$ *Roots:* $r_1 = 2, \quad r_2 = 3$
General solution of the homogeneous equation:

$$y_h(x) = Ae^{2x} + Be^{3x}, \quad A, B \in \mathbb{R}$$

2. ***Finding a Particular Solution for (1)***

Using the superposition principle, it suffices to find a particular solution of each of the following:

$$y'' - 5y' + 6y = 2e^{3x} \qquad (4)$$

$$y'' - 5y' + 6y = e^{4x} \qquad (5)$$

Particular Solution y_{p1} ***of (4)***

Let $f_1(x) = 2e^{3x}$. *Since 3 is a simple root of the characteristic equation, we try:*

$$y_{p1}(x) = \lambda x e^{3x}$$

Compute derivatives:

$$y'_{p1}(x) = \lambda e^{3x} + 3\lambda x e^{3x}$$

$$y''_{p1}(x) = 6\lambda e^{3x} + 9\lambda x e^{3x}$$

Substitute into (4):

$$(6\lambda e^{3x} + 9\lambda x e^{3x}) - 5(\lambda e^{3x} + 3\lambda x e^{3x}) + 6\lambda x e^{3x} = 2e^{3x}$$

$$\lambda e^{3x} = 2e^{3x} \Rightarrow \lambda = 2$$

Thus,

$$y_{p1}(x) = 2xe^{3x}$$

Particular Solution y_{p2} ***of (5)***

Let $f_2(x) = e^{4x}$. *Since 4 is not a root of the characteristic equation, try:*

$$y_{p2}(x) = ke^{4x}$$

Compute derivatives:

$$y'_{p2}(x) = 4ke^{4x}, \quad y''_{p2}(x) = 16ke^{4x}$$

Substitute into (5):

$$16ke^{4x} - 20ke^{4x} + 6ke^{4x} = e^{4x} \Rightarrow 2ke^{4x} = e^{4x} \Rightarrow k = \frac{1}{2}$$

Thus,

$$y_{p2}(x) = \frac{1}{2}e^{4x}$$

Final we have the general Solution is

$$y(x) = y_h(x) + y_{p1}(x) + y_{p2}(x) = Ae^{2x} + Be^{3x} + 2xe^{3x} + \frac{1}{2}e^{4x}, \quad A, B \in \mathbb{R}$$

Exercises

Exercise 1: Solve the following separable variable equations:

(a) $y' = x^2 y$

(b) $y' = y^2 \sin x$

(c) $y' = \dfrac{xy}{x^2+1}$

(d) $y' = \dfrac{1-y^2}{1+x^2}$

Exercise 2: Solve the following first-order linear equations:

(a) $y' + 2y = e^{-x}$

(b) $y' - 3y = x$

(c) $y' + y \tan x = \sin x$

Exercise 3: Solve the following Bernoulli equations:

(a) $y' + y = y^2$

(b) $y' - y = y^3$

(c) $y' + y \ln x = xy^2$

Exercise 4: Solve the following second-order linear differential equations with constant coefficients:

(a) $y'' + 4y = 0$

(b) $y'' - 2y' + y = 0$

(c) $y'' - 3y' + 2y = e^x$

(d) $y'' + y = \cos x$

(e) $y'' - 2y' + 5y = e^x$

Chapter 3

Matrices and Determinants

Introduction

In mathematics, matrices are used to express the theoretical results of linear algebra in terms of operational calculations. All disciplines that study linear phenomena use matrices.

In this chapter, $(K, +, \cdot)$ denotes a commutative field, typically $K = \mathbb{R}$ or $K = \mathbb{C}$.

3.1 Basic Concepts of Matrices

3.1.1 Definitions and Notations

Definition 3.1.1. *A **matrix** with n rows and p columns and entries in K is an object A consisting of np elements of K, written as:*

$$A = (a_{ij})_{1 \le i \le n, 1 \le j \le p} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{np} \end{pmatrix}$$

Here, $i \in \{1, 2, \ldots, n\}$ refers to rows and $j \in \{1, 2, \ldots, p\}$ refers to columns. The element a_{ij} is the entry in the i-th row and j-th column.

Notations:

1. The set of all $n \times p$ matrices with entries in K is denoted by $\mathcal{M}_{n,p}(K)$.

2. If $n = p$, the matrix A is called a **square matrix of order** n, and the set of such matrices is denoted $\mathcal{M}_n(K)$.

3. The square matrix of order n with all diagonal entries equal to 1 and all off-diagonal entries equal to 0 is called the **identity matrix** of order n, denoted by I_n:
$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

4. For all $(n, p) \in (\mathbb{N}^*)^2$, and for $1 \leq i \leq n$, $1 \leq j \leq p$, the matrix in $\mathcal{M}_{n,p}(K)$ whose (i, j)-th entry is 1 and all others are 0 is called an **elementary matrix**, denoted by E_{ij}.

5. The matrix in which all elements are zero is called the **zero matrix**, denoted by $0_{n,p}(K)$.

Example 3.1.1. *1.* ***General matrix in*** $\mathcal{M}_{n,p}(K)$*:*

Let $A = \begin{pmatrix} 2 & -1 & 0 \\ 5 & 3 & 4 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$.

This is a matrix with 2 rows and 3 columns with real entries, so $A \in \mathcal{M}_{2,3}(\mathbb{R})$.

2. ***Square matrix in*** $\mathcal{M}_n(K)$*:*

Let $B = \begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$.

This matrix is square of order 2 (2 rows and 2 columns), so it belongs to $\mathcal{M}_2(\mathbb{R})$.

3. ***Identity matrix*** I_n*:*

The identity matrix of order 3 is:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix satisfies $I_3 A = A I_3 = A$ *for any* $A \in \mathcal{M}_3(\mathbb{R})$.

4. ***Elementary matrix*** *E_{ij}:*

   Let $E_{23} \in \mathcal{M}_{3,4}(\mathbb{R})$ be the matrix where the element in the 2nd row and 3rd column is 1, and all other entries are 0:

$$E_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5. ***Zero matrix*** *$0_{n,p}(K)$:*

   The zero matrix in $\mathcal{M}_{2,3}(\mathbb{R})$ is:

$$0_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   Every entry in this matrix is zero.

3.1.2 Operations on Matrices

a) Equality of Two Matrices

Definition 3.1.2. *Two matrices $A = (a_{ij}) \in \mathcal{M}_{n,p}(K)$ and $B = (b_{ij}) \in \mathcal{M}_{n,p}(K)$ are said to be equal if and only if*

$$a_{ij} = b_{ij} \quad \forall i \in \{1, \dots, n\},\ \forall j \in \{1, \dots, p\}.$$

Note: *Two matrices must be of the same size to be compared.*

Example 3.1.2.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A = B$$

b) Sum and Difference of Matrices

Definition 3.1.3. *Let $A = (a_{ij}), B = (b_{ij}) \in \mathcal{M}_{n,p}(K)$. The sum $A+B \in \mathcal{M}_{n,p}(K)$ is defined by:*

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

for all i, j. The difference $A - B$ is defined similarly.

Necessary Condition: *The matrices must have the same number of rows and columns: $A, B \in \mathcal{M}_{n,p}(K)$.*

Example 3.1.3.

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 7 \\ 0 & 1 \end{pmatrix} \Rightarrow A + B = \begin{pmatrix} 3 & 10 \\ 4 & 6 \end{pmatrix}$$

Non-example: *If*

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

then $A + C$ is not defined because $A \in \mathcal{M}_{2,2}$ and $C \in \mathcal{M}_{2,3}$.

c) Scalar Multiplication

Definition 3.1.4. *Let $A = (a_{ij}) \in \mathcal{M}_{n,p}(K)$, and $\lambda \in K$. The scalar multiplication is defined as:*

$$\lambda A = (\lambda a_{ij}) \in \mathcal{M}_{n,p}(K)$$

Example 3.1.4. *Let*

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \quad \lambda = 3 \Rightarrow 3A = \begin{pmatrix} 6 & -3 \\ 0 & 9 \end{pmatrix}$$

d) Product of Two Matrices

Definition 3.1.5. *Let $A \in \mathcal{M}_{n,p}(K)$ and $B \in \mathcal{M}_{p,q}(K)$. The matrix product $C = AB \in \mathcal{M}_{n,q}(K)$ is defined by:*

$$c_{ik} = \sum_{j=1}^{p} a_{ij} \cdot b_{jk}$$

Necessary Condition: *The number of columns of A must equal the number of rows of B, i.e., A: $n \times \underline{p}$, B: $\underline{p} \times q$.*

Example 3.1.5.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

Non-example: *Let*

$$C = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \end{pmatrix} \in \mathcal{M}_{2,3}$$

and

$$D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in \mathcal{M}_{2,2}$$

Then CD is not defined because the number of columns in C (3) is not equal to the number of rows in D (2).

Proposition 3.1.1.

- *Matrix multiplication is associative:* $A(BC) = (AB)C$
- *It is distributive:* $A(B + C) = AB + AC$
- *It is not commutative in general:* $AB \neq BA$

Example 3.1.6.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow AB \neq BA$$

Invertible Matrices

Definition 3.1.6. *Let* $A \in \mathcal{M}_n(K)$*. The matrix* A *is called* ***invertible*** *if and only if there exists a matrix* $A^{-1} \in \mathcal{M}_n(K)$ *such that:*

$$AA^{-1} = A^{-1}A = I_n$$

In this case, A^{-1} *is called the* ***inverse*** *of* A*.*

Example 3.1.7. *Let*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

To find A^{-1}*, we suppose:*

$$A^{-1} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

We impose the condition:

$$AA^{-1} = I_3 \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This yields the system:

$$\begin{cases} x_1 + 2y_1 = 1 \\ x_2 + 2y_2 = 0 \\ x_3 + 2y_3 = 0 \\ y_1 + z_1 = 0 \\ y_2 + z_2 = 1 \\ y_3 + z_3 = 0 \\ x_1 + z_1 = 0 \\ x_2 + z_2 = 0 \\ x_3 + z_3 = 1 \end{cases}$$

Solving gives:

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

Proposition 3.1.2. *Let A and B be two invertible matrices in $\mathcal{M}_n(K)$. Then the product AB is invertible and:*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. Let $C = B^{-1}A^{-1}$. Then:

$$(AB)C = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

and similarly:

$$C(AB) = B^{-1}A^{-1}AB = B^{-1}I_nB = B^{-1}B = I_n$$

Thus, $C = (AB)^{-1}$. □

Transpose of a Matrix

Definition 3.1.7. *Let $A = (a_{ij}) \in \mathcal{M}_{n,p}(K)$. The **transpose** of A, denoted A^T (or ${}^tA$), is the matrix in $\mathcal{M}_{p,n}(K)$ defined by:*

$$A^T = (a_{ji})$$

Example 3.1.8. *Let*

$$A = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{pmatrix} \quad \textit{then} \quad A^T = \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 6 & 5 \end{pmatrix}$$

Proposition 3.1.3. *Let $A, B \in \mathcal{M}_{n,p}(K)$ and $\alpha \in K$. Then:*

1. $(A^T)^T = A$
2. $(\alpha A)^T = \alpha A^T$
3. $(A + B)^T = A^T + B^T$
4. *For $A \in \mathcal{M}_{n,p}(K)$ and $B \in \mathcal{M}_{p,q}(K)$:*
$$(AB)^T = B^T A^T$$
5. *If A is invertible, then:*
$$(A^{-1})^T = (A^T)^{-1}$$
6. $\mathrm{rank}(A) = \mathrm{rank}(A^T)$

Symmetric and Antisymmetric Matrices

Definition 3.1.8. *A square matrix A is called **symmetric** if:*

$$A^T = A$$

Example 3.1.9.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 6 \\ 3 & 5 & 4 & -1 \\ 4 & 6 & -1 & 3 \end{pmatrix} \quad \textit{is symmetric}$$

Definition 3.1.9. *A square matrix A is called **antisymmetric** if:*

$$A^T = -A$$

Triangular and Diagonal Matrices

Definition 3.1.10. *Let $A \in \mathcal{M}_n(K)$.*

1. *A is **upper triangular** if:*
$$\forall i > j, \quad a_{ij} = 0$$
2. *A is **lower triangular** if:*
$$\forall i < j, \quad a_{ij} = 0$$

3. A is ***diagonal*** *if:*

$$\forall i \neq j, \quad a_{ij} = 0$$

Example 3.1.10. *Upper triangular:*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

Lower triangular:

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 3 & 5 \end{pmatrix}$$

Diagonal:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Definition 3.1.11. *The* ***trace*** *of a square matrix $A = (a_{ij}) \in \mathcal{M}_n(K)$ is defined as:*

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

3.2 Determinants

3.2.1 Computing Determinants

Definition 3.2.1 (Determinant of a square matrix)**.** *Let $A = (a_{ij})$ be a square matrix in $\mathcal{M}_n(\mathbb{K})$. The determinant of A expanded along the i-th row is the scalar defined by:*

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det A_{ik},$$

where A_{ik} is the matrix of order $n-1$ obtained from A by removing the i-th row and the k-th column.

Similarly, the determinant of A expanded along the j-th column is given by:

$$\det A = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det A_{kj}.$$

We also use the notation:

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{pmatrix}$$

Remark 3.2.1. *Determinants apply only to square matrices. The determinant of a square matrix is unique.*

Case of order 2 determinant: Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{K}).$$

Then,

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Example 3.2.1.

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = (3)(2) - (4)(1) = 6 - 4 = 2.$$

$$\begin{vmatrix} 5 & \alpha \\ 4 & 7 \end{vmatrix} = 5 \cdot 7 - 4 \cdot \alpha = 35 - 4\alpha.$$

Case of order 3 determinant: Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{M}_3(\mathbb{K}).$$

Then,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

First method: Expansion along the first row

$$\det A = (-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Second method: Expansion along the third column

$$\det A = (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + (-1)^{2+3}a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + (-1)^{3+3}a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

$$= a_{13}(a_{21}a_{32} - a_{22}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31}) + a_{33}(a_{11}a_{22} - a_{12}a_{21}).$$

Example 3.2.2. *Let*

$$A = \begin{pmatrix} 3 & -7 & 8 \\ -3 & 4 & 2 \\ 5 & -1 & 1 \end{pmatrix}.$$

Using the first row expansion:

$\det A = 3 \cdot (4 \cdot 1 - 2 \cdot (-1)) - (-7) \cdot (-3 \cdot 1 - 2 \cdot 5) + 8 \cdot (-3 \cdot (-1) - 4 \cdot 5) = 209.$

Using the third column expansion:

$\det A = 8 \cdot (-3 \cdot (-1) - 4 \cdot 5) - 2 \cdot (3 \cdot (-1) - (-7) \cdot 5) + 1 \cdot (3 \cdot 4 - (-7) \cdot (-3)) = 209.$

Sarrus' Rule (Order 3 only)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \vdots a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & \vdots a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & \vdots a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Example 3.2.3 (Using Sarrus' Rule)**.** *Let*

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \\ -2 & 5 & 2 \end{pmatrix}.$$

We compute $\det A$ *using Sarrus' Rule:*

$$\det A = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \\ -2 & 5 & 2 \end{vmatrix} = 2 \cdot 0 \cdot 2 + (-1) \cdot 1 \cdot (-2) + 3 \cdot 4 \cdot 5 - (-2) \cdot 0 \cdot 3 - 5 \cdot 1 \cdot 2 - 2 \cdot 4 \cdot (-1)$$

Compute step by step:

$$\textit{Positive diagonals:}\quad 0,\quad 2,\quad 60$$

$$\textit{Negative diagonals:}\quad 0,\quad 10,\quad -8$$

$$\Rightarrow \det A = (0+2+60)-(0+10-8) = 62-2 = \boxed{60}.$$

Proposition 3.2.1. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$, and $\alpha \in \mathbb{K}$:*

1. $\det I_n = 1$,
2. $\det(\alpha A) = \alpha^n \det A$,
3. $\det(AB) = \det A \cdot \det B$,
4. $\det(A^m) = (\det A)^m$, *for* $m \in \mathbb{N}$,
5. *A is invertible* $\Leftrightarrow \det A \neq 0$, *and* $\det(A^{-1}) = \frac{1}{\det A}$,
6. $\det({}^tA) = \det A$,
7. *The determinant of A remains unchanged if we add to a column a linear combination of other columns,*
8. $\det A = 0$ *if one column is zero,*
9. $\det A = 0$ *if the columns (or rows) are linearly dependent,*
10. *If A is a lower or upper triangular matrix, then* $\det A = \prod_{i=1}^{n} a_{ii}$.

3.2.2 Calculation of the Inverse of a Matrix Using the Determinant

The formula $AA^{-1} = I_n$ allows us to compute the inverse A^{-1} of an invertible matrix A. A more efficient formula uses cofactors.

Definition 3.2.2 (Cofactor). *Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$. The* cofactor *of the element in position (i, j) in A, denoted by c_{ij}, is given by:*

$$c_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where A_{ij} is the matrix of order $n-1$ obtained from A by deleting the i-th row and the j-th column.

Definition 3.2.3 (Comatrix). *The* comatrix *(or matrix of cofactors) of A, denoted $com(A)$, is the square matrix of order n defined as:*

$$com(A) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix},$$

where c_{ij} is the cofactor of the element in position (i, j).

Theorem 3.2.1. *Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$. Then:*

$$A^{-1} = \frac{1}{\det(A)}\, {}^t\Big(com(A)\Big),$$

where ${}^t\Big(com(A)\Big)$ is the transpose of the comatrix.

Example 3.2.4. *Let $A \in \mathcal{M}_2(\mathbb{R})$ be the matrix:*

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}.$$

Calculate A^{-1} if it exists.

We have:

$$\det(A) = (2)(3) - (4)(1) = 6 - 4 = 2 \neq 0,$$

so A is invertible.

The comatrix of A is:

$$com(A) = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}, \quad {}^t\Big(com(A)\Big) = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}.$$

Thus:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -2 \\ \frac{-1}{2} & 1 \end{pmatrix}.$$

Example 3.2.5. *Let $B \in \mathcal{M}_3(\mathbb{R})$ be the matrix:*

$$B = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

Compute B^{-1}, if it exists.

We first compute the determinant:

$$\det(B) = 2\cdot\begin{vmatrix}2 & 2\\ 1 & 2\end{vmatrix} - 3\cdot\begin{vmatrix}3 & 2\\ 1 & 2\end{vmatrix} + 1\cdot\begin{vmatrix}3 & 2\\ 1 & 1\end{vmatrix} = 2(4-2)-3(6-2)+1(3-2) = 4-12+1 = -7 \neq 0.$$

So B is invertible.

Now we compute each cofactor $c_{ij} = (-1)^{i+j}\cdot\det(B_{ij})$:

$$com(B) = \begin{pmatrix} \begin{vmatrix}2 & 2\\ 1 & 2\end{vmatrix} & -\begin{vmatrix}3 & 2\\ 1 & 2\end{vmatrix} & \begin{vmatrix}3 & 2\\ 1 & 1\end{vmatrix} \\ -\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix} & -\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix} \\ \begin{vmatrix}3 & 1\\ 2 & 2\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix} & \begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & -4 & 1\\ -5 & 3 & 1\\ 4 & -1 & -5 \end{pmatrix}.$$

Transpose the comatrix:

$${}^t\Big(com(B)\Big) = \begin{pmatrix} 2 & -5 & 4\\ -4 & 3 & -1\\ 1 & 1 & -5 \end{pmatrix}.$$

Thus:

$$B^{-1} = \frac{1}{\det(B)}\cdot {}^t\Big(com(B)\Big) = -\frac{1}{7}\begin{pmatrix} 2 & -5 & 4\\ -4 & 3 & -1\\ 1 & 1 & -5 \end{pmatrix} = \begin{pmatrix} -\frac{2}{7} & \frac{5}{7} & -\frac{4}{7}\\ \frac{4}{7} & -\frac{3}{7} & \frac{1}{7}\\ -\frac{1}{7} & -\frac{1}{7} & \frac{5}{7} \end{pmatrix}.$$

3.3 Matrix associated with a linear application

3.3.1 Linear applications

We say that the application $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear application if:
$\forall (X,Y) \in (\mathbb{R}^n)^2,\ \forall(\alpha,\beta) \in \mathbb{R}$, we have

$$f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$$

We can express any linear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ as follows:

$$f\begin{pmatrix} x_1\\ x_2\\ \vdots\\ \vdots\\ x_n \end{pmatrix} = \begin{pmatrix} a_{1,1}x_1 + \cdots\cdots\cdots\cdots\cdots\cdots + a_{1,n}x_n\\ \vdots \qquad\qquad\qquad\qquad \vdots\\ a_{i,1}x_1 + \cdots\cdot + a_{i,j}x_j + \cdots + a_{i,n}x_n\\ \vdots \qquad\qquad\qquad\qquad \vdots\\ a_{m,1}x_1 + \cdots\cdots\cdots\cdots\cdots\cdots + a_{m,n}x_n \end{pmatrix}$$

3.3.2 Associated matrix

Let E and F be two vector spaces of finite dimension n over the same field $\mathbb{K}$. We denote by $B = \{e_1, \cdots, e_n\}$ a basis of E and $B^{'} = \{f_1, \cdots, f_m\}$ a basis of F. Let $f : E \to F$ be a linear mapping from E to F.

Definition 3.3.1. *The matrix associated with the linear application f with respect to the bases $B, B^{'}$ is the matrix $M_{B,B'}(f)$, whose columns represent the vectors $f(e_1), \cdots, f(e_n)$ expressed in the basis $B^{'}$. It is written as*

$$M_{B,B'}(f) = \begin{array}{cc} \begin{array}{ccccc} f(e_1) & \cdots & \cdots & \cdots & f(e_n) \end{array} & \\ \left(\begin{array}{ccccc} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,j} & \cdots & a_{m,n} \end{array}\right) & \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_m \end{array} \end{array}$$

Example 3.3.1. *Let f be a linear application defined by*

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x, y) \longrightarrow f(x, y) = (3x - 4y, 4x - 4y, -3x + 2y)$$

Let $B = \{e_1 = (1, 0), e_2 = (0, 1)\}$ be the canonical basis of $\mathbb{R}^2$ and $B^{'} = \{f_1 = (1, 0, 0), f_2 = (0, 1, 0), f_3 = (0, 0, 1)\}$ be the canonical basis of $\mathbb{R}^3$. Determine the matrix $M_{B,B'}(f)$, we have:

$$f(e_1) = f((1, 0)) = (3, 4, -3) = (3, 0, 0) + (0, 4, 0) + (0, 0, -3) = 3f_1 + 4f_2 - 3f_3$$
$$f(e_1) = f((0, 1)) = (-4, -4, 2) = (-4, 0, 0) + (0, -4, 0) + (0, 0, 2) = -4f_1 - 4f_2 + 2f_3$$

We conclude:

$$M_{B,B'}(f) = \begin{pmatrix} 3 & -4 \\ 4 & -4 \\ -3 & 2 \end{pmatrix}$$

Definition 3.3.2 (Matrix Rank). *The rank of a matrix $A \in M_{n,m}(\mathbb{K})$, denoted by $rk(A)$, is the rank of the linear application f associated with a matrix A. It is written as $rk(A) = rk(f)$.*

3.4 Change of basis, transition matrix

Let $B = \{e_1, \cdots, e_n\}$ and $B^{'} = \{f_1, \cdots, f_n\}$ be two bases of the same vector space E of dimension n.

Definition 3.4.1. *The matrix of change of basis from the basis* B*to the basis* B' *is called* P*, denoted as* $M_{B,B'}(id_E)$*, and it is written as*

$$\begin{aligned} id_E(e_1) &= e_1 = a_{11}f_1 + a_{21}f_2 + \cdots\cdots + a_{n1}f_n \\ id_E(e_2) &= e_2 = a_{12}f_1 + a_{22}f_2 + \cdots\cdots + a_{n2}f_n \\ &\vdots \\ id_E(e_n) &= e_n = a_{1n}f_1 + a_{2n}f_2 + \cdots\cdots + a_{nn}f_n \end{aligned}$$

which gives

$$\begin{array}{ccc} & \begin{array}{ccccc} f(e_1) & \cdots & \cdot\cdot & \cdot\cdot & f(e_n) \end{array} & \\ P = & \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \ddots & a_{2n} \\ \vdots & \cdots & a_{i,j} & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} & \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_n \end{array} \end{array}$$

Example 3.4.1. *Let* $B = \{e_1, e_2, e_3\}$ *be the canonical basis of* $\mathbb{R}^3$*. Let's choose the vectors of the basis* $B^{'} = \{e_1^{'}, e_2^{'}, e_3^{'}\}$ *defined by*

$$\begin{cases} e_1^{'} = e_1 + e_2 + 0e_3 \\ e_2^{'} = 0e_1 + e_2 + e_3 \\ e_3^{'} = e_1 + e_2 + e_3 \end{cases}$$

The transition matrix P *from* B *to* $B^{'}$ *is the matrix in* $M_3(\mathbb{K})$

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The matrix P^{-1} will be the transition matrix from $B^{'}$ to B, obtained by expressing the basis vectors of B in $B^{'}$. So:

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = \alpha_1 e_1^{'} + \beta_1 e_2^{'} + \gamma_1 e_3^{'} = \alpha_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + \beta_1 \begin{pmatrix}0\\1\\1\end{pmatrix} + \gamma_1 \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

That is,

$$\begin{cases}\alpha_1 + 0\beta_1 + \gamma_1 = 1\\ \alpha_1 + \beta_1 + \gamma_1 = 0\\ 0\alpha_1 + \beta_1 + \gamma_1 = 0\end{cases}, \quad \Longrightarrow \begin{cases}\alpha_1 = 0\\ \beta_1 = -1\\ \gamma_1 = 1\end{cases},$$

therefore,

$$e_1 = -e_2^{'} + e_3^{'}.$$

In the same way, we calculate

$$e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} = \alpha_2 e_1^{'} + \beta_2 e_2^{'} + \gamma_2 e_3^{'} = \alpha_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + \beta_1 \begin{pmatrix}0\\1\\1\end{pmatrix} + \gamma_1 \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Finally, we get

$$\begin{cases}e_1 = -e_2^{'} + e_3^{'}\\ e_2 = e_1^{'} + e_2^{'} - e_3^{'}\\ e_3 = -e_1^{'} + e_3^{'}\end{cases} \Longrightarrow P^{-1} = \begin{pmatrix}0 & 1 & -1\\ -1 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}$$

Diagonalization of a matrix: A square matrix A is said to be diagonalizable over $\mathbb{R}$ if there exists a diagonal matrix $D \in M_n(\mathbb{K})$ and an invertible matrix P such that: $A = PDP^{-1}$ as well as $D = P^{-1}AP$.

Exercise 01

Let the matrix

$$A = \begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}, \quad B = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

Answer the following:

1. calculate $A + B$ and $A - B$

2. calculate AB
3. calculate the inverse of A, if it exists, using the definition $AA^{-1} = I$
4. calculate $\det(A)$
5. calculate A^{-1} using the formula involving the determinant and comatrix

Solution

1. **Addition:**
$$A + B = \begin{pmatrix} 2+1 & 3+2 \\ 1+0 & 4+1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 5 \end{pmatrix}$$
2. **Subtraction:**
$$A - B = \begin{pmatrix} 2-1 & 3-2 \\ 1-0 & 4-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$
3. **Multiplication:**
$$AB = \begin{pmatrix} 2\cdot 1 + 3\cdot 0 & 2\cdot 2 + 3\cdot 1 \\ 1\cdot 1 + 4\cdot 0 & 1\cdot 2 + 4\cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 1 & 6 \end{pmatrix}$$
4. **Inverse by definition:** We check if there exists a matrix $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that:
$$AA^{-1} = I \Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
This gives the system:
$$\begin{cases} 2a + 3c = 1 \\ 2b + 3d = 0 \\ a + 4c = 0 \\ b + 4d = 1 \end{cases}$$
Solving:
$$a = -4c \Rightarrow 2(-4c) + 3c = 1 \Rightarrow -8c + 3c = 1 \Rightarrow -5c = 1 \Rightarrow c = -\frac{1}{5},\ a = \frac{4}{5}$$
$$b = 1-4d \Rightarrow 2(1-4d)+3d = 0 \Rightarrow 2-8d+3d = 0 \Rightarrow -5d = -2 \Rightarrow d = \frac{2}{5},\ b = 1-\frac{8}{5} = -\frac{3}{5}$$
So:
$$A^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

5. **Determinant:**

$$\det(A) = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5$$

6. **Inverse using determinant and comatrix:**

   Cofactors:

$$\text{cofactor}_{11} = \det(4) = 4, \quad \text{cofactor}_{12} = -\det(1) = -1, \quad \text{cofactor}_{21} = -\det(3) = -3, \quad \text{cofacto}$$

   Comatrix:

$$\text{com}(A) = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}, \quad \text{Transpose: } \text{com}(A)^T = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

   Then:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{com}(A)^T = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

Exercise 02

Let the matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Answer the following:

1. calculate $A + B$ and AB
2. calculate $\det(A)$
3. Compute the inverse of A, if it exists, using the determinant and the comatrix

Solution

1. **Addition:**

$$A + B = \begin{pmatrix} 1+2 & 2+0 & 3+1 \\ 0+1 & 1+2 & 4+1 \\ 5+0 & 6+1 & 0+0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 3 & 5 \\ 5 & 7 & 0 \end{pmatrix}$$

2. **Multiplication:**

$$AB = \begin{pmatrix} 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 0 & 1 \cdot 0 + 2 \cdot 2 + 3 \cdot 1 & 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 0 \\ 0 \cdot 2 + 1 \cdot 1 + 4 \cdot 0 & 0 \cdot 0 + 1 \cdot 2 + 4 \cdot 1 & 0 \cdot 1 + 1 \cdot 1 + 4 \cdot 0 \\ 5 \cdot 2 + 6 \cdot 1 + 0 \cdot 0 & 5 \cdot 0 + 6 \cdot 2 + 0 \cdot 1 & 5 \cdot 1 + 6 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 4 & 7 & 3 \\ 1 & 6 & 1 \\ 16 & 12 & 11 \end{pmatrix}$$

3. **Determinant of** A:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}$$

$= 1(1{\cdot}0-4{\cdot}6)-2(0{\cdot}0-4{\cdot}5)+3(0{\cdot}6-1{\cdot}5) = 1(-24)-2(-20)+3(-5) = -24+40-15 = 1$

4. **Inverse of** A: Since $\det(A) = 1 \neq 0$, A is invertible.

   To compute $A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{com}(A)^T$, we calculate the cofactor matrix (you may expand all 9 cofactors similarly):

$$\text{Cofactors of } A:$$

$$\mathrm{cof}_{11} = \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} = -24, \quad \mathrm{cof}_{12} = -\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} = -(-20) = 20, \quad \mathrm{cof}_{13} = \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} = -5$$

   (Continue similarly for the remaining 6 cofactors...)

   After computing all cofactors we get:

$$\mathrm{com}(A) = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix} \Rightarrow A^{-1} = \mathrm{com}(A)^T$$

   So:

$$A^{-1} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

Chapter 4

Systems of linear equations

Introduction

Linear systems are fundamental in many fields due to their wide range of applications. They form the computational backbone of linear algebra and provide essential tools for the study of finite-dimensional vector spaces. Therefore, this chapter begins with an in-depth exploration of linear equations and methods for solving them.

Specifically, we will study and compare three standard techniques used to solve linear systems:

- Cramer's Rule,
- The Inverse Matrix Method,
- Gaussian Elimination.

Each method has its own advantages and is suited for particular types of problems. Understanding these techniques is crucial for mastering the broader concepts of linear algebra.

4.1 General concepts

Definition 4.1.1. *We call a system of linear equations with m equations in n unknowns a system of the form:*

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases} \tag{4.1}$$

where the coefficients $a_{i,j}$ and b_i are given, and the x_i are unknowns in $\mathbb{R}$ or $\mathbb{C}$.

Matrix notation Can be written in matrix form:

$$A\,X = B$$

where

$$A = \begin{pmatrix} a_{1,1} & \cdots & \cdots & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & \cdots & \cdots & \cdots & a_{m,n} \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Example 4.1.1.

$$\begin{cases} -x + 2y - 12z = 14 \\ 3x + 7y = -9 \\ 5x - y - 4z = -16 \end{cases}$$

The associated matrix is:

$$A = \begin{pmatrix} -1 & 2 & -12 \\ 3 & 7 & 0 \\ 5 & -1 & -4 \end{pmatrix}, \quad \text{and } X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} 14 \\ -9 \\ -16 \end{pmatrix}.$$

Then, we have

$$\begin{pmatrix} -1 & 2 & -12 \\ 3 & 7 & 0 \\ 5 & -1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -9 \\ -16 \end{pmatrix}.$$

4.2 Methods of solving a linear system

4.2.1 Solving systems using Cramer's method

Let (S) be a square system, meaning its matrix A is square, with the matrix interpretation: $AX = B$. If the matrix A is invertible, we can solve this system using Cramer's method.

We will denote by A_i the matrix A of coefficients in which we have replaced the i-th column with the matrix B.
The solution of the system, using Cramer's method, yields

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1, \cdots\cdots\cdots, n.$$

Example 4.2.1. *Using Cramer's method, solve*

$$\begin{cases} x - 2y = 7 \\ 3x + 5y = 12 \end{cases}$$

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$$

Thus,

$$A_1 = \begin{pmatrix} 7 & -2 \\ 12 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 7 \\ 3 & 12 \end{pmatrix}$$

and

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 7 & -2 \\ 12 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{59}{11}, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 7 \\ 3 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix}} = \frac{-9}{11}$$

$(x, y) = (\frac{59}{11}, \frac{-9}{11})$ *is a unique solution to this system.*

Example 4.2.2. *Solve the following system*

$$\begin{cases} 2x - 3y + z & = 10 \\ -x - 4y + 3z & = 14 \\ 5x - y - 2z & = 12 \end{cases}$$

with Cramer's method.

Let's first identify the coefficients matrix:

$$A = \begin{pmatrix} 2 & -3 & 1 \\ -1 & -4 & 3 \\ 5 & -1 & -2 \end{pmatrix},$$

and the matrix of constants:

$$B = \begin{pmatrix} 10 \\ 14 \\ 12 \end{pmatrix}.$$

We get

$$A_1 = \begin{pmatrix} 10 & -3 & 1 \\ 14 & -4 & 3 \\ 12 & -1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 10 & 1 \\ -1 & 14 & 3 \\ 5 & 12 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & -3 & 10 \\ -1 & -4 & 14 \\ 5 & -1 & 12 \end{pmatrix},$$

thus

$$\begin{aligned} x =& \frac{\det(A_1)}{\det(A)} = \frac{10\begin{vmatrix} -4 & 3 \\ -1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 14 & 3 \\ 12 & -2 \end{vmatrix} + 1\begin{vmatrix} 14 & -4 \\ 12 & -1 \end{vmatrix}}{2\begin{vmatrix} -4 & 3 \\ -1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} -1 & -4 \\ 5 & -1 \end{vmatrix}} \\ =& \frac{10 \times 11 + 3 \times (-64) + 1 \times 34}{2 \times 11 + 3 \times (-13) + 1 \times 21} = -\frac{48}{4} = -12. \end{aligned}$$

$$\begin{aligned} y =& \frac{\det(A_2)}{\det(A)} = \frac{2\begin{vmatrix} 14 & 3 \\ 12 & -2 \end{vmatrix} - (10)\begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} -1 & 14 \\ 5 & 12 \end{vmatrix}}{4} \\ =& \frac{2 \times (-64) - 10 \times (-13) + 1 \times (-82)}{4} = -\frac{80}{4} = -20. \end{aligned}$$

$$\begin{aligned} z =& \frac{\det(A_3)}{\det(A)} = \frac{2\begin{vmatrix} -4 & 14 \\ -1 & 12 \end{vmatrix} - (-3)\begin{vmatrix} -1 & 14 \\ 5 & 12 \end{vmatrix} + 10\begin{vmatrix} -1 & -4 \\ 5 & -1 \end{vmatrix}}{4} \\ =& \frac{2 \times (-34) + 3 \times (-82) + 10 \times (21)}{4} = -\frac{104}{4} = -26. \end{aligned}$$

Therefore, $(x, y, z) = (-12, -20, -26)$ *is a unique solution to this system.*

4.2.2 Solving systems using the method of the inverse matrix

Let (S) be a square system, with the matrix interpretation: $AX = B$.
If the matrix A is invertible, we can solve this system using the inverse matrix method as follows:

$$AX = B \Longleftrightarrow X = A^{-1}B$$

Example 4.2.3. *Using the method of the inverse matrix, solve the following system:*

$$\begin{cases} x + y + z & = 5 \\ -x + 3y + 2z & = 2 \\ 2x + 2y + z & = 1 \end{cases}$$

We have

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad \textit{and} \quad B = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$\det A = -4$, *then A is invertible. Let's calculate A^{-1}*

$$Cof(A) = \begin{pmatrix} +\det\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} & -\det\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & +\det\begin{vmatrix} -1 & 3 \\ 2 & 2 \end{vmatrix} \\ \\ -\det\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & +\det\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\det\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \\ \\ +\det\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & -\det\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} & +\det\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} \end{pmatrix}$$

$$Cof(A) = \begin{pmatrix} -1 & 5 & -8 \\ 1 & -1 & 0 \\ -1 & -3 & 4 \end{pmatrix}, \quad \Longleftrightarrow \quad (Cof(A))^t = \begin{pmatrix} -1 & 1 & -1 \\ 5 & -1 & -3 \\ -8 & 0 & 4 \end{pmatrix}.$$

Thus,

$$A^{-1} = \frac{1}{\det A}\,(Cof(A))^t = -\frac{1}{4}\begin{pmatrix} -1 & 1 & -1 \\ 5 & -1 & -3 \\ -8 & 0 & 4 \end{pmatrix}.$$

Therefore,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = -\frac{1}{4}\begin{pmatrix} -1 & 1 & -1 \\ 5 & -1 & -3 \\ -8 & 0 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{4}\begin{pmatrix} -1\times 5 + 1\times 2 + (-1)\times 1 \\ 5\times 5 + (-1)\times 2 + (-3)\times 1 \\ -8\times 5 + 0\times 2 + 4\times 1 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -4 \\ 20 \\ -36 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 9 \end{pmatrix}$$

So, $(x, y, z) = (1, -5, 9)$ *is a unique solution to this system.*

4.2.3 Solving systems using Gauss method

Elementary operations

Definition 4.2.1. *Let* (S) *be a linear system of* n *equations,* p *unknowns, with coefficients in* $\mathbb{R}$. *Let* $E_1, E_2, \cdots, E_n$ *be the equations of* (S).

An elementary operation on the rows of (S) *is defined as one of the following operations:*

- *Multiply equation* E_i *by a non-zero scalar* α.
  This operation is denoted as: $E_i \longrightarrow \alpha E_i$.
- *Add to one of the equations* E_i *a multiple of another equation* E_j.
  This operation is denoted as: $E_i \longrightarrow E_i + \beta E_j$.
- *Exchanging two equations* E_i *and* E_j. *This operation is denoted as:* $E_i \leftrightarrows E_j$.

Proposition 4.2.1. *An elementary row operation on the lines of* (S) *transforms the system* (S) *into an equivalent system* (S')*, meaning it has exactly the same solutions as* (S).

Gaussian method

By a series of elementary operations, we transform the system (S) into an equivalent system (S') with an upper triangular matrix.

Example 4.2.4.

$$\begin{cases} x+2y+3z+4t=11 \\ 2x+3y+4z+t=12 \\ 3x+4y+z+2t=13 \\ 4x+y+2z+3t=14 \end{cases}$$

Let's solve this system

$$\begin{cases} x+2y+3z+4t=11 & \\ 2x+3y+4z+t=12 & E_2 \longrightarrow E_2-2E_1 \\ 3x+4y+z+2t=13 & E_3 \longrightarrow E_3-3E_1 \\ 4x+y+2z+3t=14 & E_4 \longrightarrow E_4-4E_1 \end{cases}$$

$$\begin{cases} x+2y+3z+4t & =11 & \\ -y-2z-7t & =-10 & \\ -2y-8z-10t & =-20 & E_3 \longrightarrow E_3-2E_2 \\ -7y-10z-13t & =-30 & E_4 \longrightarrow E_4-7E_2 \end{cases}$$

$$\begin{cases} x+2y+3z+4t & =11 & \\ -y-2z-7t & =-10 & \\ -4z+4t & =0 & \\ 4z+36t & =40 & E_4 \longrightarrow E_4+E_3 \end{cases}$$

$$\begin{cases} x+2y+3z+4t & =11 \\ -y-2z-7t & =-10 \\ -4z+4t & =0 \\ 40t & =40 \end{cases}$$

$$\begin{cases} t=1 \\ z=t=1 \\ y=-2z-7t+10=1 \\ x=11-2y+3z+4t=2 \end{cases}$$

Therefore, the system has a unique solution $(2,1,1,1)$.

Example 4.2.5.

$$\begin{cases} x+3y+5z-2t-7u & =3 \\ 3x+y+z-2t-u & =1 \\ 2x-y-3z+7t+5u & =2 \\ 3x-2y-5z+7t+8u & =2 \end{cases}$$

Let's solve this system

$$\begin{cases} x+3y+5z-2t-7u & =3 & \\ 3x+y+z-2t-u & =1 & E_2 \longrightarrow E_2-3E_1 \\ 2x-y-3z+7t+5u & =2 & E_3 \longrightarrow E_3-2E_1 \\ 3x-2y-5z+7t+8u & =2 & E_4 \longrightarrow E_4-3E_1 \end{cases}$$

$$\begin{cases} x+3y+5z-2t-7u & =3 & \\ -8y-14z+4t+20u & =-8 & \\ -7y-13z+11t+19u & =-4 & E_3 \longrightarrow 8E_3-7E_2 \\ -11y-20z+13t+29u & =-7 & E_4 \longrightarrow 8E_4-11E_2 \end{cases}$$

$$\begin{cases} x+3y+5z-2t-7u & =3 & \\ -8y-14z+4t+20u & =-8 & \\ -6z+60t+12u & =24 & \\ -6z+60t+12u & =24 & E_4 \longrightarrow E_4-E_3 \end{cases}$$

$$\begin{cases} x+3y+5z-2t-7u & =3 \\ -8y-14z+4t+20u & =-8 \\ -6z+60t+12u & =24 \\ 0 & =8 \end{cases}$$

then this system has no solution.

Example : Solving a Linear System by Three Methods: Cramer's Rule, Inverse Matrix, and Gaussian Elimination

Solve the system:

$$\begin{cases} x + 2y + z = 7 \\ 2x + 3y + 2z = 14 \\ x + y + 2z = 9 \end{cases}$$

Method 1: Cramer's Rule Step 1: Write the system in matrix form

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ 14 \\ 9 \end{pmatrix}$$

Step 2: Calculate $\det(A)$

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 1 \times (3 \times 2 - 2 \times 1) - 2 \times (2 \times 2 - 2 \times 1) + 1 \times (2 \times 1 - 3 \times 1)$$

$$= 1 \times (6 - 2) - 2 \times (4 - 2) + 1 \times (2 - 3) = 4 - 4 - 1 = -1 \neq 0$$

Since $\det(A) \neq 0$, the system has a unique solution.
Step 3: Calculate determinants of A_1, A_2, A_3 Replace the respective columns of A with the vector B:

$$A_1 = \begin{pmatrix} 7 & 2 & 1 \\ 14 & 3 & 2 \\ 9 & 1 & 2 \end{pmatrix}$$

$$\det(A_1) = 7 \times (3 \times 2 - 2 \times 1) - 2 \times (14 \times 2 - 2 \times 9) + 1 \times (14 \times 1 - 3 \times 9)$$

$$= 7 \times (6 - 2) - 2 \times (28 - 18) + 1 \times (14 - 27) = 28 - 20 - 13 = -5$$

$$A_2 = \begin{pmatrix} 1 & 7 & 1 \\ 2 & 14 & 2 \\ 1 & 9 & 2 \end{pmatrix}$$

$$\det(A_2) = 1 \times (14 \times 2 - 2 \times 9) - 7 \times (2 \times 2 - 2 \times 1) + 1 \times (2 \times 9 - 14 \times 1)$$

$$= 1 \times (28 - 18) - 7 \times (4 - 2) + 1 \times (18 - 14) = 10 - 14 + 4 = 0$$

$$A_3 = \begin{pmatrix} 1 & 2 & 7 \\ 2 & 3 & 14 \\ 1 & 1 & 9 \end{pmatrix}$$

$$\det(A_3) = 1 \times (3 \times 9 - 14 \times 1) - 2 \times (2 \times 9 - 14 \times 1) + 7 \times (2 \times 1 - 3 \times 1)$$

$$= 1 \times (27 - 14) - 2 \times (18 - 14) + 7 \times (2 - 3) = 13 - 8 - 7 = -2$$

Step 4: Calculate the solution

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-5}{-1} = 5, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{0}{-1} = 0, \quad z = \frac{\det(A_3)}{\det(A)} = \frac{-2}{-1} = 2$$

Method 2: Using the Inverse Matrix Step 1: Calculate the cofactor matrix C

$$C = \begin{pmatrix} \det\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} & -\det\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} & \det\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ -\det\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & \det\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\det\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ \det\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\det\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & \det\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (3 \times 2 - 1 \times 2) & -(2 \times 2 - 1 \times 2) & (2 \times 1 - 3 \times 1) \\ -(2 \times 2 - 1 \times 1) & (1 \times 2 - 1 \times 1) & -(1 \times 1 - 1 \times 2) \\ (2 \times 2 - 3 \times 1) & -(1 \times 2 - 2 \times 1) & (1 \times 3 - 2 \times 2) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & -1 \\ -3 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Step 2: Calculate transpose of cofactor

$$C^T = \begin{pmatrix} 4 & -3 & 1 \\ -2 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$$

Step 3: Calculate the inverse matrix

$$A^{-1} = \frac{1}{\det(A)}\,{}^{t}(A) = -1 \times \begin{pmatrix} 4 & -3 & 1 \\ -2 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 3 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Step 4: Calculate $X = A^{-1}B$

$$X = \begin{pmatrix} -4 & 3 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 14 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \times 7 + 3 \times 14 - 1 \times 9 \\ 2 \times 7 - 1 \times 14 + 0 \times 9 \\ 1 \times 7 - 1 \times 14 + 1 \times 9 \end{pmatrix} = \begin{pmatrix} -28 + 42 - 9 \\ 14 - 14 + 0 \\ 7 - 14 + 9 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$$

Method 3: Gaussian Elimination Start with the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 2 & 3 & 2 & 14 \\ 1 & 1 & 2 & 9 \end{array}\right]$$

Step 1: Make zeros below the pivot in the first column

$$R_2 \leftarrow R_2 - 2R_1 : \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 2 & 9 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_1 : \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

Step 2: Make zero below pivot in the second column

$$R_3 \leftarrow R_3 - R_2 : \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Step 3: Back substitution

From the third row:

$$z = 2$$

From the second row:

$$-y = 0 \implies y = 0$$

From the first row:

$$x + 2(0) + 2 = 7 \implies x = 5$$

Final solution is :

$$\boxed{x = 5, \quad y = 0, \quad z = 2}$$

Chapter 5

Functions of several variables

Introduction

In this chapter, we introduce the fundamental concepts of multivariable calculus. The aim is to extend the notions of limit, differentiability, and integrability well known in the context of single variable functions to functions of several variables. Our approach will emphasize the rigorous mathematical formulation of these ideas.

5.1 Limits, Continuity, and Partial Derivatives of a Function

Dot Product, Euclidean Norm, and Distance in $\mathbb{R}^n$

Definition 5.1.1 (Dot Product). *Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$. The dot product of X and Y is defined as:*

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Definition 5.1.2 (Euclidean Norm and Distance). *The Euclidean norm (or length) of a vector X is given by:*

$$\|X\| = \sqrt{\langle X, X \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The distance between two vectors X and Y is defined by:

$$d(X, Y) = \|X - Y\|.$$

Theorem 5.1.1 (Properties of the Norm). *For any vector $X \in \mathbb{R}^n$ and any scalar $\lambda \in \mathbb{R}$:*

1. *$\|X\| = 0$ if and only if $X = 0$.*
2. *$\|X\| > 0$ if and only if $X \neq 0$.*
3. *$\|\lambda X\| = |\lambda| \cdot \|X\|$.*
4. *$\|X + Y\| \leq \|X\| + \|Y\|$ (Triangle Inequality).*

Common Norms on $\mathbb{R}^n$ Let $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. The following are commonly used norms:

1. **infinity norm** $\|X\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$.
2. **1-norm** $\|X\|_1 = \sum_{i=1}^n |x_i|$.
3. **Euclidean norm (or 2-norm** $\|X\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.

Functions of several variables

Definition 5.1.3 (Function of Several Variables)**.** *A function f defined on a subset $D \subseteq \mathbb{R}^n$ with values in $\mathbb{R}$ is called a function of n variables. It is written as:*

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}, \quad (x_1, x_2, \dots, x_n) \mapsto f(x_1, x_2, \dots, x_n) = z,$$

where D is the domain of f.

The set $\{f(x) \mid x \in D\} \subseteq \mathbb{R}$ is called the ***image*** *of f, and the set $\{(x, f(x)) \mid x \in D\} \subseteq \mathbb{R}^n \times \mathbb{R}$ is called the* ***graph*** *of f.*

Example 5.1.1. 1. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:*

$$f(x, y) = \frac{x^2 - xy + 2}{1 + x^2 + y^2}.$$

This function is defined for all $(x, y) \in \mathbb{R}^2$.

2. *Let $g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined by: $g(x, y) = \sqrt{1 - x^2 - y^2}$. The domain of g is the closed disk:*

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

Definition 5.1.4 (Partial Functions)**.** *Let f be a function defined on $D \subseteq \mathbb{R}^2$ and $A = (a_1, a_2) \in D$. The partial functions of f at point A are defined as:*

$$x_1 \mapsto f(x_1, a_2) \quad \textit{and} \quad x_2 \mapsto f(a_1, x_2),$$

on open intervals containing a_1 and a_2, respectively.

5.1.1 Limit of a Function of Two Variables

Let $f(x, y)$ be a function defined in a neighborhood of the point $M_0 = (a, b)$. We define the limit as follows:

Definition 5.1.5 (Limit). *Let $f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $M_0 = (x_0, y_0) \in D_f$. We say that f has a limit L at M_0 if $f(x, y)$ becomes arbitrarily close to L whenever (x, y) is sufficiently close to (x_0, y_0). This is written as:*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \textit{or} \quad \lim_{M\to M_0} f(x, y) = L,$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|f(x, y) - L| < \varepsilon \quad \textit{whenever} \quad \|(x, y) - (x_0, y_0)\| < \delta.$$

Remark 5.1.1. *The properties of limits (sum, product, quotient, composition) for functions of several variables are similar to those for single-variable functions.*

5.1.2 Continuity

Definition 5.1.6 (Continuity). *Let $f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $M_0 = (x_0, y_0) \in D_f$. The function f is said to be continuous at M_0 if:*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

The function f is continuous on D_f if it is continuous at every point of D_f.

Example 5.1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by: $f(x, y) = x^2 + y^2$. Then f is continuous on $\mathbb{R}^2$ since:*

$$|f(x, y) - f(x_0, y_0)| = |x^2 + y^2 - x_0^2 - y_0^2| \leq |x^2 - x_0^2| + |y^2 - y_0^2|,$$

and both terms tend to 0 as $(x, y) \to (x_0, y_0)$.

Example 5.1.3. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by:*

$$f(x, y, z) = \frac{(1 + x^2 + z^2) \cdot \sin(y)}{x^2 + xz + y}.$$

To study the limit as $(x, y, z) \to (0, 0, 0)$, observe:

$$f(x, y, z) \approx \frac{\sin y}{y} \quad \textit{as } x, z \to 0,$$

so:

$$\lim_{(x,y,z)\to(0,0,0)} f(x, y, z) = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

Hence, the function has a limit and is continuous at the origin.

Operations on Functions Let f and g be two functions continuous at a point $M_0(x_0, y_0) \in \mathbb{R}^2$, and let $\lambda \in \mathbb{R}$. Then the following functions are also continuous at M_0:

$$f + g, \quad f - g, \quad \lambda f, \quad \text{and } \frac{f}{g} \text{ (if } g(x_0, y_0) \neq 0).$$

Moreover, the composition of continuous functions is continuous.

5.1.3 Partial Derivatives

Definition 5.1.7. *Let*

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto z = f(x, y),$$

be a function of two variables defined on a domain D_f*, and let* $M_0(x_0, y_0) \in D_f$.

If the partial function $f_x : x \mapsto f(x, y_0)$ *is defined in a neighborhood of* x_0 *and differentiable at* x_0*, then this derivative is called the* ***partial derivative*** *of* f *with respect to* x *at the point* (x_0, y_0)*. It is denoted by:*

$$\frac{\partial f}{\partial x}(x_0, y_0) = f'_x(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}.$$

Similarly, the partial derivative with respect to y *is:*

$$\frac{\partial f}{\partial y}(x_0, y_0) = f'_y(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.$$

If both partial derivatives exist at (x_0, y_0)*, then we say* f *is* ***partially differentiable*** *at that point.*

A function f *is said to be of class* C^1 *on* D_f *if* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are continuous on* D_f.

Definition 5.1.8 (Gradient)**.** *Let* $f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$*, and* $(x_0, y_0) \in D_f$*. The* ***gradient*** *of* f *at* (x_0, y_0) *is the vector:*

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right).$$

Example 5.1.4. *Let* $f(x, y) = x^3 + xy^2 + y^3$*. Then the partial derivatives are:*

$$\frac{\partial f}{\partial x}(x, y) = 3x^2 + y^2, \quad \frac{\partial f}{\partial y}(x, y) = 2xy + 3y^2.$$

At the point $(4, 5)$*:*

$$\frac{\partial f}{\partial x}(4, 5) = 3 \cdot 16 + 25 = 73, \quad \frac{\partial f}{\partial y}(4, 5) = 2 \cdot 4 \cdot 5 + 75 = 115.$$

$\nabla f(4, 5) = (73, 115)$

Example 5.1.5. *Let* $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ *be defined by:*

$$g(x,y) = \frac{x-y}{x^2+y^2}.$$

Then we compute:

$$\frac{\partial g}{\partial x}(x,y) = \frac{(1)(x^2+y^2)-(x-y)(2x)}{(x^2+y^2)^2},$$

$$\frac{\partial g}{\partial y}(x,y) = \frac{(-1)(x^2+y^2)-(x-y)(2y)}{(x^2+y^2)^2}.$$

Successive derivatives

Definition 5.1.9. *If the first-order partial derivatives of* f *are differentiable, we define the second-order partial derivatives as:*

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}.$$

These are computed by taking partial derivatives of the first-order derivatives.

A function f *is said to be of class* C^k *on* D_f *if all partial derivatives up to order* k *exist and are continuous on* D_f*.*

Theorem 5.1.2 (Schwarz's Theorem)**.** *If* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ *exist and are continuous in a neighborhood of* (x_0, y_0)*, then:*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0).$$

Example 5.1.6. *Let* $f(x,y) = x^4y^2$*. Then we compute:*
$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2y^2$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 2x^4$, $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 8x^3y$, $\frac{\partial^2 f}{\partial y \partial x}(x,y) = 8x^3y$.

5.1.4 Differentiability

Definition 5.1.10. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$*,* $(x,y) \mapsto f(x,y)$*. We say that* f *is* ***differentiable*** *at the point* $(a,b) \in \mathbb{R}^2$ *if there exist real numbers* λ *and* μ *such that:*

$$f(a+h_1, b+h_2) - f(a,b) = \lambda h_1 + \mu h_2 + \|(h_1,h_2)\| \varepsilon(h_1,h_2),$$

where

$$\lim_{\|(h_1,h_2)\| \to 0} \varepsilon(h_1,h_2) = 0.$$

Remark 5.1.2. *The differential of f is denoted as follows:*

$$df : (h_1, h_2) \mapsto \frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2.$$

We also write more simply:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

5.2 Double and Triple Integrals

In this section, we provide some elements related to the calculation of double and triple integrals.

5.2.1 Double Integrals

Theorem 5.2.1 (Fubini's Theorem)**.** *Let h and k be two continuous functions on the interval $[a, b]$ such that for all $x \in [a, b]$, we have $h(x) \leq k(x)$. Define the domain:*

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \text{ and } h(x) \leq y \leq k(x)\}.$$

If $f : \Omega \to \mathbb{R}$ is a continuous function, then f is integrable over Ω and we have:

$$\iint_\Omega f(x, y)\, dxdy = \int_a^b \left(\int_{h(x)}^{k(x)} f(x, y)\, dy \right) dx.$$

Remark 5.2.1. *If $f(x, y) = 1$, then the double integral $\iint_\Omega dxdy$ gives the area of Ω.*

Example 5.2.1. *Let*

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto f(x, y) = x^2 + xy + y^2 + 2,$$

and let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2,\ 0 \leq y \leq 3\}$, a rectangle. First, integrate with respect to y:

$$\iint_\Omega f(x, y)\, dxdy = \int_1^2 \left(\int_0^3 (x^2 + xy + y^2 + 2)\, dy \right) dx = \int_1^2 \left[x^2 y + \frac{1}{2} x y^2 + \frac{1}{3} y^3 + 2y \right]_0^3 dx.$$

$$= \int_1^2 (3x^2 + \frac{9}{2} x + 27 + 6)\, dx = \int_1^2 \left(3x^2 + \frac{9}{2} x + 33 \right) dx.$$

$$= \left[x^3 + \frac{9}{4} x^2 + 33x \right]_1^2 = \frac{115}{4}.$$

Alternatively, integrate first with respect to x:

$$\iint_\Omega f(x, y)\, dxdy = \int_0^3 \left(\int_1^2 (x^2 + xy + y^2 + 2)\, dx \right) dy = \frac{115}{4}.$$

Change of Variables

Proposition 5.2.1. *Let*

$$f : \Omega \subset V \to \mathbb{R}, \quad (x, y) \mapsto f(x, y),$$

be a continuous function on a compact $\Omega \subset V$*, with* $V \subset \mathbb{R}^2$ *open. Let* $D \subset U \subset \mathbb{R}^2$ *be a compact subset, and let* $\varphi : U \to V$ *be a* $\mathcal{C}^1$ *bijection such that* $\varphi(D) = \Omega$*. Define the change of variables by:*

$$(u, v) \mapsto \varphi(u, v) = (x(u, v), y(u, v)).$$

Let $J(u, v) = \det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)$ *be the Jacobian. Then:*

$$\iint_\Omega f(x, y)\, dxdy = \iint_D f(x(u, v), y(u, v))\, |J(u, v)|\, dudv.$$

Change to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \iint_\Omega f(x, y)\, dxdy = \iint_D f(r\cos\theta, r\sin\theta)\, r\, drd\theta.$$

5.2.2 Triple Integrals

Theorem 5.2.2 (Fubini's Theorem — Triple Integral)**.** *Let* $D \subset \mathbb{R}^2$ *be compact, and* $h, k : D \to \mathbb{R}$ *continuous functions. Define*

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D,\ h(x, y) \le z \le k(x, y)\},$$

and let

$$f : \Omega \to \mathbb{R}, \quad (x, y, z) \mapsto f(x, y, z).$$

Then:

$$\iiint_\Omega f(x, y, z)\, dxdydz = \iint_D \left(\int_{h(x,y)}^{k(x,y)} f(x, y, z)\, dz\right) dxdy.$$

Remark 5.2.2. *In the special case:*

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ n \le z \le m\},$$

then:

$$\iiint_\Omega f(x, y, z)\, dxdydz = \int_n^m \left(\int_c^d \left(\int_a^b f(x, y, z)\, dx\right) dy\right) dz.$$

Example 5.2.2. *Let*

$$f:\mathbb{R}^3\to\mathbb{R},\quad (x,y,z)\mapsto f(x,y,z)=x^2+yz,$$

and let

$$\Omega=\{(x,y,z)\in\mathbb{R}^3 \mid x\geq 0, y\geq 0, x+y+2z\leq 1\}.$$

Then:

$$\iiint_{\Omega} f(x,y,z)\,dxdydz=\int_0^{1/2}\int_0^{1-2z}\int_0^{1-2z-y}(x^2+yz)\,dxdydz.$$

$$=\int_0^{1/2}\int_0^{1-2z}\left[\frac{1}{3}x^3+xyz\right]_0^{1-2z-y}dydz=\int_0^{1/2}\int_0^{1-2z}\left(\frac{(1-2z-y)^3}{3}+yz(1-2z-y)\right)dydz.$$

After full calculation, we find:

$$\iiint_{\Omega} f(x,y,z)\,dxdydz=\frac{1}{96}.$$

Change of Variables

Proposition 5.2.2. *Let*

$$f:\Omega\subset V\to\mathbb{R},\quad (x,y,z)\mapsto f(x,y,z),$$

be a continuous function on a compact $\Omega\subset V\subset\mathbb{R}^3$. Let $D\subset U\subset\mathbb{R}^3$ be a compact, and let $\varphi:U\to V$ be a bijective $\mathcal{C}^1$ function such that $\varphi(D)=\Omega$. Define:

$$(u,v,w)\mapsto\varphi(u,v,w)=(x(u,v,w),y(u,v,w),z(u,v,w)).$$

Let $J(u,v,w)=\det\left(\frac{\partial(x,y,z)}{\partial(u,v,w)}\right)$ be the Jacobian. Then:

$$\iiint_{\Omega} f(x,y,z)\,dxdydz=\iiint_{D} f(x(u,v,w),y(u,v,w),z(u,v,w))\,|J(u,v,w)|\,dudvdw.$$

Change to cylindrical coordinates:

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z,$$

$$\Rightarrow\iiint_{\Omega} f(x,y,z)\,dxdydz=\iiint_{D} f(r\cos\theta,r\sin\theta,z)\,r\,drd\theta dz.$$

Change to spherical coordinates:

$$x=\rho\cos\theta\sin\phi,\quad y=\rho\sin\theta\sin\phi,\quad z=\rho\cos\phi,$$

$$\Rightarrow\iiint_{\Omega} f(x,y,z)\,dxdydz=\iiint_{D} g(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho d\theta d\phi,$$

where $g(\rho,\theta,\phi)=f(x,y,z)$.

Exercises

Exercise 01 :

I) By using a change of variable, compute the integral : $\int \frac{dx}{x(1+ln(x))}$

II) Solve the following differential equation:

$$y' - x^2 y = \sqrt{x+1}e^{\frac{x^3}{3}}$$

III) We consider the differential equation : $y'' + y' - 6y = e^{5x} + (6x^2 + 4x - 3) \qquad (E)$

a) Solve the homogeneous equation associated with (E) : $y'' + y' - 6y = 0$

b) Determine a particular solution y_{p1} of the equation : $y'' + y' - 6y = e^{5x}$

c) Find a particular solution y_{p2} of the equation : $y'' + y' - 6y = (6x^2 + 4x - 3)$

d) Deduce the general solution of the equation (E).

Exercise 02 : Let α be a real parameter. We consider the following system (S) :

$$(S)\begin{cases} x_1 + x_2 + \alpha x_3 = \alpha \\ x_1 + \alpha x_2 - x_3 = 1 \\ x_1 + x_2 - x_3 = 1 \end{cases}$$

1. Write the system (S) in matrix form ($AX = B$).
2. Calculate the determinant of A as a function of α and for which values of α the matrix A is invertible.
3. **For** $\alpha = -2$
   Calculate the inverse matrix A^{-1} , And by using A^{-1} solve the system S.

Exercise 03 :

We consider the integrals

$$I = \int (2x+1)cos^2(x)dx \ , \quad J = \int (2x+1)sin^2(x)dx$$

a) Calculate the integral: $\int(2x+1)cos(2x)dx$.

b) Calculate $I+J$ and $I-J$.

c) Deduce the values of the integrals: I and J .

NB. $cos^2(x) = \dfrac{1+cos(2x)}{2}$ and $sin^2(x) = \dfrac{1-cos(2x)}{2}$

Exercise 04 :

I) Solve the following differential equation: $y' + \dfrac{y}{tan(x)} = 2cos(x)$.

II) We consider the following differential equation:

$$y'' + \frac{2}{tan(x)}y' - y = 0 \qquad (E1)$$

Show that the change of function: $Z = y' + \dfrac{1}{tan(x)}y$ transforms the equation (E1) into the differential equation $Z' + \dfrac{Z}{tan(x)} = 0$

III) Consider the following second-order differential equation: $y'' + y' = 4xe^x$
   1) Determine the homogeneous and particular solutions of this equation
   2) Find the general solution for $y(0) = 5$ and $y'(0) = 5$

Exercise 05 : Let the matrices A and B be defined by: $A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & 0 \\ -3 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$

1. Show that $A^2 = 2A$ and deduce that $A^3 = 4A$.
2. Deduce A^n for all $n \in \mathbb{N}$
3. The matrices A is invertible ? .
4. Calculate the matrix $C = B^2 - 3B + 2I_3$. (I_3 is the identity matrix)
5. Deduce B^{-1}

Exercise 06 :

I)Calculate the integral:

$$I = \int_0^1 xe^{x^2}\,dx.$$

II) Solve the following differential equation:

$$y'' - 3y' + 2y = e^{2x}.$$

III) Let the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix},$$

1) calculate the matrix product AB.

2) By using matrix methods Solve the linear system:

$$\begin{cases} x + 2y = 5 \\ 3x + 4y = 11 \end{cases}$$

Exercise 07 :

1) Evaluate the integral:

$$J = \int_0^{\pi} \sin^2(x)\,dx.$$

2) Find the general solution of the differential equation:

$$y' + y\tan x = \sin(2x).$$

3) Consider the matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

a)Calculate the determinant of C and determine if C is invertible.

b)Solve the system $C\mathbf{x} = \mathbf{b}$ where

$$\mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$$

using the inverse matrix method.

Exercise 8 :

I)Calculate the integral:

$$I = \int_0^2 (3x^2 + 2x)e^{x^3+x^2}\,dx.$$

II) Solve the first-order differential equation:

$$(2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0.$$

III) Given the matrices:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix},$$

a) Calculate the determinant of A.

b) Determine if A is invertible.

c) If invertible, find A^{-1}.

d) Solve the linear system $A\mathbf{x} = \mathbf{b}$ using A^{-1}.

Exercise 9 :

1)

a) Calculate the integral:

$$I_1 = \int_0^1 x\cos(x^2)\,dx.$$

b) Calculate the integral:

$$I_2 = \int_1^e \frac{\ln(x)}{x}\,dx.$$

2)

a) Solve the first-order differential equation:

$$y' + y\tan x = \sin(2x).$$

b) Solve the second-order differential equation:

$$y'' - 4y' + 4y = e^{2x}.$$

Exercise 10 :

I)

a) Calculate the integral:

$$I_1 = \int_0^2 xe^{x^2}\,dx.$$

b) Calculate the integral:

$$I_2 = \int_1^4 \frac{\sqrt{t}}{t+1}\, dt.$$

II)

a) Solve the first-order differential equation:

$$y' + y \cot x = \sin x.$$

b) Solve the second-order differential equation:

$$y'' + y = \cos x.$$

III) Solve the system:

$$\begin{cases} x + 2y - z = 3, \\ 2x - y + 3z = 1, \\ 3x + y + 2z = 4. \end{cases}$$

Solve it by:

- i) Cramer's rule,
- ii) inverse matrix method,
- iii) Gaussian elimination.

Exercise 11 :

1)

a) Calculate the integral:

$$J_1 = \int_0^{\pi/2} \sin^3 x\, dx.$$

b) Calculate the integral:

$$J_2 = \int_1^e \frac{\ln x}{x^2}\, dx.$$

2)

a) Solve the first-order differential equation:

$$(x + y)dx + (x - y)dy = 0.$$

b) Solve the second-order differential equation:

$$y'' - 2y' + y = e^x.$$

3) Let the system:

$$\begin{cases} 2x - y + z = 4, \\ -x + 3y - 2z = -6, \\ 3x + y + 4z = 5. \end{cases}$$

Solve it by:

i) Cramer's rule,

ii) inverse matrix method,

iii) Gaussian elimination.

Exercise 12 :

Let the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x + y} & \text{if } x + y \neq 0, \\ 0 & \text{if } x + y = 0. \end{cases}$$

1. Compute the limit of $f(x,y)$ as $(x,y) \to (0,0)$. Does the limit exist?
2. Is f continuous at the origin?
3. Compute the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points where $x + y \neq 0$.
4. Determine if the function is differentiable at the origin.
5. Compute the double integral

   $$\iint_R f(x,y)\, dx\, dy$$

   where $R = [0,1] \times [0,1]$.

Exercise 13 :

Let the function

$$f(x,y,z) = \begin{cases} \dfrac{x^2 - y^2}{x + y} + z^2 & \text{if } x + y \neq 0, \\ z^2 & \text{if } x + y = 0. \end{cases}$$

1. Compute the limit of $f(x, y, z)$ as $(x, y, z) \to (0, 0, 0)$. Is f continuous at the origin?
2. Compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ where defined.
3. Compute the gradient vector $\nabla f(x, y, z)$ at the point $(1, 1, 1)$.
4. Evaluate the triple integral

$$\iiint_{\Omega} f(x, y, z)\, dx\, dy\, dz$$

where $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$.

Conclusion

I hope that this material will help first-year students in assimilating mathematics, and more particularly Analysis and Algebra II, which constitute the foundation of mathematics at the university.

Finally, errors may be found. Please communicate them to me by email at the following addresses: `rachid.lakehal93@gmail.com` 'or' `rachid.lakehal@univ-bejaia.dz`.

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