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Handout of the mathematics course

Functions and Integrals

with corrected exercises

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-Bejaia University 2024-

Contents

Preamble

This handout is a course support and corrected exercises on functions and integrals, a support with many examples which allows students to become familiar with mathematics and to go beyond the stage of "mathematics is difficult" and to let them know that mathematics is simple and accessible to all, whatever their level.

The document is structured in two parts, the first part constitutes a reminder on numerical functions with one real variable (domains of definition, limits, continuity, derivability, etc.) and the second covers the calculation of integrals (definite and indefinite integrals, improper integrals, integration methods, etc.).

Chapter 1

Numerical functions of a real variable

1 General Overview

1.1 Basic concepts and notations

1. $\mathbb{R}$ is the set of real numbers.

2. $\mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$ is the set of non-zero real numbers.

3. $\mathbb{R}_+ = [0, +\infty[$ is the set of positive real numbers.

4. $\mathbb{R}_+^* =]0, +\infty[$ is the set of non-zero positive real numbers.

5. $\mathbb{R}_- =]-\infty, 0]$ is the set of negative real numbers.

6. $\mathbb{R}_-^* =]-\infty, 0[$ is the set of non-zero negative real numbers.

Definition 1.1. A numerical function of a real variable is a defined application of a set E of $\mathbb{R}$ with values ??in $\mathbb{R}$, to any antecedent x (an element of its starting set E) we associate at most a single image, denoted $f(x)$, in its arrival set. We note

$$f \ : \ E \to \mathbb{R}$$
$$x \mapsto f(x).$$

The graph of f is the set denoted G_f formed by the points $(x, f(x))$ in the plane, defined by

$$G_f = \{(x, f(x)), x \in E\}.$$

Identity function :

The identity function is a function which returns the same value.

$$I_D \ : \ E \to E$$
$$x \mapsto I_D(x) = x.$$

Bounded function :

The function f, defined from E to $\mathbb{R}$ is said to be bounded if

$$\exists \ a, A \in \mathbb{R} : a \leq f(x) \leq A, \forall x \in E.$$

Example 1.1. The functions $\cos(x)$ and $\sin(x)$ are bounded in $\mathbb{R}$, $\forall x \in \mathbb{R}$ we have

$$-1 \leq \cos(x) \leq 1 \text{ and } -1 \leq \sin(x) \leq 1.$$

1.2 Domain of a function

A numerical function is not necessarily defined for all real numbers.

Definition 1.2. The set of real numbers for which the function f is defined is called the definition set or the domain of the function f and is denoted D_f.

Remark 1.1. When the domain is not indicated, it is sufficient to examine the expression of the function to determine the conditions of existence of $f(x)$:

- Is the function defined by a polynomial? (this is defined on $\mathbb{R}$);
- Is there a denominator? (this must be non-zero);
- Is there a square root (the interior of the root must be positive or zero);
- Is there a particular function not defined on $\mathbb{R}$, for example the function ln ($ln(x)$ is defined on $\mathbb{R}_+^*$).

Example 1.2. Give the domains of the functions defined by:

$$f(x) = \tfrac{x-1}{x+1}, \ g(x) = \sqrt{x-6} \ \text{ and } \ h(x) = x + 3 + \tfrac{2}{x}.$$

Answers :
The areas of definition or the domains are as follows:
1. $D_f =]-\infty, -1[\cup]-1, +\infty[$.
   f is defined if and only if the denominator of $f(x)$ is non-zero, i.e. $x + 1 \neq 0$, which

means $x \neq -1$. Then the domain will be $\mathbb{R}/\{-1\}$.
2. $D_g = [6, +\infty[$.

Indeed, g is defined if and only if $x - 6 \geq 0$ (the interior of the root must be positive or zero), which is true for $x \geq 6$.
3. $D_h = \mathbb{R}^*$.

h is defined by the sum of a polynomial ($x + 3$ defined on $D_{h_1} = \mathbb{R}$) and a quotient ($\frac{2}{x}$ defined on $D_{h_2} = \mathbb{R}^*$). The domain of definition of h is

$$D_h = D_{h_1} \cap D_{h_2} = \mathbb{R} \cap \mathbb{R}^* = \mathbb{R}^*.$$

1.3 Parity function (even and odd functions)

Let f be a function defined from a set E of $\mathbb{R}$ to $\mathbb{R}$, where E is symmetric about the origin. ($\forall x \in E$ we have $-x \in E$).

1. f is said to be even on E if and only if $f(-x) = f(x)$, $\forall x \in E$.
2. f is said to be odd on E if and only if $f(-x) = -f(x)$, $\forall x \in E$.

Remark 1.2. If the graph of the function is symmetric about the ordinate axis (the y axis) the function is even and if the graph is symmetric about the origin $O(0,0)$ the function is odd.

Example 1.3. Let f, g and h three functions defined by:

$$\begin{aligned} f &: [-1,1] \to \mathbb{R} \\ & \quad x \mapsto f(x) = x^2. \\ g &: [0,1] \to \mathbb{R} \\ & \quad x \mapsto g(x) = x^2. \\ h &: \mathbb{R} \to \mathbb{R} \\ & \quad x \mapsto h(x) = x^3. \end{aligned}$$

- f is an even function, because $\forall x \in [-1,1]$, we have $f(-x) = (-x)^2 = x^2 = f(x)$.
- The domain $[0,1]$ of the function g is not symmetrical about 0, then we cannot study the parity of this function, so g is neither even nor odd.
- $\forall x \in \mathbb{R}$ we have $h(-x) = (-x)^3 = -x^3 = -h(x)$, so h is an odd function.

1.4 Periodicity of functions

Let p be a non-zero real number. The function f is said to be periodic if and only if

$$f(x+p) = f(x), \ \forall x \in E.$$

The smallest positive value of p is called the period of f.

Example 1.4. The functions $f(x) = \cos(x)$ and $g(x) = \sin(x)$ are periodic with period 2π.

1. The function $f(x) = \cos(x)$ is an even periodic function with the period 2π. Indeed,
   • For any x in $\mathbb{R}$ and $k \in \mathbb{Z}$ we have $\cos(x + 2k\pi) = cos(x)$.
   $p = 2\pi$ is the period of the function $\cos(x)$ defined on $\mathbb{R}$ (the smallest non-zero positive value of $2k\pi$).

$$\begin{aligned} f(x+2\pi) &= \cos(x+2\pi) = \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi) \\ &= \cos(x) \times 1 - \sin(x) \times 0 \\ &= \cos(x) = f(x), \end{aligned}$$

   then f is periodic.
   • $\forall x \in \mathbb{R}$, $f(-x) = \cos(-x) = \cos(x) = f(x)$, so f is an even function on $\mathbb{R}$.

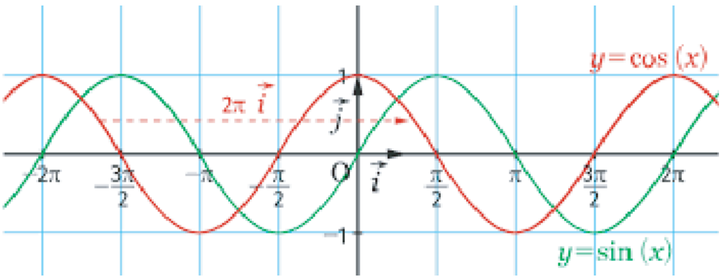

Figure 1.1: Functions curves $\cos(x)$ and $\sin(x)$.

2. The function $g(x) = \sin(x)$ is an odd function and periodic with the period 2π. In fact,
   • For any x in $\mathbb{R}$ we have

$$\begin{aligned} g(x+2\pi) &= \sin(x+2\pi) = \sin(x)\cos(2\pi) + \cos(x)\sin(2\pi) \\ &= \sin(x) \times 1 - \sin(x) \times 0 \\ &= \sin(x) = g(x), \end{aligned}$$

so g is periodic with period 2π.
• $\forall x \in \mathbb{R}$, $g(-x) = \sin(-x) = -\sin(x) = -g(x)$, then g is an odd function on $\mathbb{R}$.

1.5 Compound (composite) function

Definition 1.3. Let f: $E \to F$ and g: $F \to G$ two real-variable functions. The compound function of g and f is the function $gof : E \to F$ defined by $gof(x) = g(f(x))$.

Properties

1. $(fog)oh = fo(goh) = fogoh$.
2. In general, the fog function is different from gof $(fog(x) \neq gof(x))$.

Example 1.5. Let f and g be two real-variable functions defined by

$$\begin{array}{rl} f : \mathbb{R} \to \mathbb{R}_+ \quad , & g : \mathbb{R}_+ \to [-1, 1] \\ x \mapsto f(x) = x^2 & x \mapsto g(x) = \sin(x). \end{array}$$

The function gof is defined as

$$\begin{aligned} gof \quad : \quad & \mathbb{R} \to [-1, 1] \\ & x \mapsto gof(x) = g(f(x)) = g(x^2) = \sin(x^2). \end{aligned}$$

1.6 Bijection functions

Let f be a function defined from E to F $(f : E \to F)$.

1. **Injection:**
   The function f is said to be injective if and only if every element of F is the image of at most one element of E by f, i.e.:

   $$f \text{ is injective} \Leftrightarrow \forall x, x^{'} \in E,\ f(x) = f(x^{'}) \Rightarrow x = x^{'}.$$

   In other words,

   $$f \text{ is injective} \Leftrightarrow \forall x, x^{'} \in E,\ x \neq x^{'} \Rightarrow f(x) \neq f(x^{'}).$$

2. **Surjection:**
   f is said to be injective if and only if every element of F is the image of at least one element of E by f, that is:

   $$f \text{ is surjective} \Leftrightarrow \forall y \in F,\ \exists x \in E \text{ such that } y = f(x).$$

3. **Bijection:**
If f is both injective and surjective, then f is said to be bijective. The function f associates with every element of F a unique antecedent in E.

$$f \text{ is bijective} \Leftrightarrow \forall y \in F, \exists \,!\, x \in E \text{ such that } y = f(x).$$

($\exists$! means "there is a unique").

Example 1.6. A small change in the starting or ending point of the same function can affect the nature of that function. Take the case of $f(x) = x^2$, for example.

a. The function f_1 defined by

$$\begin{aligned} f_1 \quad : \quad & \mathbb{R} \to \mathbb{R}_+ \\ & x \mapsto f_1(x) = x^2. \end{aligned}$$

is not injective (for $x = -2 \neq x' = 2$ we have $f(-2) = f(2)$, i.e. $f(x) = f(x')$), but it is surjective.

b. The function f_2 defined below

$$\begin{aligned} f_2 \quad : \quad & \mathbb{R}_+ \to \mathbb{R} \\ & x \mapsto f_2(x) = x^2. \end{aligned}$$

is not surjective (for $y = -2$ we have no antecedent), but it is injective.

c. The function f_3 defined as follows

$$\begin{aligned} f_3 \quad : \quad & \mathbb{R}_+ \to \mathbb{R}_+ \\ & x \mapsto f_3(x) = x^2. \end{aligned}$$

is bijective, because every real $y \geq 0$ admits a unique antecedent $x = \sqrt{y}$ in $\mathbb{R}_+$ by f.

1.7 The inverse function (the reciprocal function)

Let f be a bijective function defined from a set E to a set F ($f : E \to F$) and $y = f(x)$ is the image of an antecedent x by f. Knowing that f is bijective then there exists a unique $x \in E$ such that $y = f(x)$, let us denote this unique antecedent by $x = f^{-1}(y)$.
The function denoted f^{-1}, defined by

$$\begin{aligned} f^{-1} \quad : \quad & F \to E \\ & y \mapsto x = f^{-1}(y) \end{aligned}$$

is called the reciprocal (or inverse) function of f, which is also bijective, and verifies:

$$\forall x \in E,\ f^{-1}of(x) = x;$$
$$\forall y \in F,\ fof^{-1}(y) = y.$$

$f^{-1}of = I_E$ (the identity function in E) and $fof^{-1} = I_F$ (the identity function in F). If $E = F$, we will have $f^{-1}of = fof^{-1} = I_E$.

Example 1.7. Let f be a function defined by

$$f \ : \ \mathbb{R}_+ \to \mathbb{R}_+$$
$$x \mapsto f(x) = x^2.$$

The function f is bijective and its reciprocal function is

$$f^{-1} \ : \ \mathbb{R}_+ \to \mathbb{R}_+$$
$$x \mapsto f^{-1}(x) = \sqrt{x}.$$

The following table is a summary of inverses of some known functions:

f	$ln(x) :]0, +\infty[$	$\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}]$	$\cos(x) : [0, \pi]$	$tg(x) :]\frac{\pi}{2}, \frac{\pi}{2}[$
f^{-1}	$e^x : \mathbb{R}$	$arcsin(x) : [-1, 1]$	$arccos(x) : [-1, 1]$	$arctg(x) : \mathbb{R}$

2 Limits

2.1 Limits at a point x_0

Let x_0 be a real number and f be a function defined in a domain D containing x_0 (except perhaps in x_0 itself).

Definition 1.4. We say that f admits the number l as limit at the point x_0 if:

$\forall \epsilon > 0$, $\exists \eta > 0$ (dependent on ϵ) such that for all $x \neq x_0$ verifying $\mid x - x_0 \mid < \eta$ we have
$$\mid f(x) - l \mid < \epsilon.$$

We note
$$\lim_{x \to x_0} f(x) = l \text{ or } \lim_{x \to x_0} f(x) \to l.$$

In simpler terms, $f(x)$ tends to l as x tends to x_0; in other words, as x approaches x_0, $f(x)$ approaches l.

Remark 1.3. When the limit of f at x_0 exists, it is unique.

Example 1.8. The function $f(x) = 3x + 2$ tends to 8 when x tends to 2.
By definition

$$\lim_{x \to 2} f(x) = 8 \Leftrightarrow \forall \epsilon > 0, \exists \eta > 0, \text{ such that } \forall x \neq 2 : \mid x - 2 \mid < \eta \Rightarrow \mid f(x) - 8 \mid < \epsilon,$$

This means that

$$\lim_{x \to 2} 3x + 2 = 8 \Leftrightarrow \forall \epsilon > 0, \exists \eta > 0 \text{ such that } \mid x - 2 \mid < \eta \Rightarrow \mid f(x) - 8 \mid < \epsilon.$$

We have

$$f(x) - 8 = 3x + 2 - 8 = 3x - 6 = 3(x - 2).$$

So for a ϵ (any positive real), we get

$$\begin{aligned} \mid f(x) - 8 \mid < \epsilon &\Leftrightarrow 3 \mid x - 2 \mid < \epsilon \\ &\Leftrightarrow \mid x - 2 \mid < \frac{\epsilon}{3}. \end{aligned}$$

So just take $\eta < \frac{\epsilon}{3}$.

1. **Left and Right-hand Limits at a point (sided limits)**

   (a) **Left-hand limit at x_0:**
   Let x_0 be a real number and f a function defined at least to the left of x_0. We say that f admits the number l_L as a limit to the left of x_0 if:

   $$\lim_{x \to x_0} f(x) = l_L \text{ with } x < x_0.$$

   In other words,

   $$\forall \epsilon > 0, \exists \eta > 0 \text{ such that for any } x \neq x_0 \text{ verifying } x_0 - x < \eta \text{ we get } \mid f(x) - l_L \mid < \epsilon.$$

   We note

   $$\lim_{x \to x_0^-} f(x) = l_L \text{ or } \lim_{x \to^< x_0} f(x) = l_L.$$

   (b) **Right-hand limit at x_0:**
   Let x_0 be a real number and f be a function defined at least to the right of x_0. We say that f admits the number l_R as a right-hand limit of x_0 if:

   $$\lim_{x \to x_0} f(x) = l_R \text{ with } x > x_0.$$

   In other words,

$$\forall \epsilon > 0,\ \exists \eta > 0 \text{ such that for all } x \neq x_0 \text{ vérifiant } x - x_0 < \eta \text{ we get } \mid f(x) - l_R \mid < \epsilon.$$

We note

$$\lim_{x \to x_0^+} f(x) = l_R \text{ or } \lim_{x \to^> x_0} f(x) = l_R.$$

Remark 1.4. The function f has a limit at $x0$ if both the left-hand and right-hand limits exist and are equal.
When the function f is defined at x_0, then the limit of f as x tends to x_0 is equal to the value of $f(x_0)$.

2. **Infinite limit at x_0**
In this case, the limit of f as x tends to x_0 tends to infinity ($+\infty$ or $-\infty$).

   (a) **First case:**

$$\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists \eta > 0 \text{ such that } \mid x - x_0 \mid < \eta \text{ involves } f(x) > A.$$

That is, for a positive real A arbitrarily chosen as large as desired, then all values of $f(x)$ exceed this real, as soon as x is close enough to x_0.

Example 1.9. The function $f(x) = \frac{1}{x^2}$ tends to $+\infty$ when x tends to 0.
Par définition

$$\lim_{x \to 0} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists \eta > 0 \text{ such that } \mid x - 0 \mid < \eta \text{ implies } f(x) > A$$

i.e.

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty \Leftrightarrow \forall A > 0, \exists \eta > 0 \text{ such that } \mid x - 0 \mid < \eta \Rightarrow \frac{1}{x^2} > A.$$

Let A be any positive real.
If $\mid x - 0 \mid < \eta$, so $x^2 < \eta^2$, which implies $\frac{1}{x^2} > \frac{1}{\eta^2}$, i.e., $f(x) > \frac{1}{\eta^2}$,
then just take $\frac{1}{\eta^2} > A$, that is $\eta < \frac{1}{\sqrt{A}}$.

   (b) **Second case:**

$$\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists \eta > 0 \text{ such that } \mid x - x_0 \mid < \eta \text{ implies } f(x) < -A.$$

Example 1.10. The function $f(x) = \frac{-1}{x^2}$ tends to $-\infty$ when x tends to 0.
By definition

$$\lim_{x \to 0} \frac{-1}{x^2} = -\infty \Leftrightarrow \forall A > 0, \exists \eta > 0 \text{ such that } \mid x - 0 \mid < \eta \Rightarrow \frac{-1}{x^2} < -A.$$

Let A be any positive real.
If $\mid x-0 \mid < \eta$, so $x^2 < \eta^2$, which involves $\frac{1}{x^2} > \frac{1}{\eta^2}$, and multiplying both sides of the inequality by "-1" we get $\frac{-1}{x^2} < \frac{-1}{\eta^2}$, i.e. $f(x) < \frac{-1}{\eta^2}$.
Then simply take $\frac{-1}{\eta^2} < -A$, or in other words $\frac{1}{\eta^2} > A$ that is $\eta < \frac{1}{\sqrt{A}}$.

2.2 Limits at infinity ($+\infty$ or $-\infty$)

These limits can be defined in the same way as above:

$$\lim_{x\to+\infty} f(x) = l \quad \Leftrightarrow \quad \forall\epsilon > 0, \exists B > 0, \text{ such that } \forall x, x > B \Rightarrow \mid f(x) - l \mid < \epsilon.$$
$$\lim_{x\to-\infty} f(x) = l \quad \Leftrightarrow \quad \forall\epsilon > 0, \exists B > 0, \text{ such that } \forall x, x < -B \Rightarrow \mid f(x) - l \mid < \epsilon.$$

In the two previous limits, it is a matter of finding and writing B as a function of ϵ.

$$\lim_{x\to+\infty} f(x) = +\infty \quad \Leftrightarrow \quad \forall A > 0, \exists B > 0, \text{ such that } \forall x, x > B \Rightarrow f(x) > A.$$
$$\lim_{x\to-\infty} f(x) = +\infty \quad \Leftrightarrow \quad \forall A > 0, \exists B > 0, \text{ such that } \forall x, x < -B \Rightarrow f(x) > A.$$
$$\lim_{x\to+\infty} f(x) = -\infty \quad \Leftrightarrow \quad \forall A > 0, \exists B > 0, \text{ such that } \forall x, x > B \Rightarrow f(x) < -A.$$
$$\lim_{x\to-\infty} f(x) = -\infty \quad \Leftrightarrow \quad \forall A > 0, \exists B > 0, \text{ such that } \forall x, x < -B \Rightarrow f(x) < -A.$$

In these last previous limits, it is a question of expressing B as a function of A.

Example 1.11. The function $f(x) = x^2 + 1$ tends to $+\infty$ when x tends to $+\infty$.
This involves choosing a positive real number A as large as desired and then finding a positive number B such that $x > B$ induces $f(x) > A$.

$$\begin{aligned} f(x) > A \quad &\Leftrightarrow \quad x^2 + 1 > A \\ &\Leftrightarrow \quad x^2 > A - 1 \\ &\Leftrightarrow \quad x > \sqrt{A-1}. \end{aligned}$$

Just take $B = \sqrt{A-1}$.

Remark 1.5. In practice, we obviously have a number of methods and theorems which allow us to calculate the limits directly without having to resort to these definitions each time.

2.3 Operations on limits

Determinate Forms:

Let f and g be two functions and λ a real number.

1. The limits of f and g are constants which we denote respectively by l_f and l_g:

   (a) λf tends to λl_f, $\forall \lambda \in \mathbb{R}$.

   (b) $f + g$ tends to $l_f + l_g$.

   (c) fg tends to $l_f l_g$.

   (d) $\frac{f}{g}$ tends to $\frac{l_f}{l_g}$, with $l_g \neq 0$.

2. The limit of f is a real constant l_f and the limit of g is infinite:

   (a) λg tends to ∞ ($+\infty$ or $-\infty$ according to the rule of signs), with $\lambda \neq 0$.

   (b) $f + g$ tends to ∞ ($+\infty$ if g tends to $+\infty$ or $-\infty$ if g tends to $-\infty$).

   (c) fg tends to ∞ ($+\infty$ or $-\infty$ according to the sign rule), for $l_f \neq 0$.

   (d) $\frac{f}{g}$ tends to 0 (0^+ or 0^- according to the sign rule).

   (e) $\frac{g}{f}$ tends to ∞ ($+\infty$ or $-\infty$ according to the sign rule).

3. Les limites de f et g sont infinies:

   (a) fg tends to ∞ ($+\infty$ or $-\infty$ according to the sign rule).

   (b) $f + g$:
   - $f + g$ tends to $+\infty$ if f and g tendent vers $+\infty$.
   - $f + g$ tends to $-\infty$ if f and g tendent vers $-\infty$.
   - is presented in the form of an indeterminate $+\infty - \infty$, when one of the functions tends to $+\infty$ and the other to $-\infty$.

   (c) $\frac{f}{g}$ is an indeterminate form $\frac{\infty}{\infty}$.

Indeterminate forms:

The known indeterminate forms are: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $+\infty - \infty$, 1^{∞}, 0^0 and ∞^0.

2.4 Some theorems and methods for calculating limits

In practice, the calculation of limits that requires development is much more concerned with cases where indeterminate forms are present. Here we present a few fundamental theorems that allow us to remove these indeterminate forms.

1. **Limit of a polynomial when x tends to infinity:**

   Theorem 1.1. *The limit of a polynomial as x tends to infinity is equal to the limit of its highest degree monomial.*

Example 1.12. The polynomial $4x^3+3x^2-x+6$ has the same limit as the polynomial $4x^3$ when x tends to $+\infty$.
Indeed,

$$\lim_{x\to+\infty} 4x^3+3x^2-x+6 = x^3(4+\frac{3}{x}-\frac{1}{x^2}+\frac{6}{x^3}) = +\infty(4+0-0+0) = +\infty$$

and

$$\lim_{x\to+\infty} 4x^3 = +\infty.$$

2. **Limit of a ratio of two polynomials when x tends to infinity :**

Theorem 1.2. *The limit of a ratio of two polynomials $\frac{P(x)}{Q(x)}$ when x tends to infinity is equal to the limit of the ratio of the highest degree monomials of $P(x)$ and $Q(x)$ respectively.*

Example 1.13. Using this theorem, the calculation of the limits of a ratio of two polynomials as x tends to infinity becomes very simple.

(a) The expression $\frac{3x^2+4x-5}{2x^2-3x+7}$ admits the same limit as $\frac{3x^2}{2x^2}$ that is $\frac{3}{2}$ when x tends to $+\infty$ or $-\infty$.

(b) $\frac{x^3+5x-2}{2x^2-x+6}$ admits the same limit as $\frac{x^3}{2x^2}=\frac{x}{2}$ when x tends to ∞.

$$\lim_{x\to+\infty} \frac{x^3+5x-2}{2x^2-x+6} = \lim_{x\to+\infty} \frac{x^3}{2x^2} = \lim_{x\to+\infty} \frac{x}{2} = +\infty$$

and

$$\lim_{x\to-\infty} \frac{x^3+5x-2}{2x^2-x+6} = \lim_{x\to-\infty} \frac{x^3}{2x^2} = \lim_{x\to-\infty} \frac{x}{2} = -\infty.$$

(c) The limit of $f(x)=\frac{4x^3+x+1}{x^5-x}$ when x tends to $+\infty$ is

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{4x^3+x+1}{x^5-x} = \lim_{x\to+\infty} \frac{4x^3}{x^5} = \lim_{x\to+\infty} \frac{4}{x^2} = 0^+.$$

3. **Indeterminate form $\frac{0}{0}$ (ratio of two polynomials) :**
If a ratio $\frac{P(x)}{Q(x)}$ of two polynomials is in the indeterminate form $\frac{0}{0}$, when x tends towards a constant a, we can remove this indeterminacy by factoring $P(x)$ and $Q(x)$ by $(x-a)$.

Example 1.14. Let us consider

$$R(x) = \frac{x^2+5x-6}{x^2+x-2}.$$

$R(x)$ takes the indeterminate form $\frac{0}{0}$ when x tends to 1.
We can then factorize each of the polynomials x^2+5x-6 (numerator of $R(x)$) and x^2+x-2 (denominator of $R(x)$) by $(x-1)$ and we obtain:

$$x^2+5x-6=(x-1)(x+6) \text{ and } x^2+x-2=(x-1)(x+2).$$

$$R(x)=\frac{x^2+5x-6}{x^2+x-2}=\frac{(x-1)(x+6)}{(x-1)(x+2)}=\frac{x+6}{x+2}.$$

Then

$$\lim_{x\to 1} R(x)=\lim_{x\to 1}\frac{x^2+5x-6}{x^2+x-2}=\lim_{x\to 1}\frac{x+6}{x+2}=\frac{7}{3}.$$

Example 1.15. Limits calculus:

$$\lim_{x\to 2}\frac{x^3-5x+2}{2x^4-6x^2-2x-12},\qquad \lim_{x\to -1}\frac{x^3-3x-2}{x^3+5x^2+7x+3},\qquad \lim_{x\to 2}\frac{x^2-4}{x-2}.$$

The three limits correspond to the indeterminate form $\frac{0}{0}$ and the indeterminacy is removed after factoring, division and simplification.

(a) $$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4.$$

(b) $$\lim_{x\to 2}\frac{x^3-5x+2}{2x^4-6x^2-2x-12}=\lim_{x\to 2}\frac{(x-2)(x^2+2x-1)}{(x-2)(2x^3+4x^2+2x+6)}=\lim_{x\to 2}\frac{x^2+2x-1}{2x^3+4x^2+2x+6}=\frac{7}{42}=\frac{1}{6}.$$

(c) $$\lim_{x\to -1}\frac{x^3-3x-2}{x^3+5x^2+7x+3}=\lim_{x\to -1}\frac{(x+1)^2(x-2)}{(x+1)^2(x+3)}=\lim_{x\to -1}\frac{x-2}{x+3}=\frac{-3}{2}.$$

4. **Indeterminate forms $\frac{0}{0}$ (irrational expressions) :**
Generally, when an irrational expression is in the indeterminate form $\frac{0}{0}$, the indeterminacy can be removed by multiplying the expression by the conjugate expression.

Expression	Conjugate expression
$\sqrt{A}+B$	$\sqrt{A}-B$
$A+\sqrt{B}$	$A-\sqrt{B}$
$\sqrt{A}-\sqrt{B}$	$\sqrt{A}+\sqrt{B}$

Let us recall the usual product:

$$(A+B)(A-B)=A^2–B^2,$$

thus

$$(\sqrt{A}-\sqrt{B})(\sqrt{A}-\sqrt{B})=A-B.$$

Example 1.16. The limit of $f(x) = \frac{\sqrt{x+3}-2}{x^2+x-2}$, when x tends to 1 is in the indeterminate form $\frac{0}{0}$.

$$f(x) = \frac{\sqrt{x+3}-2}{x^2+x-2} = \frac{\sqrt{x+3}-2}{(x-1)(x+2)}.$$

Let us multiply the numerator and denominator of $f(x)$ by the conjugate expression of $\sqrt{x+3}-2$, i.e. by $\sqrt{x+3}+2$.
We obtain

$$\begin{aligned} f(x) &= \frac{\sqrt{x+3}-2}{(x-1)(x+2)}\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \frac{(\sqrt{x+3})^2-2^2}{(x-1)(x+2)(\sqrt{x+3}+2)} \\ &= \frac{x+3-4}{(x-1)(x+2)(\sqrt{x+3}+2)} = \frac{x-1}{(x-1)(x+2)(\sqrt{x+3}+2)} \\ &= \frac{1}{(x+2)(\sqrt{x+3}+2)} \end{aligned}$$

Thus, the indeterminate form is lifted and the limit of $f(x)$ as x tends to 1 is:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{(x+2)(\sqrt{x+3}+2)} = \frac{1}{12}.$$

Example 1.17.

$$h(x) = \frac{\sqrt{x+6}-3}{\sqrt{2x-5}-1}$$

$$\lim_{x\to 3} h(x) = \lim_{x\to 3} \frac{\sqrt{x+6}-3}{\sqrt{2x-5}-1} = \frac{0}{0} \quad \text{is an indeterminate form.}$$

To remove the indeterminacy, we multiply and divide the two terms of this expression by the product of the conjugate expression of the numerator and the conjugate expression of the denominator, which are respectively: $\sqrt{x+6}+3$ and $\sqrt{2x-5}+1$ and we get:

$$\begin{aligned} h(x) &= \frac{\sqrt{x+6}-3}{\sqrt{2x-5}-1}\left(\frac{\sqrt{x+6}+3}{\sqrt{2x-5}+1}\right)\left(\frac{\sqrt{2x-5}+1}{\sqrt{x+6}+3}\right) = \frac{x+6-9}{2x-5-1}\left(\frac{\sqrt{2x-5}+1}{\sqrt{x+6}+3}\right) \\ &= \frac{x-3}{2x-6}\left(\frac{\sqrt{2x-5}+1}{\sqrt{x+6}+3}\right) = \frac{x-3}{2(x-3)}\left(\frac{\sqrt{2x-5}+1}{\sqrt{x+6}+3}\right) = \frac{1}{2}\frac{\sqrt{2x-5}+1}{\sqrt{x+6}+3}. \end{aligned}$$

So

$$\lim_{x\to 3} h(x) = \lim_{x\to 3} \frac{1}{2}\frac{\sqrt{2x-5}+1}{\sqrt{x+6}+3} = \frac{2}{12} = \frac{1}{6}.$$

5. **Indeterminate forms "$+\infty-\infty$" and "$0\times\infty$":**
These forms are often written as $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 1.18. Let f and g be two functions defined by:

$$f(x) = \sqrt{x+6} - x, \qquad g(x) = \frac{1}{x} - \frac{1}{x^2}.$$

(a) $f(x) = \sqrt{x+6} - x$

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt{x+6} - x$ is in the form of an indeterminate $+\infty - \infty$.

Multiplying and dividing $f(x)$ by the conjugate expression of $\sqrt{x+6} - x$ gives:

$$f(x) = \sqrt{x+6} - x = (\sqrt{x+6} - x)\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x} = \frac{x+6-x^2}{\sqrt{x+6}+x}.$$

So

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x+6-x^2}{\sqrt{x+6}+x} = \lim_{x\to+\infty} \frac{-x^2}{x} \lim_{x\to+\infty}(-x) = -\infty.$$

(b) $g(x) = \frac{1}{x} - \frac{1}{x^2}$

$$\lim_{x\to0^+} g(x) = \lim_{x\to0^+} (\frac{1}{x} - \frac{1}{x^2}) \text{ indeterminate form } +\infty - \infty.$$

$$\lim_{x\to0^+} g(x) = \lim_{x\to0^+} (\frac{1}{x} - \frac{1}{x^2}) = \lim_{x\to0^+} \frac{1}{x^2}(x-1) = +\infty(0-1) = -\infty.$$

Example 1.19.

$$\lim_{x\to0^+} (x^3+x^2+x)\frac{1}{xe^x} = 0 \times (+\infty) \text{ indeterminate form.}$$

$$\lim_{x\to0^+} (x^3+x^2+x)\frac{1}{xe^x} = \lim_{x\to0^+} \frac{x^3+x^2+x}{xe^x} = \lim_{x\to0^+} \frac{x^2+x+1}{e^x} = 1.$$

6. **Indeterminate form "0^0":**
In the case of the indeterminate form 0^0, we return to the exponential and logarithmic forms.

Example 1.20.

$$\lim_{x\to0^+} x^x = 0^0 \text{ indeterminate form.}$$

$$\lim_{x\to0^+} x^x = \lim_{x\to0^+} e^{ln(x^x)} = \lim_{x\to0^+} e^{xln(x)} = e^0 = 1.$$

3 Continuity

3.1 Definitions

Let f be a function defined on E and x_0 a point on E.

Continuity at a point x_0 :

We say that f is continuous at the point x_0, if and only if:

$$\lim_{x \to x_0} f(x) = f(x_0),$$

i.e. the function is defined at x_0 and its limit when x tends to x_0 is equal to $f(x_0)$. In other words, f is continuous at x_0 is equivalent to

$$\forall \epsilon > 0, \exists \eta > 0 \text{ such that } \mid x - x_0 \mid < \eta \Rightarrow \mid f(x) - f(x_0) \mid < \epsilon.$$

Left continuity at a point x_0 :

We say that f is left continuous at the point x_0, if and only if the limit of f to the left of x_0 is equal to $f(x_0)$:

$$\lim_{x \to^{<} x_0} f(x) = f(x_0).$$

Right continuity at a point x_0 :

We say that f is right continuous at the point x_0, if and only if the limit of f to the right of x_0 is equal to $f(x_0)$:

$$\lim_{x \to^{>} x_0} f(x) = f(x_0).$$

Theorem 1.3. *The function f is continuous at the point x_0, belonging to its domain of definition, if and only if it is continuous to the left and to the right of x_0.*

If f is continuous at every point in its domain of definition E, then it is said to be continuous on E.

Example 1.21. Let f and g be two functions defined on $\mathbb{R}$:

$$f(x) = x^3 \text{ and } g(x) = \begin{cases} \frac{\sqrt{x^2}}{x}, & \text{for } x \neq 0; \\ 1, & \text{for } x = 0. \end{cases}$$

1. The function f is continuous at every point of $\mathbb{R}$ (because $f(x)$ is a polynomial of degree three and all polynomials are continuous on $\mathbb{R}$).

2. The function g is not continuous at the point $x = 0$.
   In fact,

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1.$$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1.$$

   The two limits are different ($\lim_{x\to 0^+} g(x) \neq \lim_{x\to 0^-} g(x)$), so $\lim_{x\to 0} g(x)$ does not exist.
   Therefore, g is not continuous at $x = 0$.

3.2 Continuous extension

Definition 1.5. Let I be an interval of $\mathbb{R}$, $a \in I$ and f a continuous function on $I \setminus \{a\}$. If $\lim_{x\to a} f(x) = \alpha$ (with $\alpha \in \mathbb{R}$), so f admits an extension by continuity at $x = a$, denoted $\widetilde{f}$, defined on$I$ by :

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{for } x \neq a; \\ \alpha, & \text{for } x = a. \end{cases}$$

Example 1.22. Let f be the function defined by

$$\begin{aligned} f \ : \ & \mathbb{R}^* \to \mathbb{R} \\ & x \mapsto f(x) = \frac{\sin(x)}{x}. \end{aligned}$$

The function f admits an extension by continuity at $x = 0$, because the limit of f when x tends to 0 exists.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{x}{x} = 1.$$

The extension by continuity of f is the function denoted $\widetilde{f}$ defined on $\mathbb{R}$ by:

$$\widetilde{f}(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{for } x \neq 0; \\ 1, & \text{for } x = 0. \end{cases}$$

3.3 Properties of continuous functions

1. If f and g are two continuous functions at a point x_0, then :
   - The function $f + g$ is continuous at x_0.
   - The function $f.g$ is continuous at x_0.
   - If $g(x_0) \neq 0$, so the function $\frac{f}{g}$ is continuous at x_0.

- The function λf is continuous at x_0, for any value of $\lambda \in \mathbb{R}$.

2. If f is a continuous function at a point x_0 and g is a continuous function at the point $f(x_0)$, then the function gof is continuous at x_0.

3.4 Continuous Functions Theorems

Here are some important theorems for continuous numerical functions.

Theorem 1.4. ***Intermediate Value theorem (Bolzano's theorem)***
Let f be a continuous function on an interval $[a, b]$. For any real number u between $f(a)$ and $f(b)$, the equation $f(x) = u$ has at least one solution on $[a, b]$.
Thus, the segment with endpoints $f(a)$ and $f(b)$, is included in $f([a, b])$.

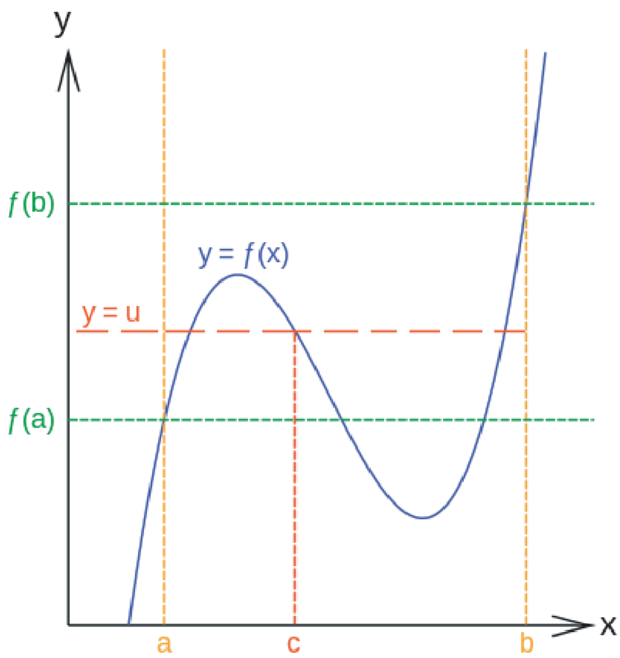

Figure 1.2: Bolzano's theorem illustration ($\exists c \in [a, b] \setminus f(x) = u$).

Theorem 1.5. *Any continuous function on an interval $[a, b]$ and taking values of opposite signs at the limits of this interval ($f(a)f(b) < 0$), vanishes at least once on this segment; in other words, the equation $f(x) = 0$ admits at least one solution in $[a, b]$, that is to say $\exists c \in [a, b]$ such that $f(c) = 0$. The number of existing solutions is always odd.*

Remark 1.6. Any continuous and strictly monotone function (strictly increasing or strictly decreasing) on an interval $[a, b]$ and $f(a).f(b) < 0$, vanishes once and only once on this interval, that is to say that the equation $f(x) = 0$ admits a unique solution in $[a, b]$.

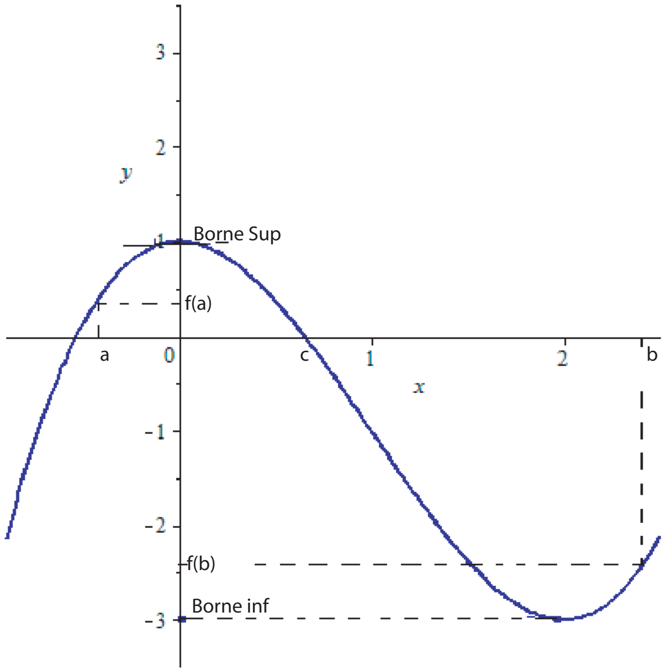

Figure 1.3: In the figure $f(a)f(b) < 0$ and $\exists c \in [a, b]$, such that $f(c) = 0$.

Theorem 1.6. ***Weierstrass's theorem (or bounds theorem)***
Any continuous function on a bounded (non-empty) closed interval is bounded and reaches its limits there. In other words, if f is a continuous function on an interval $[a, b]$, then there exists α and $\beta \in [a, b]$ such that

$$\inf(f([a, b])) = f(\alpha) \text{ and } \sup(f([a, b])) = f(\beta),$$

or again, taking into account the previous theorem $f([a, b]) = [f(\alpha), f(\beta)]$, with $\inf(f([a, b]))$ is the lower bound of $f([a, b])$ and $\sup(f([a, b]))$ is the upper bound of $f([a, b])$.

Example 1.23. The function $f(x) = x^3 - 3x^2 + 1$ is defined on $\mathbb{R}$ and has three roots, i.e. the equation $f(x) = 0$ has three solutions. The points of intersection of the curve of f with the x axis (see figure 1.4), c1, c2 and c3, are the solutions of this equation:

- A solution c1 in $]-1, 0[$.
  The function f is strictly increasing in the interval $]-1, 0[$, and $f(-1) = -3$ has a sign that is not the same as $f(0) = 1$. $(f(-1).f(0) < 0)$.

- A solution c2 in $]0,1[$.

  The function f is strictly decreasing in the interval $]0,1[$, and $f(0) = 1$ has a sign that is not the same as $f(1) = -1$ $(f(0).f(1) < 0)$.

- Une solution c3 dans $]2,3[$.

  The function f is strictly increasing in the interval $]2,3[$, and $f(2) = -3$ has a sign that is not the same as $f(3) = 1$ $(f(2).f(3) < 0)$.

If we want more precision on these solutions, for example c1, it will be enough to choose a smaller interval $]a,b[\subset]-1,0[$ such that $f(a).f(b) < 0$.

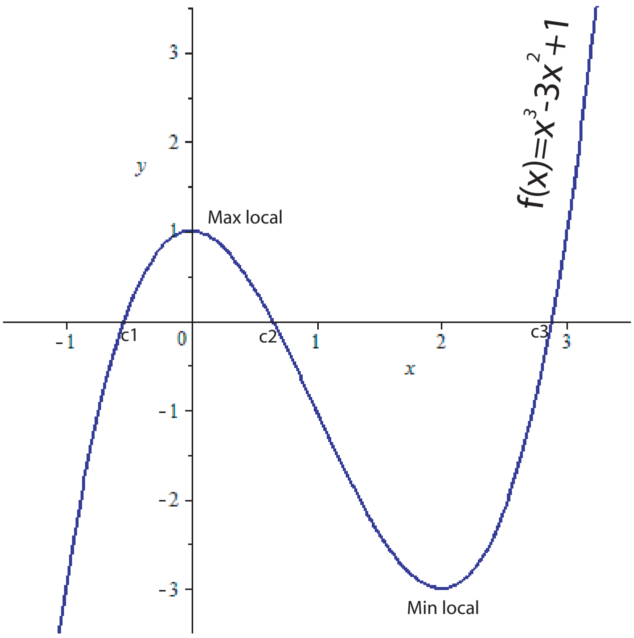

Figure 1.4: Graph of the function $f(x) = x^3 - 3x^2 + 1$.

4 Differentiability of a function

4.1 Definitions

Let f be a function defined and continuous on an interval $]a,b[$ and x_0 a point of this interval.

Differentiability at a point x_0:

We say that f is differentiable at x_0, if and only if:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

This number, when it exists, is called the derivative of the function f at x_0 and denoted $f^{'}(x_0)$.

Example 1.24. The derivative $f(x) = x^2$ at $x_0 = 1$ is 2.
Indeed,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1}(x+1) = 2.$$

Left-hand derivative at a point x_0 :
We say that f is left-derivative at the point x_0, if and only if:

$$\lim_{x \to^< x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

This number, if it exists, is denoted by $f^{'}_L(x_0)$.

Right-hand derivative at a point x_0 :
We say that f is right-derivative at the point x_0, if and only if:

$$\lim_{x \to^> x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

This number, if it exists, is denoted by $f^{'}_R(x_0)$.

Theorem 1.7. *A function f which is continuous at x_0 is derivable at x_0 if and only if it is derivable to the left and right of x_0.*

Derivability on a set :
We say that f is derivable on a set E of $\mathbb{R}$ if it is derivable at any point of E, and we denote the derivative function associated with any point x of E by $f'(x)$.

Remark 1.7. Any function differentiable at a point x_0 (or on an interval I of $\mathbb{R}$) is continuous at this point (on this interval), but the reverse is not always true.
Any function that is not continuous at a point x_0 (or on an interval I) is not differentiable at this point (on this interval).

Example 1.25. The function $f(x) = \frac{1}{x}$ is not differentiable at $x = 0$.
In fact,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = +\infty \text{ and } \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

The two limits (left and right of 0) tend to infinity and are different ($\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$),
so $\lim_{x \to 0} f(x)$ does not exist.
Therefore, f is not continuous and is not differentiable at $x = 0$.

4.2 Differentiable functions' properties

1. If f and g are two functions that can be differentiated at a point x_0 or on an interval I, then:

   - The function $f+g$ is differentiable at x_0 (on the interval I). Its derivative function at any point x of I, when it exists, is defined by:

   $$\Big(f(x)+g(x)\Big)^{'} = f^{'}(x)+g^{'}(x).$$

   - The function $f.g$ is differentiable at x_0 (on the interval I). Its derivative function at any point x is:

   $$\Big(f(x)g(x)\Big)^{'} = f^{'}(x)g(x)+g^{'}(x)f(x).$$

   - The function f^n is differentiable at x_0 (on the interval I), with $n \in \mathbb{R}_+^*$. Its derivative function is:

   $$\Big(f^n(x)\Big)^{'} = nf^{'}(x)f^{n-1}(x).$$

   - If $g(x_0) \neq 0$ ($g(x) \neq 0$, $\forall x \in I$), then the function $\frac{f}{g}$ is differentiable at x_0 (on the interval I). Its derivative function at any point x, when it exists, is:

   $$\Big(\frac{f(x)}{g(x)}\Big)^{'} = \frac{f^{'}(x)g(x)-g^{'}(x)f(x)}{(g(x))^2}.$$

   - The function λf is differentiable at x_0 (on the interval I), for all $\lambda \in \mathbb{R}$ and its derivative is:

   $$\Big(\lambda f(x)\Big)^{'} = \lambda f^{'}(x).$$

   Example 1.26. Calculation of the derivatives of the following functions: real constant k, x, x^n, $\frac{1}{x}$, $\sqrt{x}$ and $\sqrt{f(x)}$.

   $$\begin{aligned}
   k^{'} &= 0 \text{ with } k \in \mathbb{R}.\\
   x^{'} &= 1.\\
   \Big(x^n\Big)^{'} &= nx^{n-1}.\\
   \Big(\frac{1}{x}\Big)^{'} &= \frac{-1}{x^2} \text{ with } x \neq 0.\\
   \Big(\sqrt{x}\Big)^{'} &= \frac{1}{2\sqrt{x}} \text{ with } x > 0.\\
   \Big(\sqrt{f(x)}\Big)^{'} &= \frac{f'(x)}{2\sqrt{f(x)}} \text{ with } f(x) > 0.
   \end{aligned}$$

2. If f is differentiable at a point x_0 and g is differentiable at $f(x_0)$, then the composite function gof is differentiable at x_0 and the derivative of the function gof at x_0 is:

$$\Big(gof(x_0)\Big)^{'} = g^{'}(f(x_0))f^{'}(x_0).$$

Example 1.27. Calculation of the derivatives of the functions:

$$ln(x), ln(f(x)), e^x \text{ and } e^{f(x)}.$$

1. $\Big(ln(x)\Big)^{'} = \frac{1}{x}$, with $x > 0$.
2. $\Big(ln(f(x))\Big)^{'} = \frac{f^{'}(x)}{f(x)}$, with $f(x) > 0$.
3. $\Big(e^x\Big)^{'} = e^x$.
4. $\Big(e^{f(x)}\Big)^{'} = f^{'}(x)e^{f(x)}$.

Example 1.28. The derivatives of the following functions:

$$f(x) = 3x^3 + 4x^2 - 5,\ g(x) = (2x^2+3)^3,\ h(x) = (x^2+1)(2x-1) \text{ and } k(x) = \frac{2x^2+3x}{x-1}.$$

$$\begin{aligned}
f^{'}(x) &= 9x^2 + 8x. \\
g^{'}(x) &= 3(2x^2+3)^{'}(2x^2+3)^{3-1} = 3(4x)(2x^2+3)^2 = 12x(2x^2+3)^2. \\
h^{'}(x) &= (x^2+1)^{'}(2x-1) + (x^2+1)(2x-1)^{'} = 2x(2x-1) + 2(x^2+1) = 6x^2 - 2x + 2.
\end{aligned}$$

$$\begin{aligned}
k^{'}(x) &= \frac{(2x^2+3x)^{'}(x-1) - (x-1)^{'}(2x^2+3x)}{(x-1)^2} = \frac{(4x+3)(x-1) - 1(2x^2+3x)}{(x-1)^2} \\
&= \frac{2x^2-4x-3}{(x-1)^2}, \quad \text{with } x \neq 1.
\end{aligned}$$

4.3 The derivative, direction of variation, and extremas.

A function's monotonicity

A function f is said to be monotonic on an interval I if it is increasing or decreasing on this interval.

Theorem 1.8. *A function is increasing (respectively decreasing) on an interval I if and only if its derivative is positive (respectively negative) on this interval.*

Example 1.29. Let us study the monotonicity of the function $f(x) = x^2$ on the interval $]-2, 2[$.
The derivative of f is $f'(x) = 2x$, which is strictly positive for $x > 0$ and strictly negative for $x < 0$.

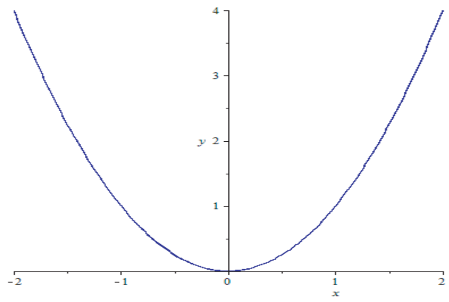

Figure 1.5: Curve of $f(x) = x^2$ on the interval $]-2, 2[$.

The figure 1.5 shows us that for $x < 0$ (where $f'(x) < 0$) the function f is strictly decreasing and for $x > 0$ (where $f'(x) > 0$) the function f is strictly increasing.

Function extremas

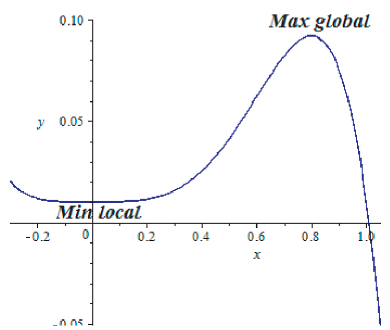

Figure 1.6: Local minimum and global maximum.

Let $f : I \to \mathbb{R}$ be a function defined on an interval I and let $a \in I$.
We say that f has a maximum at a, if

$$\forall x \in I, f(x) \leq f(a);$$

and f admits a minimum in a, if

$$\forall x \in I, f(x) \geq f(a).$$

We sometimes speak of a global maximum or global minimum of the function, and we say that $f(a)$ is the maximum (respectively the minimum) of f on I.
f has a local maximum at a if there exists an open interval J containing a such that, for all $x \in J \cap I$, we have $f(x) \leq f(a)$. A local minimum is defined similarly by reversing the direction of inequality. Local extremum refers to a local maximum or minimum.

Theorem 1.9. *The abscissas of the extrema can be identified by the values of x where the derivative vanishes and changes sign :*

1. *If $f^{'}(x) = 0$, $f^{'}(x) > 0$ for $x > a$ and $f^{'}(x) < 0$ for $x < a$, then a represents the abscissa of a maximum.*

2. *If $f^{'}(x) = 0$, $f^{'}(x) < 0$ for $x > a$ and $f^{'}(x) > 0$ for $x < a$, then a represents he abscissa of a minimum.*

Example 1.30. Let us look for the extrema of the function $f(x) = x^3 - 3x^2 + 1$.
The derivative function of $f(x)$ is $f^{'}(x) = 3x^2 - 6x = 3x(x-2)$, which vanishes for $x = 0$ and $x = 2$.

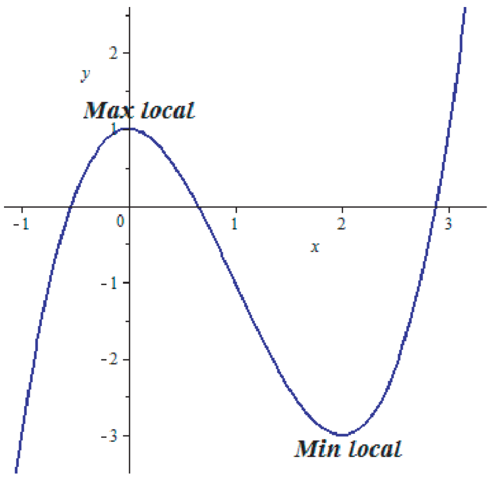

Figure 1.7: Extrema of the function $f(x) = x^3 - 3x^2 + 1$.

- $x = 0$ is the abscissa of the local maximum $(0, 1)$ of the function f (see the figure 1.7):

$$\begin{cases} f^{'}(0) = 0; \\ f^{'}(x) > 0, \quad x < 0; \\ f^{'}(x) < 0, \quad 0 < x < 2. \end{cases}$$

- $x = 2$ is the abscissa of the local minimum $(2, -3)$ of the function f (see the figure 1.7):

$$\begin{cases} f^{'}(2) = 0; \\ f^{'}(x) < 0, \quad 0 < x < 2; \\ f^{'}(x) > 0, \quad x > 2. \end{cases}$$

4.4 L'Hôpital's rule

L'Hôpital's rule can only be applied in the case where direct substitution yields an indeterminate form, meaning $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
Soient f et g deux fonctions continues au voisinage d'un point a, tels que $\lim\limits_{x\to a} f(x) = 0$ (respectivement ∞) et $\lim\limits_{x\to a} g(x) = 0$ (respectivement ∞).

According to L'Hôpital's rule, if the limit $\lim\limits_{x\to a} \frac{f^{'}(x)}{g^{'}(x)}$ exists, then :

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^{'}(x)}{g^{'}(x)}.$$

Example 1.31. $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0}$ is an indeterminate form.
Given that the functions $\sin(x)$ and x are continuous at the point $x = 0$, then we can use the L'Hôpital's rule and we obtain:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{(\sin(x))^{'}}{x^{'}} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1.$$

4.5 Successive derivatives and Leibniz's formula

Given an open interval I, we say that f is differentiable on I, if it is differentiable at every point of I.
Let f be a function differentiable on I. Its derivative $f^{'}$ can itself be differentiable. We call the second derivative of f the derivative of $f^{'}$, and we denote it $f^{''}$. This function can itself be differentiable, etc.
If f is k times differentiable, we denote $f^{(k)}$ its derivative of order k. By definition, the derivative of order 0 is the function itself.

Definition 1.6. Let f be a function defined on an interval I of $\mathbb{R}$. We say that f is of class $\mathcal{C}^k$ on I, or again f is k times continuously differentiable, if it admits a k-th continuous derivative on I.
We say that f is of class $\mathcal{C}^\infty$ on I, if it admits successive derivatives of any order (they are necessarily continuous since they are differentiable).

Remark 1.8. The following functions are of class $\mathcal{C}^\infty$ on the open intervals where they are defined: polynomial, power, exponential, sine, cosine functions.

Leibniz's formula

Leibniz's formula, very close to Newton's binomial formula, expresses the n-th derivative of a product using the successive derivatives of the components.

Proposition 1.1. *If f and g are two functions defined from $\mathbb{R}$ to $\mathbb{R}$, n times differentiable on an interval I, then the product fg is n times differentiable on I and:*

$$\Big(f(x)g(x)\Big)^{(n)} = \sum_{k=0}^{n} C_n^k \Big(f(x)\Big)^{(k)} \Big(g(x)\Big)^{(n-k)}$$

where

$$C_n^k = \frac{n!}{(n-k)!k!} \text{ and } n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n.$$

By convention $0! = 1$ *and* $1! = 1$.

Example 1.32. Calculation of the nth derivative of $f(x) = e^{2x}(x+1)^2$, with $n > 3$.

$$(f(x))^{(n)} = \Big(e^{2x}(x+1)^2\Big)^{(n)} = \sum_{k=0}^{n} C_n^k \Big((x+1)^2\Big)^{(k)} \Big(e^{2x}\Big)^{(n-k)}.$$

We have

$$\begin{aligned}
\Big((x+1)^2\Big)^{(0)} &= (x+1)^2 \\
\Big((x+1)^2\Big)^{(1)} &= 2(x+1) \\
\Big((x+1)^2\Big)^{(2)} &= 2 \\
\Big((x+1)^2\Big)^{(3)} &= 0 \\
\vdots \quad & \quad \vdots \\
\Big((x+1)^2\Big)^{(n)} &= 0
\end{aligned}$$

$$\begin{aligned}
\left(e^{2x}\right)^{(0)} &= e^{2x}\\
\left(e^{2x}\right)^{(1)} &= 2e^{2x}\\
\left(e^{2x}\right)^{(2)} &= 2^2e^{2x}\\
\left(e^{2x}\right)^{(3)} &= 2^3e^{2x}\\
\vdots\quad & \quad\vdots\\
\left(e^{2x}\right)^{(n)} &= 2^ne^{2x}
\end{aligned}$$

Since the k-th derivatives of $(x+1)^2$ for $k \geq 3$ are zero ($\left((x+1)^2\right)^{(k)} = 0,\ k \geq 3$), then all terms of the Leibniz form, which corresponds to a derivative $k \geq 3$ will be equal to 0.
Let us calculate C_n^0, C_n^1 and C_n^2.

$$\begin{aligned}
C_n^0 &= \frac{n!}{(n-0)!0!} = \frac{n!}{n!} = 1\\
C_n^1 &= \frac{n!}{(n-1)!1!} = \frac{n \times (n-1)!}{(n-1)!} = n\\
C_n^2 &= \frac{n!}{(n-2)!2!} = \frac{n(n-1) \times (n-2)!}{(n-2)!2} = \frac{n(n-1)}{2}
\end{aligned}$$

The nth derivative of $f(x)$ is:

$$\begin{aligned}
\left(f(x)\right)^{(n)} &= C_n^0\left((x+1)^2\right)^{(0)}\left(e^{2x}\right)^{(n)} + C_n^1\left((x+1)^2\right)^{(1)}\left(e^{2x}\right)^{(n-1)} + C_n^2\left((x+1)^2\right)^{(2)}\left(e^{2x}\right)^{(n-2)} + 0\\
&= 1(x+1)^2 2^n e^{2x} + 2n(x+1)2^{n-1}e^{2x} + 2\frac{n(n-1)}{2}2^{n-2}e^{2x}\\
&= 2^n(x+1)^2e^{2x} + n2^n(x+1)e^{2x} + n(n-1)2^{n-2}e^{2x}
\end{aligned}$$

5 Exercises with answers

5.1 Exercises

Exercise 1: Study the parity of the following functions:

$$f(x) = \sin(\frac{1}{x}),\ f(x) = \sqrt{1-x},\ f(x) = \tan(x),\ f(x) = \sqrt{\ln(2x)},\ f(x) = \frac{e^x}{\sqrt{\cos(x)}}.$$

Exercise 2: Calculate the following limits:

$$\lim_{x\longrightarrow 2} \frac{x^3-8}{x^2-4},\ \lim_{x\longrightarrow 1} \sin(\frac{1}{x-1}),\ \lim_{x\longrightarrow 0} \frac{\sin(3x)}{\tan(3x)},\ \lim_{x\longrightarrow 0} \frac{\sqrt{1+x}-1}{x}.$$

Exercise 3: Let f, g and h three functions defined by:

$$f(x) = \frac{1}{1-x} - \frac{3}{1-x^3},\ g(x) = x\sin(\frac{1}{x}) \text{ and } h(x) = \begin{cases} e^x - b, & \text{si } x < 0; \\ 2\cos(x), & \text{si } x > 0. \end{cases}$$

1. Study the continuity of the three functions on their domains of definition.
2. Study the existence of extension by continuity of functions f, g and h.

Exercise 4: Let f be a function defined on $\mathbb{R}^*$ by:

$$f(x) = x^2 \cos\frac{1}{x}.$$

1. Show that the function f admits an extension by continuity on $\mathbb{R}$.
2. Let $\widetilde{f}$ be the continuous extension of f on $\mathbb{R}$. Study the continuity and derivability of $\widetilde{f}$ on $\mathbb{R}$.

Exercise 5: Calculate the derivatives of the following functions:

$$f_1(x) = e^{2x+3},\ f_2(x) = e^x \sin(\frac{x}{2}),\ f_3(x) = (x^2+2)^4,\ f_4(x) = \frac{x^4}{(x^2+2)^4},$$

$$f_5(x) = \ln(\cos(x)),\ f_6(x) = \sqrt[3]{x^2 + \ln(x)} \text{ et } f_7(x) = x^{\cos(x)}.$$

Exercise 6: Determine the 10th-order derivative of the following function:

$$h(x) = (x^3+2)e^{3x}.$$

5.2 Correction

Exercise 1: Parity for the following functions:

$$f_1(x) = \sin(\frac{1}{x}),\ f_2(x) = \sqrt{1-x},\ f_3(x) = \tan(x),\ f_4(x) = \sqrt{\ln(2x)},\ f_5(x) = \frac{e^x}{\sqrt{\cos(x)}}.$$

1. $f_1(x) = \sin(\frac{1}{x})$.
   $D_{f_1} = \mathbb{R}^*$.
   For any $x \in \mathbb{R}^*$, we have $f_1(-x) = \sin(\frac{-1}{x}) = -\sin(\frac{1}{x}) = -f_1(x)$, so f_1 is an odd function.
2. $f_2(x) = \sqrt{1-x}$.
   $D_{f_2} = \{x \in \mathbb{R}/1 - x \geq 0\} = \{x \in \mathbb{R}/1 \geq x\} =]-\infty, 1]$.
   The domain of definition D_{f_2} of the function f_2 is not symmetrical about 0, so the parity of this function cannot be studied. Consequently, f_2 is neither even nor odd.
3. $f_3(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$.
   $D_{f_3} = \{x \in \mathbb{R}/\cos(x) \neq 0\} = \{x \in \mathbb{R}/x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$.
   For any $x \in D_{f_3}$, we have $f_3(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -f_3(x)$, then f_3 is an odd function.
4. $f_4(x) = \sqrt{\ln(2x)}$.
   $$\begin{aligned} D_{f_4} &= \{x \in \mathbb{R}/2x > 0 \text{ et} \ln(2x) \geq 0\} = \{x \in \mathbb{R}/x > 0 \text{ et } 2x \geq 1\} \\ &= \{x \in \mathbb{R}/x \geq \frac{1}{2}\} = [\frac{1}{2}, +\infty[. \end{aligned}$$
   f_4 is neither even nor odd, because its domain of definition is not symmetrical about 0.
5. $f_5(x) = \frac{e^x}{\sqrt{\cos(x)}}$.
   $D_{f_5} = \{x \in \mathbb{R}/\cos(x) > 0\} = \bigcup_{k\in\mathbb{Z}}]-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi[$.
   f_5 is neither even nor odd, because $\forall x \in D_{f_5}$, we have:
   $$f_5(-x) = \frac{e^{-x}}{\sqrt{\cos(-x)}} = \frac{e^{-x}}{\sqrt{\cos(x)}} \neq f_5(x) \text{ et } f_5(-x) \neq -\frac{e^x}{\sqrt{\cos(x)}} = -f_5(x).$$

Exercise 2: Calcul des limites:

$$\lim_{x\longrightarrow 2} \frac{x^3-8}{x^2-4},\ \lim_{x\longrightarrow 1} \sin(\frac{1}{x-1}),\ \lim_{x\longrightarrow 0} \frac{\sin(3x)}{\tan(3x)},\ \lim_{x\longrightarrow 0} \frac{\sqrt{1+x}-1}{x}.$$

1. $\lim\limits_{x\longrightarrow 2}\frac{x^3-8}{x^2-4}=\frac{0}{0}$ **(IF)**.

$$\lim_{x\longrightarrow 2}\frac{x^3-8}{x^2-4}=\lim_{x\longrightarrow 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}=\lim_{x\longrightarrow 2}\frac{x^2+2x+4}{x+2}=3.$$

2. $\lim\limits_{x\longrightarrow 1}\sin(\frac{1}{x-1})$ n'existe pas.
Sachant que $\lim\limits_{x\longrightarrow 1}\frac{1}{x-1}$ n'existe pas, car les limites à gauche et à droite de 1 sont différentes ($\lim\limits_{x\longrightarrow^> 1}\frac{1}{x-1}=+\infty$ et $\lim\limits_{x\longrightarrow^< 1}\frac{1}{x-1}=-\infty$), alors $\lim\limits_{x\longrightarrow 1}\sin(\frac{1}{x-1})$ n'existe pas aussi.
En particulier, pour $x\longrightarrow^> 1$, on a $\lim\limits_{x\longrightarrow^> 1}\frac{1}{x-1}=+\infty$. Si on pose $t=\frac{1}{x-1}$, alors $\lim\limits_{x\longrightarrow^> 1}\sin(\frac{1}{x-1})=\lim\limits_{t\longrightarrow+\infty}\sin(t)$ et cette limite n'existe pas, car elle n'est pas unique, voir:

$$\begin{cases}\lim\limits_{k\longrightarrow+\infty}\sin(2k\pi)=0;\\ \lim\limits_{k\longrightarrow+\infty}\sin(\frac{\pi}{2}+2k\pi)=1;\\ \lim\limits_{k\longrightarrow+\infty}\sin(\frac{-\pi}{2}+2k\pi)=-1.\end{cases}$$

3. $\lim\limits_{x\longrightarrow 0}\frac{\sin(3x)}{\tan(3x)}=\frac{0}{0}$ **(IF)**.

On a $$\lim_{x\longrightarrow 0}\frac{\sin(3x)}{\tan(3x)}=\lim_{x\longrightarrow 0}\frac{\sin(3x)}{\frac{\sin(3x)}{\cos(3x)}}=\lim_{x\longrightarrow 0}\frac{\sin(3x)\cos(3x)}{\sin(3x)}=\lim_{x\longrightarrow 0}\cos(3x)=1.$$

4. $\lim\limits_{x\longrightarrow 0}\frac{\sqrt{x+1}-1}{x}=\frac{0}{0}$ **(IF)**.

On a $$\begin{aligned}\lim_{x\longrightarrow 0}\frac{\sqrt{x+1}-1}{x}&=\lim_{x\longrightarrow 0}\frac{\sqrt{x+1}-1}{x}\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)=\lim_{x\longrightarrow 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)}\\&=\lim_{x\longrightarrow 0}\frac{x}{x(\sqrt{x+1}+1)}=\lim_{x\longrightarrow 0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{2}.\end{aligned}$$

Exercise 3: Study the continuity and extension by continuity of the functions f, g and h defined by:

$$f(x)=\frac{1}{1-x}-\frac{3}{1-x^3},\quad g(x)=x\sin(\frac{1}{x})\text{ and } h(x)=\begin{cases}e^x-b, & \text{si } x<0;\\ 2\cos(x), & \text{si } x>0.\end{cases}$$

1. $f(x)=\frac{1}{1-x}-\frac{3}{1-x^3}$.
   (a) **Domain of f**: $D_f=]-\infty,1[\cup]1,+\infty[=\mathbb{R}/\{1\}$.

(b) **Continuity of the function f on D_f:** f is continuous on $\mathbb{R}/\{1\}$ by construction (sum of two continuous functions on $\mathbb{R}/\{1\}$).

(c) **Continuous extension of f on $\mathbb{R}$ (in $x = 1$):** f can be extended by continuity at $x = 1$ if and only if $\lim_{x\longrightarrow 1} f(x)$ exists and finite.

$$\begin{aligned}\lim_{x\longrightarrow 1} f(x) &= \lim_{x\longrightarrow 1}(\frac{1}{1-x} - \frac{3}{1-x^3}) = \lim_{x\longrightarrow 1}(\frac{1}{1-x} - \frac{3}{(1-x)(x^2+x+1)}) \\ &= \lim_{x\longrightarrow 1}\frac{x^2+x+1-3}{(1-x)(x^2+x+1)} = \lim_{x\longrightarrow 1}\frac{x^2+x-2}{(1-x)(x^2+x+1)} \\ &= \lim_{x\longrightarrow 1}\frac{(x-1)(x+2)}{(1-x)(x^2+x+1)} = \lim_{x\longrightarrow 1}\frac{-(x+2)}{(x^2+x+1} = \frac{-3}{3} = -1.\end{aligned}$$

$\lim_{x\longrightarrow 1} f(x) = -1$, so f can be extended by continuity to $\mathbb{R}$. This extension, denoted $\widetilde{f}$, is defined by:

$$\widetilde{f}(x) = \begin{cases} \frac{1}{1-x} - \frac{3}{1-x^3}, & \text{for } x \neq 1; \\ -1, & \text{for } x = 1. \end{cases}$$

2. $g(x) = x\sin(\frac{1}{x})$.

(a) **Domain of g:** $D_g =]-\infty, 0[\cup]0, +\infty[= \mathbb{R}^*$.

(b) **Continuity of g on D_g:** The function g is the product and composite of continuous functions on $\mathbb{R}^*$, so g is continuous on $\mathbb{R}^*$.

(c) **Continuity extension of g on $\mathbb{R}$ (at $x = 0$):** Given that

$$\lim_{x\longrightarrow 0} g(x) = \lim_{x\longrightarrow 0} x\sin(\frac{1}{x}) = 0,$$

then g admits an extension by continuity, denoted by $\widetilde{g}$, on $\mathbb{R}$ (at $x = 0$):

$$\widetilde{g}(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

3. $h(x) = \begin{cases} e^x - b, & \text{for } x < 0; \\ 2\cos(x), & \text{for } x > 0. \end{cases}$

(a) **Domain of h:** $D_h =]-\infty, 0[\cup]0, +\infty[= \mathbb{R}^*$.

(b) **Continuity of h on D_h:** The function $e^x - b$ is continuous on $]-\infty, 0[$ and the function $2\cos(x)$ is continuous on $]0, +\infty[$, so h is continuous over $\mathbb{R}^*$.

(c) **Extension by continuity of h on $\mathbb{R}$ (at $x = 0$):** The function h can be extended by continuity to $x = 0$ if:

$$\lim_{x\longrightarrow 0} h(x) = \lim_{x\longrightarrow 0^-}(e^x - b) = \lim_{x\longrightarrow 0^+} 2\cos(x) = 2,$$

in other words $\lim_{x\longrightarrow 0^-}(e^x - b) = 1 - b = 2$, which is verified for $b = -1$. Consequently, h can be extended by continuity to $x = 0$ for $b = -1$ and this extension is :

$$\widetilde{h}(x) = \begin{cases} h(x), & \text{for } x \neq 0; \\ 2, & \text{for } x = 0. \end{cases}$$

For $b = -1$, we have : $h(x) = \begin{cases} e^x + 1, & \text{for } x < 0; \\ 2\cos(x), & \text{for } x > 0. \end{cases}$

Exercise 4: Let f be a function defined on $\mathbb{R}^*$ by:

$$f(x) = x^2 \cos(\frac{1}{x}).$$

1. Showing that the function f has an extension by continuity on $\mathbb{R}$.
   (a) $D_f = \mathbb{R}^*$;
   (b) f is the product and compound of continuous functions on $\mathbb{R}^*$, it is also continuous on $\mathbb{R}^*$.
   (c) f f can be extended by continuity on $\mathbb{R}$ and in particular at $x = 0$, because $\lim_{x\longrightarrow 0} f(x) = \lim_{x\longrightarrow 0} x^2\cos(\frac{1}{x}) = 0$.
2. The continuity extension, noted $\widetilde{f}$, of f on $\mathbb{R}$ is:

$$\widetilde{f}(x) = \begin{cases} x^2\cos(\frac{1}{x}), & \text{si } x \neq 0; \\ 0, & \text{si } x = 0. \end{cases}$$

   $\widetilde{f}$ is continuous by construction (and by definition) of an extension by continuity on $\mathbb{R}$.

Exercise 5: Calculation of the derivatives of the following functions: $f_1(x) = e^{2x+3}$, $f_2(x) = e^x\sin(\frac{x}{2})$, $f_3(x) = (x^2+2)^4$, $f_4(x) = \frac{x^4}{(x^2+2)^4}$, $f_5(x) = \ln(\cos(x))$, $f_6(x) = \sqrt[3]{x^2 + \ln(x)}$ and $f_7(x) = x^{\cos(x)}$.

1. $f_1'(x) = \left(e^{2x+3}\right)' = 2e^{2x+3}$.
2. $f_2'(x) = \left(e^x\sin(\frac{x}{2})\right)' = e^x\sin(\frac{x}{2}) + \frac{1}{2}e^x\cos(\frac{x}{2})$.
3. $f_3'(x) = \left((x^2+2)^4\right)' = 4(2x)(x^2+2)^{(4-1)} = 8x(x^2+2)^3$.
4. $f_4'(x) = \left(\frac{x^4}{(x^2+2)^4}\right)' = \frac{4x^3(x^2+2)^4 - 8x^5(x^2+2)^3}{(x^2+2)^8} = \frac{4x^3(x^2+2) - 8x^5}{(x^2+2)^5} = \frac{8x^3 - 4x^5}{(x^2+2)^5}$.
5. $f_5'(x) = \left(\ln(\cos(x))\right)' = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$.
6. $f_6'(x) = \left(\sqrt[3]{x^2+\ln(x)}\right)' = \left((x^2+\ln(x))^{\frac{1}{3}}\right)' = \frac{1}{3}(2x + \frac{1}{x})\left(x^2+\ln(x)\right)^{\frac{-2}{3}}$.

7. $f_7'(x) = \left(x^{\cos(x)}\right)' = \left(\exp^{ln(x^{\cos(x)})}\right)' = \left(\exp^{\cos(x)ln(x)}\right)'$
$f_7'(x) = \left(-\sin(x)ln(x) + \frac{\cos(x)}{x}\right)\exp^{\cos(x)ln(x)} = \left(-\sin(x)ln(x) + \frac{\cos(x)}{x}\right)x^{\cos(x)}.$

Exercise 6: Determination of the 10th order derivative of the function $h(x) = (x^3+2)e^{3x}$.
The calculation of the tenth derivative of h will be done using Leibnitz's formula:

$$h^{(10)}(x) = \sum_{p=0}^{10} C_{10}^p \left(x^3+2\right)^{(p)}\left(e^{3x}\right)^{(10-p)}.$$

$(x^3+2)^{(0)} = x^3+2$	$(e^{3x})^{(0)} = e^{3x}$	$C_n^p = \frac{n!}{(n-p)!p!}$
$(x^3+2)' = 3x^2$	$(e^{3x})' = 3e^{3x}$	$C_{10}^0 = \frac{10!}{(10-0)!0!} = 1$
$(x^3+2)" = 6x$	$(e^{3x})" = 3^2e^{3x}$	$C_{10}^1 = \frac{10!}{(10-1)!1!} = 10$
$(x^3+2)^{(3)} = 6$	$(e^{3x})^{(3)} = 3^3e^{3x}$	$C_{10}^2 = \frac{10!}{(10-2)!2!} = 45$
$(x^3+2)^{(4)} = 0$	$(e^{3x})^{(4)} = 3^4e^{3x}$	$C_{10}^3 = \frac{10!}{(10-3)!3!} = 120$
$\vdots$	$\vdots$	
$(x^3+2)^{(10)} = 0$	$(e^{3x})^{(n)} = 3^ne^{3x}$	

$$\begin{aligned}
h^{(10)}(x) &= \sum_{p=0}^{10} C_{10}^p \left(x^3+2\right)^{(p)}\left(e^{3x}\right)^{(10-p)} \\
h^{(10)}(x) &= C_{10}^0\left(x^3+2\right)\left(e^{3x}\right)^{(10)} + C_{10}^1\left(x^3+2\right)'\left(e^{3x}\right)^{(9)} + C_{10}^2\left(x^3+2\right)"\left(e^{3x}\right)^{(8)} \\
&\quad + C_{10}^3\left(x^3+2\right)^{(3)}\left(e^{3x}\right)^{(7)} + 0 \\
&= \left(1(x^3+2)3^{10}e^{3x}\right) + \left(10(3x^2)3^9e^{3x}\right) + \left(45(6x)3^8e^{3x}\right) + \left(120(6)3^7e^{3x}\right) \\
&= e^{3x}\left[3^{10}(x^3+2) + 3^9(30x^2) + 3^8(270x) + 3^7(720)\right].
\end{aligned}$$

Chapter 2

Integration of a functions

1 Antiderivatives (primitives) and Integrals

First, let's recall the definition of an antiderivative (a primitive).

Definition 2.1. A primitive of a function f, defined on an interval $]a, b[$, is any function F derivable on $]a, b[$, whose derivative coincides with f on $]a, b[$.

$$F^{'}(x) = f(x),\ \forall x \in]a, b[.$$

Any continuous function on an interval I has a primitive on this interval.

Remark 2.1. If F is a primitive of the function $f : I \to \mathbb{R}$, any primitive G of f on I is of the form

$$\begin{aligned} G\ :\ & I \to \mathbb{R} \\ & x \mapsto G(x) = F(x) + c, \end{aligned}$$

where c is a real constant.

Two primitives of the same function differ by one real constant, whose derivative is zero.

Theorem 2.1. *Let f be a continuous function on $[a, b]$, and c a point on the interval $[a, b]$. Consider the function $F_c(x)$, which to $x \in [a, b]$ associates $F_c(x) = \int_c^x f(t)dt$. Then F_c is the only primitive of f that cancels at the point c.*

Any primitive F of f can be used to calculate an integral.

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Remark 2.2. If F is a primitive of f, we write

$$\int f(x)dx = F(x) + c,$$

where c is a real constant.

Example 2.1. The primitives of $\cos(x)$ are all functions of the form $\sin(x) + c$, where c is a real constant.

$$\int \cos(x)\mathrm{d}x = \sin(x) + c.$$

1.1 Primitives of usual functions

Function	Primitive	Area of validity
a (real number)	$ax + c,\ c \in \mathbb{R}$	$\mathbb{R}$
$x^n,\ n \in \mathbb{N}$	$\frac{x^{n+1}}{n+1} + c,\ c \in \mathbb{R}$	$\mathbb{R}$
$\frac{1}{x^n},\ n \in \mathbb{N}^*/\{1\}$	$\frac{-1}{(n-1)x^{n-1}} + c,\ c \in \mathbb{R}$	$\mathbb{R}^*$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + c,\ c \in \mathbb{R}$	$\mathbb{R}_+^*$
$x^\alpha,\ \alpha \in \mathbb{R}/\mathbb{Z}$	$\frac{x^{\alpha+1}}{\alpha+1} + c,\ c \in \mathbb{R}$	$\mathbb{R}_+^*$
$\frac{1}{x}$	$\ln(\lvert x\rvert) + c,\ c \in \mathbb{R}$	$\mathbb{R}^*$
e^x	$e^x + c,\ c \in \mathbb{R}$	$\mathbb{R}$
$\cos(x)$	$\sin(x) + c,\ c \in \mathbb{R}$	$\mathbb{R}$
$\sin(x)$	$-\cos(x) + c,\ c \in \mathbb{R}$	$\mathbb{R}$
$\frac{1}{\cos^2(x)} = 1 + tan^2(x)$	$tan(x) + c,\ c \in \mathbb{R}$	$\mathbb{R}/\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$
$\frac{1}{1+x^2}$	$\arctan(x) + c,\ c \in \mathbb{R}$	$\mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + c,\ c \in \mathbb{R}$	$]-1, 1[$
$\frac{-1}{\sqrt{1-x^2}}$	$\arccos(x) + c,\ c \in \mathbb{R}$	$]-1, 1[$
$a^x,\ a \in \mathbb{R}_+^*/\{1\}$	$\frac{a^x}{\ln(a)} + c,\ c \in \mathbb{R}$	$\mathbb{R}$

1. If f and g are two functions defined on an interval I and respectively admitting F and G as primitives on I and λ any real, then:

   - $F + G$ is a primitive of the function $f + g$ on I.
   - λF is a primitive of the function λf on I.

2. Otherwise, we have the following table in which f designates a function derivable on an interval I whose derivative f' is continuous on I.
   Recall that, if f is a function derivable on an interval I and g is derivable on $f(I)$, $gof(x)$ is a primitive of $f'(x)\big(g'(f(x))\big)$ on I.

Function	**Primitive (antiderivative)**
$f^{'}(x)f^n(x)$, $n \in \mathbb{R}/\{-1\}$	$\frac{f^{n+1}(x)}{n+1} + c$, $c \in \mathbb{R}$
$\frac{f^{'}(x)}{\sqrt{f(x)}}$, $f(x) > 0$	$2\sqrt{f(x)} + c$, $c \in \mathbb{R}$
$\frac{f^{'}(x)}{f(x)}$, $f(x) \neq 0$	$\ln(\lvert f(x)\rvert) + c$, $c \in \mathbb{R}$
$f^{'}(x)e^{f(x)}$	$e^{f(x)} + c$, $c \in \mathbb{R}$
$f^{'}(x)\cos(f(x))$	$\sin(f(x)) + c$, $c \in \mathbb{R}$
$f^{'}(x)\sin(f(x))$	$-\cos(f(x)) + c$, $c \in \mathbb{R}$
$\frac{f^{'}(x)}{\cos^2(f(x))} = \frac{f^{'}(x)}{(1+\tan^2(f(x)))}$	$\tan(x) + c$, $c \in \mathbb{R}$
$\frac{f^{'}(x)}{1+(f(x))^2}$	$\arctan(x) + c$, $c \in \mathbb{R}$
$\frac{f^{'}(x)}{\sqrt{1-(f(x))^2}}$	$\arcsin(x) + c$, $c \in \mathbb{R}$

Example 2.2. Calculation the antiderivatives (primitives) of the two functions:

$$f_1(x) = \frac{x}{(x^2+1)^3} \text{ et } f_2(x) = \frac{x}{(x^2+1)}$$

1. $\int f_1(x)dx = \int \frac{x}{(x^2+1)^3}dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^3}dx$
   Knowing that $(x^2+1)^{'} = 2x$, then the primitive of $f_1(x) = \frac{1}{2}\frac{(x^2+1)^{'}}{(x^2+1)^3}$ will be calculated using the primitive of $f_1(x) = \frac{1}{2}(f(x))^{'}(f(x))^n$ with $f(x) = x^2+1$ et $n = -3$.
   So the primitive $F_1(x) = \int f_1(x)dx$, can be deduced as follows
   $$F_1(x) = \frac{1}{2}\frac{(f(x))^{n+1}}{n+1} + c = \frac{1}{2}\frac{(x^2+1)^{-3+1}}{-3+1} + c = \frac{1}{2}\frac{(x^2+1)^{-2}}{-2} + c = \frac{-1}{4(x^2+1)^2} + c,$$
   with c is a real constant.

2. $\int f_2(x)dx = \int \frac{x}{(x^2+1)}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx$
   In the same way, Knowing that $(x^2+1)^{'} = 2x$, then the primitive of $f_2(x)$ is
   $$F_2(x) = \frac{1}{2}\ln(\mid x^2+1 \mid) + c = \frac{1}{2}\ln(x^2+1) + c,$$
   where c is a real constant.
   (The usual primitive used is $\int \frac{f^{'}(x)}{f(x)}dx = \ln(\lvert f(x)\rvert) + c$)

1.2 Integrals

Any continuous function on an interval $[a, b]$ is integrable on $[a, b]$.

1.3 Properties of definite integrals

1. If f is continuous on an interval I, then for any real a, b and c of I (with $a < b < c$), we have:
   - $\int_a^a f(x)dx = 0$;
   - $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
   - $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.
2. If f and g are integrable on $[a, b]$, then $(f + g)$ is integrable on $[a, b]$ and:
$$\int_a^b \big(f(x) + g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$
3. If f is intégrable on $[a, b]$ and λ a real number, then (λf) is integrable on $[a, b]$ with
$$\int_a^b (\lambda f(x))dx = \lambda \int_a^b f(x)dx.$$
4. If f is integrable on $[a, b]$ and for any x belonging to $[a, b]$ we have $f(x) \geq 0$, then
$$\int_a^b f(x)dx \geq 0.$$
5. If f and g are two integrable functions on $[a, b]$ and for any x belonging to $[a, b]$ we have $f(x) \geq g(x)$, then:
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

1.4 Application of integrals to area calculation

If f is a continuous, positive real function taking its values in a segment $I = [a, b]$, then the integral of f on I, denoted by
$$\int_a^b f(x)dx,$$
is the area of a surface bounded by the graphical representation of f and by the three straight lines of equation $x = a$, $x = b$ and $y = 0$ (see the blue-colored surface in figure

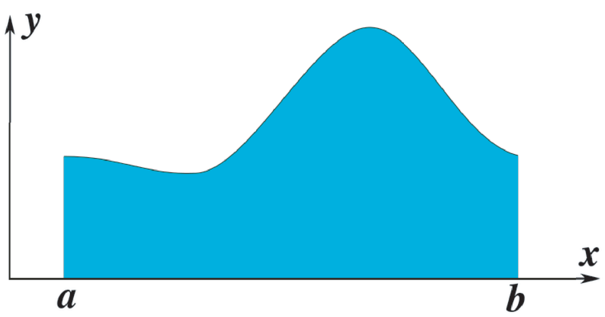

Figure 2.1: The area of the blue zone is $S_f = \int_a^b f(x)dx$, when f is positive.

2.1).

The area of surfaces above the x-axis is given a positive sign. To be able to deal with negative functions, we give a negative sign to the areas below this axis.

So, to define the area of the surface bounded by the curve of f and by the three straight lines of equation $x = a$, $x = b$ and $y = 0$ (see figure 2.2), we cut the integral over the different parts where the function is positive and negative, but don't forget to put a minus sign in front of it when the function is negative. The area of this surface is given by this integral:

$$S_f = \int_{x\in[a,b]} f^+(x)dx - \int_{x\in[a,b]} f^-(x)dx,$$

with $f^+ = \begin{cases} f(x), & \text{for } f(x) > 0; \\ 0, & \text{otherwise.} \end{cases}$ et $f^- = \begin{cases} -f(x), & \text{for } f(x) < 0; \\ 0, & \text{otherwise.} \end{cases}$

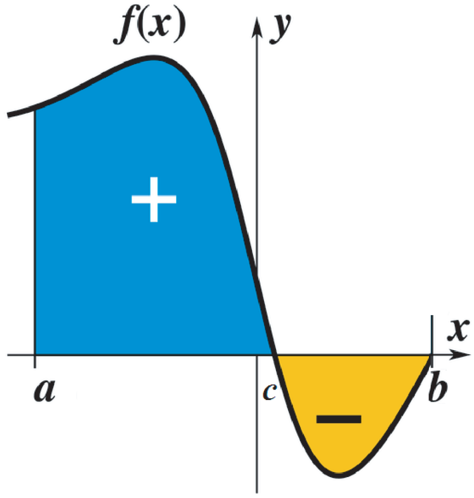

Figure 2.2: The surface area is $S_f = \int_a^c f(x)dx - \int_c^b f(x)dx$.

2 Integration methods

In this section, we give some integration methods that allow us to transform the problem of calculating an integral into the calculation of another, simpler integral.

2.1 Formula for integration by parts

Definition 2.2. Let f be a function defined on an interval I of $\mathbb{R}$ with values in $\mathbb{R}$. We say that f is of class C^1 on I if and only if f is derivable on I and its derivative f' is continuous on I.

Let U and V be two functions of class C^1 on an interval $[a,b]$ with values in $\mathbb{R}$.
By hypothesis, the functions U and V are derivable on $[a,b]$, and so is the function UV:

$$(UV)' = U'V + UV'.$$

On intègre les deux membres de cette équation sur le segment $[a,b]$ et on obtient par linéarité de l'intégrale:

$$\int_a^b U'(x)V(x)dx + \int_a^b U(x)V'(x)dx = \int_a^b \Big(U(x)V(x)\Big)' dx = \Big[U(x)V(x)\Big]_a^b.$$

Theorem 2.2. *Let a and b be two real numbers such that $a < b$, U and V be two functions of class C^1 on $[a,b]$ with values in $\mathbb{R}$. Then*

$$\int_a^b U'(x)V(x)dx = \Big[U(x)V(x)\Big]_a^b - \int_a^b U(x)V'(x)dx.$$

Example 2.3. Calculate $I = \int_0^1 xe^x$.
For $x \in [0,1]$, we put

$$U'(x) = e^x \quad \Rightarrow \quad U(x) = e^x$$

$$V(x) = x \quad \Rightarrow \quad V'(x) = 1$$

The two functions U and V are of class C^1 on $[0,1]$, so we can integrate by parts.

$$\begin{aligned}\int_0^1 xe^x &= \int_0^1 U'(x)V(x)dx = \Big[U(x)V(x)\Big]_0^1 - \int_0^1 U(x)V'(x)dx \\ &= \big[xe^x\big]_0^1 - \int_0^1 1e^x dx = e - 0 - \int_0^1 e^x dx \\ &= e - \big[e^x\big]_0^1 = e - (e-1) = 1.\end{aligned}$$

So $\int_0^1 xe^x = 1$.

Remark 2.3. Integration by parts is also very useful for finding primitives.

$$\int U^{'}(x)V(x)dx = U(x)V(x) - \int U(x)V^{'}(x)dx$$

Example 2.4. Let's calculate the primitives of the function $f(x) = \ln(x)$ on $]0, +\infty[$, in other words the indefinite integral $\int \ln(x)dx$.
We have $\int \ln(x)dx = \int 1\ln(x)dx$.

Integration by parts:
$$U^{'}(x) = 1 \quad \Rightarrow \quad U(x) = x$$
$$V(x) = \ln(x) \quad \Rightarrow \quad V^{'}(x) = \frac{1}{x}$$

$$\begin{aligned}\int \ln(x)dx &= \int 1\ln(x)dx = x\ln(x) - \int (x \times \frac{1}{x})dx \\ &= x\ln(x) - \int 1dx = x\ln(x) - x + c.\end{aligned}$$

2.2 Method for integration by substitution

The principle is the same as integration by parts, to reduce an integral (difficult or complicated if you like) to an integral that is simpler to calculate.

Theorem 2.3. *Let φ be an application of class C^1 on an interval I with values in $\mathbb{R}$. Let f be a continuous function on $\varphi(I)$ with values in $\mathbb{R}$. Then, for all $(a, b) \in I \times I$,*

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f(\varphi(t))\varphi^{'}(t)dt,$$

where $x = \varphi(t)$ and $dx = \varphi^{'}(t)dt$.

Example 2.5. Calculating the integrals: $I_1 = \int_0^{\frac{\pi}{2}} \sin(x)\cos(x)dx$ et $I_2 = \int_0^1 \frac{2x}{x^2+1}dx$.

1. $I_1 = \int_0^{\frac{\pi}{2}} \sin(x)\cos(x)dx$

   Given $t = \sin(x)$, then $dt = cos(x)dx$ and
   $$x = 0 \quad \Rightarrow \quad t = \sin(0) = 0,$$
   $$x = \frac{\pi}{2} \quad \Rightarrow \quad t = \sin(\frac{\pi}{2}) = 1.$$

   By changing the variable, we have

   $$\begin{aligned}I_1 &= \int_0^{\frac{\pi}{2}} \sin(x)\cos(x)dx = \int_0^1 tdt = \Big[\frac{1}{2}t^2\Big]_0^1 \\ &= \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}.\end{aligned}$$

2. $I_2 = \int_0^1 \frac{2x}{x^2+1}dx$

   Posing $t = x^2 + 1$, then $dt = 2xdx$ and $\begin{array}{lll} x=0 & \Rightarrow & t = 0^2+1 = 1, \\ x = 1 & \Rightarrow & t = 1^2+1 = 2. \end{array}$

   After the change of variable, we have

$$\begin{aligned} I_2 &= \int_0^1 \frac{2x}{x^2+1}dx = \int_1^2 \frac{1}{t}dt = \big[\ln(t)\big]_1^2 \\ &= \ln(2) - \ln(1) = \ln(2) - 0. \end{aligned}$$

   So $\int_0^1 \frac{2x}{x^2+1}dx = \ln(2)$.

Example 2.6. Calculate $\int_0^1 \frac{1}{1+e^x}$.

We have

$$\begin{aligned} \int_0^1 \frac{1}{1+e^x}dx &= \int_0^1 \frac{1+e^x-e^x}{1+e^x}dx = \int_0^1 (\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x})dx \\ &= \int_0^1 (1 - \frac{e^x}{1+e^x})dx = \int_0^1 1dx - \int_0^1 \frac{e^x}{1+e^x}dx \end{aligned}$$

$\int_0^1 \frac{1}{1+e^x}$ is the sum of two integrals, where $\int_0^1 1dx = \big[x\big]_0^1 = 1$ and $\int_0^1 \frac{e^x}{1+e^x}dx$ which can be calculated using a change of variable, by posing $t = e^x$ with $dt = e^x dx$ (for $x = 0$ we have $t = e^0 = 1$ and for $x = 1$ we have $t = e^1 = e$):

$$\int_0^1 \frac{e^x}{1+e^x}dx = \int_1^e \frac{1}{1+t}dt = \big[\ln(1+t)\big]_1^e = \ln(1+e) - \ln(2).$$

Then

$$\int_0^1 \frac{1}{1+e^x}dx = \int_0^1 1dx - \int_0^1 \frac{e^x}{1+e^x}dx = 1 - \ln(1+e) + \ln(2).$$

Remark 2.4. The change of variable is also useful for finding primitives.

Example 2.7. Let's go back to the 2.5 example and look at the primitives of functions: $f_1(x) = \sin(x)\cos(x)$ et $f_2(x) = \frac{2x}{x^2+1}$.

1. $F_1(x) = \int \sin(x)\cos(x)dx$

   By changing the variable $t = \sin(x)$ avec $dt = \cos(x)dx$, we get

$$\begin{aligned} F_1(x) &= \int \sin(x)\cos(x)dx = \int tdt = \frac{1}{2}t^2 + c \\ &= \frac{1}{2}\sin^2(x) + c, \end{aligned}$$

   where c is a real constant.

   Hence the antiderivative of the function $f_1(x) = \sin(x)\cos(x)$ is $F_1(x) = \frac{1}{2}\sin^2(x) + c$.

2. $F_2(x) = \int \frac{2x}{x^2+1} dx$
   After making the variable change $t = x^2 + 1$ where $dt = 2xdx$, we obtain

$$\begin{aligned} F_2(x) &= \int \frac{2x}{x^2+1} dx = \int \frac{1}{t} dt = \ln(\mid t \mid) + c \\ &= \ln(\mid x^2 + 1 \mid) + c = \ln(x^2 + 1) + c. \end{aligned}$$

   So the primitive of $f_2(x) = \frac{2x}{x^2+1}$ is $F_2(x) = \ln(x^2 + 1) + c$.

2.3 Decomposition into simple elements

The decomposition into simple elements (sometimes called decomposition into partial fractions) is not a technique specific to integral calculus. It allows to decompose a rational fraction of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are two polynomials with $Q(x) \neq 0$, into a sum of elementary fractions that we know how to integrate.

Integration of a simple element of the first kind:

The general form of a simple element of the first kind is as follows (with k and a real constants): $\frac{k}{(x-a)^n}$.
So you need to know how to integrate fractions of the following form: $\frac{1}{(x-a)^n}$.
In the case where $n = 1$ the primitive is:

$$\int \frac{1}{(x-a)} dx = \ln(\mid x - a \mid) + c,$$

where c is a real constant.

Remark 2.5. In this case the function to be integrated is simply in the form $\frac{U'}{U}$ with $U \neq 0$, hence direct integration.

In the case where $n > 1$ the primitive obtained is:

$$\int \frac{1}{(x-a)^n} = \frac{1}{1-n} . \frac{1}{(x-a)^{n-1}} + c.$$

Remark 2.6. In this case the function to be integrated is of the form $\frac{U'}{U^n}$ with $U \neq 0$ non-zero, hence direct integration.
Generally, the integration of a simple element of the first kind is immediate.

Integration of a simple element of the second kind:

The general form of a simple element of the second kind is as follows:

$$\frac{px+q}{(ax^2+bx+c)^n},$$

with p, q, a, b and c real constants.
It can always be broken down into two fractions:

$$\frac{px+q}{(ax^2+bx+c)^n} = p\frac{x}{(ax^2+bx+c)^n} + \frac{q}{(ax^2+bx+c)^n}.$$

After the change of variable, we return to the calculation of some primitives that we need to know:

1. $\int \frac{1}{x^2+1}dx = \arctan(x) + c$;
2. $\int \frac{1}{(x+a)^2+b^2}dx = \frac{1}{b}\arctan(\frac{x+a}{b}) + c$, with $b \neq 0$;
3. $\int \frac{f'(x)f(x)}{f^2(x)+1}dx = \frac{1}{2}\ln(f^2(x)+1) + c$, in particular $\int \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1) + c$.

Example 2.8. Calculating primitives:

$$R_1(x) = \frac{1}{x^2-x-2},\quad R_2(x) = \frac{x^2}{x+1},\quad R_3(x) = \frac{x^3+5x^2+8x}{x^2+5x^2+6},\quad R_4(x) = \frac{x^2-2}{(x^2+1)(x-1)}$$

1. $R_1(x) = \frac{1}{x^2-x-2}$
   The denominator of $R_1(x)$, $x^2-x-2=0$ for $x=-1$ and $x=2$, so its decomposition into simple elements has two simple elements of the first kind:

   $$R_1(x) = \frac{1}{x^2-x-2} = \frac{1}{(x+1)(x-2)} = \frac{a}{(x+1)} + \frac{b}{(x-2)},$$

   with a and b are two real constants that we now need to determine.
   We could very well put the decomposition to the same denominator as the original form of $R_1(x)$ and then deduce a and b by identification by solving a system with two equations and two unknowns.

   $$\frac{1}{(x+1)(x-2)} = \frac{a}{(x+1)} + \frac{b}{(x-2)} = \frac{a(x-2)+b(x+1)}{(x+1)(x-2)} = \frac{(a+b)x+(b-2a)}{(x+1)(x-2)}$$

   So

   $$\begin{cases} a+b=0, \\ b-2a=1, \end{cases} \Rightarrow \begin{cases} a=\frac{-1}{3}, \\ b=\frac{1}{3}, \end{cases} \quad \text{then } R_1(x) = \frac{1}{x^2-x-1} = \frac{-1}{3(x+1)} + \frac{1}{3(x-2)}.$$

The integration of simple elements of the first kind being immediate, we deduce a primitive of $R_1(x)$:

$$\begin{aligned}\int R_1(x)dx &= \int \frac{1}{x^2-x-1}dx = \int \Big(\frac{-1}{3(x+1)} + \frac{1}{3(x-2)}\Big)dx \\ &= \frac{-1}{3}\int \frac{1}{(x+1)}dx + \frac{1}{3}\int \frac{1}{(x-2)}dx \\ &= \frac{-1}{3}\ln(|\ x+1\ |) + \frac{1}{3}\ln(|\ x-2\ |) + c = \frac{1}{3}\ln(|\ \frac{x-2}{x+1}\ |) + c.\end{aligned}$$

2. $R_2(x) = \frac{x^2}{x+1}$
   Reducing to a simple and easy integral, we have:

$$\begin{aligned}R_2(x) &= \frac{x^2}{x+1} = \frac{x^2-1+1}{x+1} = \frac{(x+1)(x-1)+1}{x+1} \quad \text{(sachant que } x^2-1=(x+1)(x-1)\text{)} \\ &= \frac{(x+1)(x-1)}{x+1} + \frac{1}{x+1} = x-1+\frac{1}{x+1}.\end{aligned}$$

   The integration of $R_2(x)$ is:

$$\int R_2(x)dx = \int (x-1)dx + \int \frac{dx}{x+1}dx = \frac{x^2}{2} - x + \ln(|\ x+1\ |) + c.$$

   Example of a numerical application:

$$\int_0^1 \frac{x^2}{x+1}dx = [\frac{x^2}{2} - x + \ln(|\ x+1\ |)]_0^1 = \ln(2) - \frac{1}{2}.$$

3. $R_3(x) = \frac{x^3+5x^2+8x}{x^2+5x^2+6}$

$$\begin{aligned}R_3(x) &= \frac{x^3+5x^2+8x}{x^2+5x^2+6} = \frac{x(x^2+5x+6)+2x}{x^2+5x^2+6} \\ &= x + \frac{2x}{x^2+5x^2+6} = x + \frac{2x}{(x+2)(x+3)} = x + \frac{a}{(x+2)} + \frac{b}{(x+3)}\end{aligned}$$

   By reduction to the same denominator and identification we have: $a = -4$ et $b = 6$, therefore

$$R_3(x) = \frac{x^3+5x^2+8x}{x^2+5x^2+6} = x - \frac{4}{(x+2)} + \frac{6}{(x+3)}$$

   The antiderivative of $R_3(x)$ is:

$$\int R_3(x)dx = \int xdx - 4\int \frac{dx}{x+2}dx + 6\int \frac{dx}{x+3} = \frac{x^2}{2} - 4\ln(|\ x+2\ |) + 6\ln(|\ x+3\ |) + c.$$

4. $R_4(x) = \frac{x^2-2}{(x^2+1)(x-1)}$

$$R_4(x) = \frac{x^2-2}{(x^2+1)(x-1)} = \frac{ax+b}{(x^2+1)} + \frac{c}{(x-1)}.$$

By reduction to the same denominator and identification we have:

$$R_4(x) = \frac{3(x+1)}{2(x^2+1)} - \frac{1}{2(x-1)} = \frac{3x}{2(x^2+1)} + \frac{3}{(2(x^2+1)} - \frac{1}{2(x-1)}$$

The primitive of $R_4(x)$ is:

$$\begin{aligned} \int R_4(x)dx &= \frac{3}{2}\int \frac{x}{(x^2+1)}dx + \frac{3}{2}\int \frac{1}{(x^2+1)}dx - \int \frac{1}{2(x-1)}dx \\ &= \frac{3}{4}\ln(x^2+1) + \frac{3}{2}\arctan(x) - \frac{1}{2}\ln(\mid x-1 \mid) + constant. \end{aligned}$$

3 Integrals of type $\int R(sin(x), cos(x))dx$

Reminder:
$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$;
$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$;
$\cos^2(a) + \sin^2(a) = 1$.
Using $\sin(a+b)$ and $\cos(a+b)$, we deduce:

1. $\sin(x)$ and $\cos(x)$ depending on $\sin(\frac{x}{2})$ and $\cos(\frac{x}{2})$:

$$\begin{aligned} \sin(x) &= \sin(\frac{x}{2}+\frac{x}{2}) = \sin(\frac{x}{2})\cos(\frac{x}{2}) + \cos(\frac{x}{2})\sin(\frac{x}{2}) \\ &= 2\sin(\frac{x}{2})\cos(\frac{x}{2}); \\ \cos(x) &= \cos(\frac{x}{2}+\frac{x}{2}) = \cos(\frac{x}{2})\cos(\frac{x}{2}) - \sin(\frac{x}{2})\sin(\frac{x}{2}) \\ &= \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}). \end{aligned}$$

2. $\sin(x)$ and $\cos(x)$ depending on $\tan(\frac{x}{2})$:

   (a) $\sin(x) = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}$
   Indeed,

$$\sin(x) = \frac{\sin(x)}{1} = \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})}{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})} = \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})/\cos^2(\frac{x}{2})}{\left(\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})\right)/\cos^2(\frac{x}{2})} = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}.$$

(b) $\cos(x) = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}$ We have

$$\cos(x) = \frac{\cos(x)}{1} = \frac{\cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})}{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})} = \frac{\left(\cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})\right)/\cos^2(\frac{x}{2})}{\left(\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})\right)/\cos^2(\frac{x}{2})} = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}.$$

3. If we set $t = \tan(\frac{x}{2})$, we will have:

$$\begin{aligned} \sin(x) &= \frac{2t}{1+t^2} \\ \cos(x) &= \frac{1-t^2}{1+t^2} \\ dt &= \frac{1}{2}(1+\tan^2(\frac{x}{2}))dx = \frac{1}{2}(1+t^2)dx \Rightarrow dx = \frac{2dt}{1+t^2}. \end{aligned}$$

Example 2.9. Let's calculate the primitives:

$$\int \frac{1}{\sin(x)}dx \quad \text{and} \quad \int \frac{1}{1+\sin(x)+\cos(x)}dx.$$

1. $F_1(x) = \int \frac{1}{\sin(x)}dx$
   Calculating this primitive becomes very easy, using the change of variable $t = \tan(\frac{x}{2})$, $dx = \frac{2dt}{1+t^2}$ and

$$\sin(x) = \frac{2t}{1+t^2} \Rightarrow \frac{1}{\sin(x)} = \frac{1+t^2}{2t}.$$

$$\begin{aligned} F_1(x) &= \int \frac{1}{\sin(x)}dx = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}}dt = \int \frac{2(1+t^2)}{2t(1+t^2)}dt = \int \frac{2}{2t}dt \\ &= \int \frac{1}{t}dt = \ln(\mid t \mid) + c = \ln(\mid \tan(\frac{x}{2}) \mid) + c, \ c \in \mathbb{R}. \end{aligned}$$

2. $F_2(x) = \int \frac{1}{1+\sin(x)+\cos(x)}dx$
   This primitive is very simple to calculate with the change of variable $t = \tan(\frac{x}{2})$, with

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad dx = \frac{2dt}{1+t^2}.$$

$$\begin{aligned} F_2(x) &= \int \frac{1}{1+\sin(x)+\cos(x)}dx = \int \frac{\frac{2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}dt = \int \frac{2}{1+t^2+2t+1-t^2}dt \\ &= \int \frac{2}{2+2t}dt = \int \frac{1}{1+t}dt = \ln(\mid 1+t \mid) + c = \ln(\mid 1+\tan(\frac{x}{2}) \mid) + c, \ c \in \mathbb{R}. \end{aligned}$$

4 Improper integrals

Case of a function f locally integrable on a semi-open interval $[a, b[$ (respectively $]a, b]$), with $b \in \mathbb{R}$ ou $b = +\infty$ (respectively $a \in \mathbb{R}$ or $a = -\infty$).

Definition 2.3. We say that the integral $\int_a^b f(x)dx$ is convergent (or exists) if the function $t \mapsto \int_a^t f(x)dx$ has a limit (in the sense of a finite limit) when t tends to b. We then state:

$$\int_a^b f(x)dx = \lim_{t \to b} \int_a^t f(x)dx.$$

This real number is called the improper integral of f on $[a, b[$.
If the function $t \mapsto \int_a^t f(x)dx$ has no limit when t tends to b, we say that the integral $\int_a^b f(x)dx$ is divergent.

Example 2.10. Calculating improper integrals

$$I_1 = \int_0^{+\infty} e^{-x}dx,\ I_2 = \int_0^1 \ln(x)dx \text{ and} I_3 = \int_1^{+\infty} \ln(x)dx$$

1. $I_1 = \int_0^{+\infty} e^{-x}dx$

$$\begin{aligned} I_1 &= \int_0^{+\infty} e^{-x}dx = \lim_{b \to +\infty} \int_0^b e^{-x}dx = \lim_{b \to +\infty} \big[-e^{-x}\big]_0^b \\ &= \lim_{b \to +\infty} (-e^{-b} + 1) = 1. \end{aligned}$$

   Hence the integral I_1 is convergent and equals 1.

2. $I_2 = \int_0^1 \ln(x)dx$
   First, let's recall the primitive of the function $\ln(x)$: $\int \ln(x)dx = x\ln(x) - x + c$.

$$\begin{aligned} I_2 &= \int_0^1 \ln(x)dx = \lim_{a \to 0^+} \int_a^1 \ln(x)dx = \lim_{a \to 0^+} \big[x\ln(x) - x\big]_a^1 \\ &= \lim_{a \to 0^+} (0 - 1 - a\ln(a) + a) = -1. \end{aligned}$$

   The integral I_2 is therefore convergent and equals -1.

3. $I_3 = \int_1^{+\infty} \ln(x)dx$

$$\begin{aligned} I_3 &= \int_1^{+\infty} \ln(x)dx = \lim_{b \to +\infty} \int_1^b \ln(x)dx = \lim_{b \to +\infty} \big[x\ln(x) - x\big]_1^b \\ &= \lim_{b \to +\infty} (b\ln(b) - b - 0 + 1) = \lim_{b \to +\infty} b(\ln(b) - 1) + 1 = +\infty. \end{aligned}$$

   The integral I_3 is divergent.

4. $I_4 = \int_{-1}^{0} \frac{1}{1+x} dx$

$$\begin{aligned} I_4 &= \int_{-1}^{0} \frac{1}{1+x} dx = \lim_{a \to -1} \int_{a}^{0} \frac{1}{1+x} dx = \lim_{a \to -1} \big[\ln(\mid 1+x \mid) \big]_a^0 \\ &= \lim_{a \to -1} \Big(\ln(1) - \ln(\mid 1+a \mid) \Big) = \lim_{a \to -1} \Big(-\ln(\mid 1+a \mid) \Big) = +\infty. \end{aligned}$$

So I_4 is divergent.

5 Corrected exercises

5.1 Exercises

Exercise 1: Calculate the following integrals:

$$I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3(x)}{\sin^4(x)}dx, \qquad I_2 = \int_2^4 \frac{1}{x(\ln(x))^3}dx, \qquad I_3 = \int_0^1 e^{\sqrt{x+1}}dx,$$

$$I_4 = \int_1^2 \frac{\ln(1+2x)}{x^2}dx, \qquad I_5 = \int_1^2 \frac{x^2}{x^2-3x-4}dx, \qquad I_6 = \int_0^\pi e^x sin(x)dx.$$

Exercise 2: Calculate the primitives of the following functions:

$$\int (\ln(x))^2 dx, \quad \int \frac{\sin(x)}{1+\sin(x)}dx, \quad \int e^{2x}\ln(1+e^x)dx, \quad \int \frac{1}{1-\sqrt{x+2}}dx.$$

Exercise 3: Compute the following improper integrals:

$$\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx, \quad \int_0^{+\infty} \frac{1}{1+e^x}dx, \quad \int_1^{+\infty} \frac{\ln(x)}{x^2}.$$

Exercise 4: For all $n \in \mathbb{N}$, we put

$$I_n = \int_0^1 x^n e^{2x} dx.$$

1. Calculate I_0.
2. Show that for all $n \in \mathbb{N}$, we have

$$I_{n+1} = \frac{e^2}{2} - \frac{n+1}{2} I_n.$$

3. Deduce the value of I_2.

5.2 Corrections

Exercise 1: Calculating integrals:

$$I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3(x)}{\sin^4(x)}dx, \qquad I_2 = \int_2^4 \frac{1}{x(\ln(x))^3}dx, \qquad I_3 = \int_0^1 e^{\sqrt{x+1}}dx,$$

$$I_4 = \int_1^2 \frac{\ln(1+2x)}{x^2}dx, \qquad I_5 = \int_1^2 \frac{x^2}{x^2-3x-4}dx, \qquad I_6 = \int_0^\pi e^x sin(x)dx.$$

1. $I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3(x)}{\sin^4(x)}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\sin^2(x)}{\sin^4(x)}\cos(x)dx$

   We put $t = \sin(x)$ et $dt = \cos(x)dx$ and $\begin{array}{l} x = \frac{\pi}{6} \quad \Rightarrow \quad t = \sin(\frac{\pi}{6}) = \frac{1}{2}, \\ x = \frac{\pi}{2} \quad \Rightarrow \quad t = \sin(\frac{\pi}{2}) = 1. \end{array}$

   With this change of variable, the calculation of the integral I_1, will be:

$$\begin{aligned} I_1 &= \int_{\frac{1}{2}}^{1} \frac{1-t^2}{t^4}dt = \int_{\frac{1}{2}}^{1} \frac{1}{t^4}dt - \int_{\frac{1}{2}}^{1} \frac{1}{t^2}dt \\ &= [\frac{-1}{3t^3}]_{\frac{1}{2}}^{1} - [\frac{-1}{t}]_{\frac{1}{2}}^{1} = (-\frac{1}{3} + \frac{8}{3}) - (1-2) = \frac{4}{3}. \end{aligned}$$

2. $I_2 = \int_2^4 \frac{1}{x(\ln(x))^3}dx$

   Posing $t = \ln(x)$ then $dt = \frac{1}{x}dx$ and $\begin{array}{l} x = 2 \quad \Rightarrow \quad t = \ln(2), \\ x = 4 \quad \Rightarrow \quad t = \ln(4). \end{array}$

   Hence

$$\begin{aligned} I_2 &= \int_{\ln(2)}^{\ln(4)} \frac{1}{t^3}dt = [\frac{-1}{2t^2}]_{\ln(2)}^{\ln(4)} \\ &= \frac{-1}{2\big(\ln(4)\big)^2} - \frac{-1}{2\big(\ln(2)\big)^2} = \frac{3}{8\ln^2(x)}. \end{aligned}$$

3. $I_3 = \int_0^1 e^{\sqrt{x+1}}dx$

   Change of variable: $t = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$ et $dt = \frac{1}{2\sqrt{x+1}}dx$.
   From the derivative dt we deduce $dx = 2\sqrt{x+1}dt = 2tdt$. So

$$I_3 = \int_0^1 e^{\sqrt{x+1}}dx = \int_1^{\sqrt{2}} 2te^t dt$$

   Integration by parts: $\begin{array}{l} U = 2t \quad \Rightarrow \quad U' = 2, \\ V' = e^t \quad \Rightarrow \quad V = e^t. \end{array}$

$$\begin{aligned} I_3 &= [U(t)V(t)]_1^{\sqrt{2}} - \int_1^{\sqrt{2}} U'(t)V(t)dt = [2te^t]_1^{\sqrt{2}} - 2\int_1^{\sqrt{2}} e^t dt \\ &= 2[te^t - e^t]_1^{\sqrt{2}} = 2[e^t(t-1)]_1^{\sqrt{2}} = 2e^{\sqrt{2}}(\sqrt{2}-1). \end{aligned}$$

4. $I_4 = \int_1^2 \frac{\ln(1+2x)}{x^2} dx$

Integration by parts:
$$U = \ln(1+2x) \quad \Rightarrow \quad U' = \tfrac{2}{1+2x},$$
$$V' = \tfrac{1}{x^2} \quad \Rightarrow \quad V = \tfrac{-1}{x}.$$

$$\begin{aligned} I_4 &= [\frac{-1}{x}\ln(1+2x)]_1^2 - \int_1^2 \frac{2}{x(1+x)}dx = [\frac{-1}{x}\ln(1+2x)]_1^2 - 2\int_1^2 (\frac{1}{x} - \frac{1}{x+1}) \\ &= [\frac{-1}{x}\ln(1+2x)]_1^2 - 2[\ln(x) - \ln(1+x)]_1^2 \\ &= \big(\frac{-1}{2}\ln(5) + \ln(3)\big) + 2\big(\ln(2) - \ln(3) - \ln(1) + \ln(2)\big) = \ln(\frac{16}{3\sqrt{5}}). \end{aligned}$$

5. $I_5 = \int_1^2 \frac{x^2}{x^2-3x-4}dx = \int_1^2 \frac{x^2-3x-4+3x+4}{x^2-3x-4}dx = \int_1^2 (1 + \frac{3x+4}{x^2-3x-4})dx$

Identically equivalent to

$$\begin{aligned} I_5 &= \int_1^2 1dx + \int_1^2 \frac{3x+4}{x^2-3x-4}dx = \int_1^2 1dx + \int_1^2 \frac{3x+4}{(x-4)(x+1)}dx \\ &= \int_1^2 1dx + \int_1^2 \big(\frac{16}{5}(\frac{1}{x-4}) - \frac{1}{5}(\frac{1}{x+1})\big)dx \\ &= [x]_1^2 + \big[\frac{16}{5}\ln(\mid x-4 \mid) - \frac{1}{5}\ln(x+1)\big]_1^2 \\ &= [2-1] + [\big(\frac{16}{5}\ln(2) - \frac{1}{5}\ln(3)\big) - \big(\frac{16}{5}\ln(3) - \frac{1}{5}\ln(2)\big)] = 1 + \frac{17}{5}\big(\ln(2) - \ln(3)\big). \end{aligned}$$

6. $I_6 = \int_0^\pi e^x \sin(x)dx$. (twice the integration by parts)

First integration by parts:
$$U = \sin(x) \quad \Rightarrow \quad U' = \cos(x),$$
$$V' = e^x \quad \Rightarrow \quad V = e^x.$$

$$I_6 = [e^x \sin(x)]_0^\pi - \int_0^\pi e^x \cos(x)dx = 0 - \int_0^\pi e^x \cos(x)dx$$

Seconde integration by parts:
$$U = \cos(x) \quad \Rightarrow \quad U' = -\sin(x),$$
$$V' = e^x \quad \Rightarrow \quad V = e^x.$$

$$I_6 = -\Big([e^x \cos(x)]_0^\pi + \int_0^\pi e^x \sin(x)dx\Big) = -\Big([-e^\pi - 1] + \int_0^\pi e^x \sin(x)dx\Big).$$

Remember that $I_6 = \int_0^\pi e^x \sin(x)dx$, so $I_6 = -\Big([-e^\pi - 1] + I_6\Big) = e^\pi + 1 - I_6$,
In other words, $2I_6 = e^\pi + 1$. So

$$I_6 = \frac{1}{2}(e^\pi + 1).$$

Exercise 2: Primitives (antiderivatives) calculus:

$$\int (\ln(x))^2 dx, \quad \int \frac{\sin(x)}{1+\sin(x)} dx, \quad \int e^{2x}\ln(1+e^x)dx, \quad \int \frac{1}{1-\sqrt{x+2}} dx.$$

1. $F_1(x) = \int (\ln(x))^2 dx$
   Integration by parts:
   $U = (\ln(x))^2 \quad \Rightarrow \quad U' = 2\frac{\ln(x)}{x}$,

   $V' = 1 \qquad \Rightarrow \qquad V = x.$
   Hence $F_1(x) = \int (\ln(x))^2 dx = x(\ln(x))^2 - 2\int \ln(x)dx, \ c \in \mathbb{R}.$
   Integration by parts to calculate $\int \ln(x)$:
   $U = \ln(x) \quad \Rightarrow \quad U' = \frac{1}{x}$,

   $V' = 1 \quad \Rightarrow \quad V = x.$
   we will have

   $$F_1(x) = x(\ln(x))^2 - 2\big(x\ln(x) - \int dx\big) = x(\ln(x))^2 - 2x\big(\ln(x) - x\big) + c$$

   The primitive of $(\ln(x))^2$ is

   $$F_1(x) = x(\ln(x))^2 - 2x\ln(x) + 2x + c = x((\ln(x))^2 - 2\ln(x) + 2) + c.$$

2. $F_2(x) = \int \frac{\sin(x)}{1+\sin(x)} dx$

   $$\begin{aligned} F_2(x) &= \int \frac{\sin(x)}{1+\sin(x)} dx = \int \frac{\sin(x)+1-1}{1+\sin(x)} dx = \int \big(1 - \frac{1}{1+\sin(x)}\big) dx \\ &= \int 1dx - \int \frac{1}{1+\sin(x)} dx = x - \int \frac{1}{1+\sin(x)} dx. \end{aligned}$$

   To calculate $\int \frac{1}{1+\sin(x)} dx$, we put $t = \tan(\frac{x}{2})$, $\sin(x) = \frac{2t}{1+t^2}$.

   $$dt = \frac{1}{2}(1+\tan^2(\frac{x}{2}))dx = \frac{1}{2}(1+t^2)dx \Rightarrow dx = \frac{2dt}{1+t^2}.$$

   $$\begin{aligned} \int \frac{1}{1+\sin(x)} dx &= \int \frac{1}{(1+\frac{2t}{1+t^2})} \frac{2}{1+t^2} dt = \int \frac{(1+t^2)}{(1+t^2+2t)} \frac{2}{(1+t^2)} dt \\ &= \int \frac{2}{(1+t^2+2t)} dt = \int \frac{2}{(1+t)^2} dt \\ &= \frac{-2}{1+t} + c = \frac{-2}{1+\tan(\frac{x}{2})} + c. \end{aligned}$$

Therefore

$$F_2(x) = x + \frac{2}{1+\tan(\frac{x}{2})} + c,\ c \in \mathbb{R}..$$

3. $F_3(x) = \int e^{2x}\ln(1+e^x)dx$
   we pose $t = e^x$ and $dt = e^x dx = tdx \Rightarrow dx = \frac{dt}{t}$.
   By making this change of variable, we will have

$$F_3(x) = \int e^{2x}\ln(1+e^x)dx = \int t^2\ln(1+t)\frac{dt}{t} = \int t\ln(1+t)dt$$

   Integration by parts:
   $U = \ln(1+t) \quad \Rightarrow \quad U' = \frac{1}{1+t}$,

   $V' = t \qquad \Rightarrow \quad V = \frac{1}{2}t^2$.

$$\begin{aligned}
F_3(x) &= \frac{1}{2}t^2\ln(1+t) - \frac{1}{2}\int \frac{t^2}{1+t}dt = \frac{1}{2}t^2\ln(1+t) - \frac{1}{2}\int \frac{t^2-1+1}{1+t}dt \\
&= \frac{1}{2}t^2\ln(1+t) - \frac{1}{2}\int \frac{(t-1)(t+1)+1}{1+t}dt \\
&= \frac{1}{2}t^2\ln(1+t) - \frac{1}{2}\int \Big((t-1) + \frac{1}{1+t}\Big)dt \\
&= \frac{1}{2}t^2\ln(1+t) - \frac{1}{2}\Big(\frac{t^2}{2} - t + \ln(\mid 1+t \mid)\Big) + c \\
&= \frac{1}{2}e^{2x}\ln(1+e^x) - \frac{1}{2}\Big(\frac{e^{2x}}{2} - e^x + \ln(1+e^x)\Big) + c,\ c \in \mathbb{R}.
\end{aligned}$$

4. $F_4(x) = \int \frac{1}{1-\sqrt{x+2}}dx$
   We pose $t = \sqrt{x+2}$, then $dt = \frac{1}{2\sqrt{x+2}}dx = \frac{1}{2t}dx \Rightarrow dx = 2tdt$.

$$\begin{aligned}
F_4(x) &= \int \frac{1}{1-\sqrt{x+2}}dx = \int \frac{2t}{1-t}dt = \int \frac{2t-2+2}{1-t}dt \\
&= \int \Big(-2 + \frac{2}{1-t}\Big)dt = -2t - 2\ln(\mid 1-t \mid) + c \\
&= -2\sqrt{x+2} - 2\ln(\mid 1-\sqrt{x+2} \mid) + c,\ c \in \mathbb{R}.
\end{aligned}$$

Exercise 3: Improper integrals calculus:

$$\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx, \quad \int_0^{+\infty} \frac{1}{1+e^x}dx, \quad \int_1^{+\infty} \frac{\ln(x)}{x^2}dx.$$

1. $\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx$ is improper in $\frac{\pi}{2}$.

$$\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx = \lim_{a\to\frac{\pi}{2}} \int_a^{\pi} \tan(x)dx = \lim_{a\to\frac{\pi}{2}} \int_a^{\pi} \frac{\sin(x)}{\cos(x)}dx = \lim_{a\to\frac{\pi}{2}} \Big(-\int_a^{\pi} \frac{-\sin(x)}{\cos(x)}dx\Big).$$

Such that $\int \frac{f'(x)}{f(x)}dx = \ln(\mid f(x) \mid) + c,\ c \in \mathbb{R}$, then

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx &= \lim_{a\to\frac{\pi}{2}} \Big(-\int_a^{\pi} \frac{-\sin(x)}{\cos(x)}dx \Big) = \lim_{a\to\frac{\pi}{2}} \Big[-\ln(\mid \cos(x) \mid) \Big]_a^{\pi} \\ &= \lim_{a\to\frac{\pi}{2}} \Big(-\ln(1) + \ln(\mid \cos(a) \mid \Big) = \lim_{a\to\frac{\pi}{2}} \ln(\mid \cos(a) \mid) = -\infty.\end{aligned}$$

Hence $\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx$ is divergent.

2. $\int_0^{+\infty} \frac{1}{1+e^x}dx = \lim_{b\to+\infty} \int_0^b \frac{1}{1+e^x}dx$
Change of variable: we put $t = e^x$ et $dt = e^x dx \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t}$.

$$\begin{aligned}\int_0^{+\infty} \frac{1}{1+e^x}dx &= \lim_{b\to+\infty} \int_0^b \frac{1}{1+e^x}dx = \lim_{b\to+\infty} \int_1^{e^b} \frac{1}{t(1+t)}dt \\ &= \lim_{b\to+\infty} \int_1^{e^b} \Big(\frac{1}{t} - \frac{1}{(1+t)}\Big)dt = \lim_{b\to+\infty} \Big[\ln(t) - \ln(1+t)\Big]_1^{e^b} \\ &= \lim_{b\to+\infty} \Big[\ln(\frac{t}{1+t})\Big]_1^{e^b} = \lim_{b\to+\infty} \Big(\ln(\frac{e^b}{1+e^b}) - \ln(\frac{1}{2})\Big) = -\ln(\frac{1}{2}) = \ln(2).\end{aligned}$$

So $\int_0^{+\infty} \frac{1}{1+e^x}dx$ is convergent.

3. $\int_1^{+\infty} \frac{\ln(x)}{x^2}dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^2}\ln(x)dx$

Integration by parts:
$$U = \ln(x) \quad \Rightarrow \quad U' = \frac{1}{x},$$
$$V' = \frac{1}{x^2} \quad \Rightarrow \quad V = \frac{-1}{x}.$$

$$\begin{aligned}\int_1^{+\infty} \frac{\ln(x)}{x^2}dx &= \lim_{b\to+\infty} \int_1^b \frac{1}{x^2}\ln(x)dx = \lim_{b\to+\infty} \Big(\Big[\frac{-\ln(x)}{x}\Big]_1^b + \int_1^b \frac{1}{x^2}dx\Big) \\ &= \lim_{b\to+\infty} \Big(\Big[\frac{-\ln(x)}{x}\Big]_1^b + \Big[\frac{-1}{x}\Big]_1^b\Big) = \lim_{b\to+\infty} \Big(\frac{-\ln(b)}{b} + \frac{\ln(1)}{1} - \frac{1}{b} + 1\Big) \\ &= \lim_{b\to+\infty} \frac{-(1+\ln(b))}{b} + 1 = 1.\end{aligned}$$

Consequently, $\int_1^{+\infty} \frac{\ln(x)}{x^2}dx$ is convergent.

Exercise 4:

$$I_n = \int_0^1 x^n e^{2x}dx,\ n \in \mathbb{N}.$$

1. $I_0 = \int_0^1 e^{2x}dx = \Big[\frac{1}{2}e^{2x}\Big]_0^1 = \frac{1}{2}(e^2 - 1)$.

2. Showing (proving) that for any $n \in \mathbb{N}$, we have $I_{n+1} = \frac{e^2}{2} - \frac{n+1}{2} I_n$.

$$I_{n+1} = \int_0^1 x^{n+1} e^{2x} dx$$

Integration by parts:
$$\begin{array}{lcl} U = x^{n+1} & \Rightarrow & U' = (n+1)x^n, \\ V' = e^{2x} & \Rightarrow & V = \frac{1}{2}e^{2x}. \end{array}$$

$$\begin{aligned} I_{n+1} &= \int_0^1 x^{n+1} e^{2x} dx = \left[\frac{1}{2} x^{n+1} e^{2x}\right]_0^1 - \frac{(n+1)}{2} \overbrace{\int_0^1 x^n e^{2x} dx}^{I_n} \\ &= \frac{1}{2} e^2 - \frac{(n+1)}{2} I_n. \end{aligned}$$

3. Deduction of I_2:

$$I_2 = \frac{e^2}{2} - \frac{2}{2} I_1 = \frac{e^2}{2} - (\frac{e^2}{2} - \frac{1}{2} I_0) = \frac{1}{2} I_0 = \frac{1}{4}(e^2 - 1).$$

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