Chapter 2: Representation of Control Systems

1. Laplace Transform:

The Laplace transform converts time-domain functions into a complex frequency-domain representation, enabling the analysis and solution of linear time-invariant systems, differential equations, and control problems. Its main advantage lies in turning differentiation and integration operations into algebraic manipulations, greatly simplifying problem-solving.

1.1. Definition

For a given function $f(t)$, defined for $t \geq 0$, the Laplace transform $F(s)$ is

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t,$$

Where s is a complex variable, $s = \sigma + j\omega$. Convergence of this integral depends on the growth of $f(t)$ and the real part of s.

1.2. Existence and Region of Convergence

- A transform exists if $\int_0^\infty |f(t)|\, e^{-\sigma t}\,\mathrm{d}t < \infty$ for some σ.
- The **Region of Convergence (ROC)** is the set of s-values for which the integral converges.
  - For right-sided signals (zero for $t < 0$), the ROC is a half-plane $\Re(s) > \sigma_0$.
  - The ROC determines system stability and causality.

1.3. Properties of Laplace Transform

Property	Formula
Linearity	$\mathcal{L}\{af(t)+bg(t)\} = aF(s)+bG(s)$
First Derivative	$\mathcal{L}\{f'(t)\} = sF(s)-f(0)$
Second Derivative	$\mathcal{L}\{f''(t)\} = s^2F(s)-sf(0)-f'(0)$
nth Derivative	$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
Integration	$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$
s-shift	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
t-shift	$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$
Scaling	$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$
Convolution	$\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$

Table 1.: Properties of Laplace Transform

Example:

Solve $y''+3y'+2y=u(t)$ with zero initial conditions:

- Laplace Transform: $(s^2Y+3sY+2Y)=\frac{1}{s}$.
- Algebraic solution: $Y(s)=\frac{1}{s(s^2+3s+2)}=\frac{1}{s}\left(\frac{1}{s+1}-\frac{1}{s+2}\right)$.
- Inverse Laplace transform: $y(t)=(1-e^{-t})-(1-e^{-2t})=e^{-2t}-e^{-t}$.

1.4. Laplace Transform Table:

Function $f(t)$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n (n positive integer)	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\delta(t)$ (Dirac delta)	1
$\delta(t-a)$	e^{-as}
$u(t-a)$ (Unit step)	$\frac{e^{-as}}{s}$
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$
$2\sqrt{\frac{t}{\pi}}$	$\frac{1}{s\sqrt{s}}$

Table 2: Basic Laplace Transforms

2. Inverse Laplace transform:

The inverse transform restores a frequency domain (s-domain) function $F(s)$, and converts it back to its corresponding time-domain function $f(t)$.

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

In practice, we almost always use simpler methods like partial fraction decomposition and a table of known Laplace transforms (Table 2).

2.1. Partial Fraction Decomposition (PFD):

The most common method for finding an inverse Laplace transform is to manipulate F(s) until it looks like a sum of simpler functions that appear in a standard Laplace transform table. Then "looking up" the inverse of each piece.

The form of the decomposition depends on the factors in the denominator.

2.1.1. Case 1: Distinct Real Roots

If the denominator can be factored into distinct linear factors, you decompose into terms with constant numerators.

Example: Find $\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+3s+2}\right\}$.

1. Factor the denominator: $s^2+3s+2=(s+1)(s+2)$
2. Set up the partial fractions: $\frac{s+3}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$
3. Solve for A and B: $s+3=A(s+2)+B(s+1)$

   Let $s=-1$: $2=A(1) \Rightarrow A=2$

   Let $s=-2$: $1=B(-1) \Rightarrow B=-1$
4. So $F(s)=\frac{2}{s+1}-\frac{1}{s+2}$
5. Now take the inverse Laplace transform term-by-term using the table:

$(\mathcal{L}^{-1}\{1/(s-a)\}=e^{at})$: $f(t)=2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$ $f(t)=2e^{-t}-e^{-2t}$

2.1.2. Case 2: Repeated Real Roots

For a repeated factor like $(s-a)^n$, you need a term for each power.

Example: Find $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}$.

1. Set up the partial fractions: $\frac{s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$
2. Solve for A and B: $s = A(s+1) + B$ $\quad s = As + (A+B)$ Equate coefficients:
   - For s: $1 = A$
   - For constant: $0 = A + B \Rightarrow B = -1$
3. So, $F(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2}$
4. Take the inverse Laplace transform. From the table, $\mathcal{L}^{-1}\{1/(s+1)\} = e^{-t}$ and, $\mathcal{L}^{-1}\{1/(s+1)^2\} = t \cdot e^{-t}$. $f(t) = e^{-t} - te^{-t}$

2.1.3. Case 3: Complex Conjugate Roots

For an irreducible quadratic factor $s^2 + \beta s + \gamma$, complete the square to make it look like the transforms for sine and cosine.

Example: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+13}\right\}$.

1. Complete the square in the denominator:
   $s^2 + 4s + 13 = (s^2 + 4s + 4) + 9 = (s+2)^2 + 3^2$
2. Now, $F(s) = \frac{1}{(s+2)^2+3^2}$
3. Look at the table. This matches the form for $e^{at}\sin(\omega t)$, which is $\frac{\omega}{(s-a)^2+\omega^2}$. Here, $a = -2$ and $\omega = 3$.
4. Therefore, the inverse transform is: $f(t) = e^{-2t}\sin(3t)$.

2.1.4. A More Complex Example (Combining Cases):

Let's find $\mathcal{L}^{-1}\left\{\frac{s^2+1}{s(s-1)^2}\right\}$.

1. **Set up PFD:** The denominator has a distinct root ***s*** and a repeated root ***(s-1)²***.
$$\frac{s^2+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$
2. **Solve for constants:** $s^2 + 1 = A(s-1)^2 + Bs(s-1) + Cs$ Let's solve efficiently:
   - Let $s = 0$: $1 = A(1) \Rightarrow A = 1$
   - Let $s = 1$: $2 = C(1) \Rightarrow C = 2$
   - Let $s = 2$: $5 = A(1) + B(2) + C(2) = 1 + 2B + 4 \quad 5 = 5 + 2B \Rightarrow 2B = 0 \Rightarrow B = 0$
3. **Rewrite F(s):** $F(s) = \frac{1}{s} + \frac{0}{s-1} + \frac{2}{(s-1)^2} = \frac{1}{s} + \frac{2}{(s-1)^2}$

4. **Take the inverse transform:**
   - $\mathcal{L}^{-1}\{1/s\} = 1$ (or $u(t)$, the unit step)
   - $\mathcal{L}^{-1}\{2/(s-1)^2\} = 2t \cdot e^t$ (using $\mathcal{L}\{t \cdot e^{at}\} = 1/(s-a)^2$)
5. **Final Answer:** $f(t) = 1 + 2te^t$

2.2. Inverse Laplace Transforms Table:

$F(s)$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$
$\dfrac{1}{s}$	1
$\dfrac{1}{s^2}$	t
$\dfrac{1}{s^n}$ (n positive integer)	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^n}$	$\dfrac{t^{n-1}e^{at}}{(n-1)!}$
$\dfrac{1}{s^2+a^2}$	$\dfrac{1}{a}\sin(at)$
$\dfrac{s}{s^2+a^2}$	$\cos(at)$
$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh(at)$
$\dfrac{s}{s^2-a^2}$	$\cosh(at)$
$\dfrac{1}{(s-a)^2+b^2}$	$\dfrac{1}{b}e^{at}\sin(bt)$
$\dfrac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos(bt)$
e^{-as}	$\delta(t-a)$
$\dfrac{e^{-as}}{s}$	$u(t-a)$

Table 3: Inverse Laplace Transforms

3. Mathematical model of a dynamic system

A critical aspect in the analysis and design of control systems is the mathematical modelling of these systems. The predominant techniques for modelling linear systems are **the transfer function** method and **the state-space representation** method. The transfer function is applicable solely to linear time-invariant systems, while state equations can be utilised for both linear and nonlinear systems.

A control system may consist of diverse components such as mechanical, thermal, fluid, pneumatic, and electrical elements; sensors and actuators; and computational devices. This section examines systems represented by ordinary **differential equations**.

• The modelling of electrical systems.

3.1. Representation by Differential Equations

Physical systems are typically modelled using ordinary or partial differential equations that describe how system variables change with time and space. For linear time-invariant systems, the transfer function approach provides a powerful tool, relating the Laplace transform of the output to that of the input and enabling frequency-domain analysis.

3.2. Representation by Transfer Functions

A transfer function $H(s) = \frac{Y(s)}{U(s)}$ characterizes the input-output relationship in the Laplace domain, where poles and zeros determine system dynamics, stability, and frequency response. This representation facilitates control system design and system interconnection analysis.

3.3. Modelling of mechanical systems:

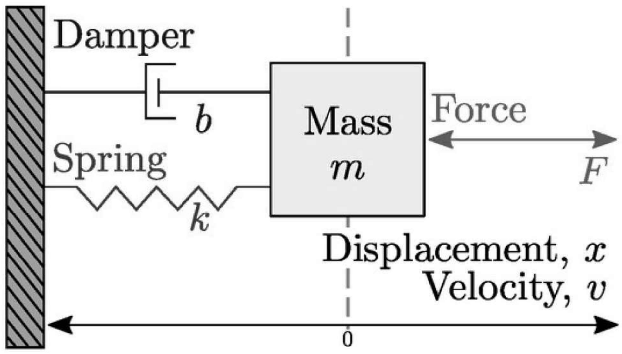
Mechanical systems can be represented as systems of lumped masses (rigid bodies) or as distributed mass (continuous) systems. The latter are described by partial differential equations, while the former are characterised by ordinary differential equations. Translational Motion The motion of translation is described as a motion that takes place along a straight or curved path.

The parameters utilised to characterise translational motion include acceleration, velocity, and displacement. Newton's rule of motion states that the algebraic total of external forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction. The law can be represented as:

$$\sum_{ext} forces = M.a$$

Where ***M*** denotes the mass, and ***a*** is the acceleration in the direction considered.

Horizontal mass-spring-damper system:



- **Mass:** m (kg)
- **Spring Constant:** k (N/m) - ideal and linear (obeys Hooke's Law)
- **Damping Coefficient:** c (N·s/m) - viscous damping (e.g., a dashpot)
- **Input:** An external force $F(t)$ (N) applied to the mass.
- **Output:** The displacement of the mass $x(t)$ (m) from its equilibrium (rest) position. We define the positive direction for x and F to be to the right.
- **Spring Force (F_k)**: The spring resists being stretched or compressed. According to **Hooke's Law**, this force is proportional to the displacement x and acts in the *opposite* direction: $F_k = -kx$. (To the left if x is positive).
- **Damper Force (F_c)**: The damper (or dashpot) resists motion. For a viscous damper, this force is proportional to the *velocity* $v = \dot{x}$ and acts in the *opposite* direction: $F_c = -c\dot{x}$. (To the left if the mass is moving to the right).

Summing the forces from the Free-Body Diagram (FBD) (taking right as positive):

$$\sum F = F(t) + F_k + F_c$$

$$\sum F = F(t) + (-kx) + (-c\dot{x})$$

$$\sum F = F(t) - kx - c\dot{x}$$

Now, equating this to $\boldsymbol{m\ddot{x}}$: $$m\ddot{x} = F(t) - kx - c\dot{x}$$

This is the linear **<u>ordinary differential equation (ODE)</u>** that governs the dynamics of the mass-spring-damper system.

$$\boldsymbol{m\ddot{x} + c\dot{x} + kx = F(t)}$$

Applying **the Laplace Transform** to each term. With the following properties (with zero initial conditions $x(0) = 0, \dot{x}(0) = 0$):

- $\mathcal{L}\{x(t)\} = X(s)$
- $\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0) = sX(s)$
- $\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s)$
- $\mathcal{L}\{F(t)\} = F(s)$

Applying the transform to the entire equation:

$$\mathcal{L}\{m\ddot{x}(t)\} + \mathcal{L}\{c\dot{x}(t)\} + \mathcal{L}\{kx(t)\} = \mathcal{L}\{F(t)\}$$

$$m \cdot \mathcal{L}\{\ddot{x}(t)\} + c \cdot \mathcal{L}\{\dot{x}(t)\} + k \cdot \mathcal{L}\{x(t)\} = F(s)$$

$$m \cdot s^2X(s) + c \cdot sX(s) + k \cdot X(s) = F(s)$$

Factor $X(s)$ out of the left-hand side:

$$(ms^2 + cs + k)X(s) = F(s)$$

Dividing both sides by $F(s)$ and by $(ms^2 + cs + k)$ to isolate the transfer function:

This is the transfer function of the system.

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

3.4. Modelling of electrical systems

First we discuss modelling of electrical networks using simple passive elements such as resistors, inductors, and capacitors. Here is a table summarizing the R, L, and C components with their voltage formulas and impedance:

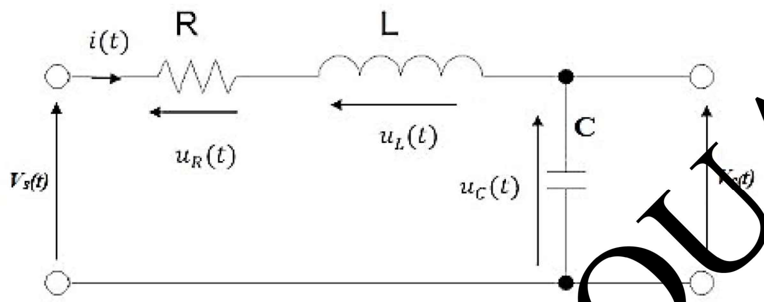
Component	Voltage Formula	Impedance Formula	Description
Resistor (R)	$V_R = I \cdot R$	$Z_R = R$	Voltage and current are in phase. Impedance is purely real resistance.
Inductor (L)	$V_L = L\frac{dI}{dt} = I \cdot j\omega L$	$Z_L = j\omega L$	Voltage leads current by 90°. Impedance is purely inductive reactance.
Capacitor (C)	$V_C = \frac{1}{C}\int I dt = \frac{I}{j\omega C}$	$Z_C = \frac{1}{j\omega C}$	Voltage lags current by 90°. Impedance is purely capacitive reactance.

Where:

- I is the current,
- $\omega = 2\pi f$ is the angular frequency,
- j is the imaginary unit representing a 90° phase shift.

Series RLC circuit:

We will consider the most common case: a voltage source $v_s(t)$ as the input, and the voltage across the capacitor $v_c(t)$ as the output.



Input: $v_s(t)$ *Output:* $v_c(t)$

Applying Kirchhoff's Voltage Law (KVL)

$$v_s(t) = v_R(t) + v_L(t) + v_C(t)$$

Writing Voltage-Current Relationships for Each Component

- **Resistor (Ohm's Law):** $v_R(t) = R \cdot i(t)$
- **Inductor:** $v_L(t) = L\frac{di(t)}{dt}$
- **Capacitor:** $v_C(t) = \frac{1}{C}\int i(t)\,dt$

Substituting Relationships into KVL Equation $v_s(t) = R \cdot i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$

This is an **integro-differential equation**. We need to convert it into a pure differential equation.

Eliminating the Integral by Relating $i(t)$ to $v_c(t)$ We know the current through the capacitor is $i(t) = C\frac{dv_c(t)}{dt}$.

- $v_C(t) = v_c(t)$ (This is the output)
- $v_R(t) = R \cdot i(t) = R \cdot \left(C\frac{dv_c(t)}{dt}\right) = RC\frac{dv_c(t)}{dt}$
- $v_L(t) = L\frac{di(t)}{dt} = L\frac{d}{dt}\left(C\frac{dv_c(t)}{dt}\right) = LC\frac{d^2v_c(t)}{dt^2}$

Form the Final Differential Equation

$v_s(t) = v_R(t) + v_L(t) + v_C(t)$: $v_s(t) = LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t)$

Final Differential Equation: $\boxed{LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = v_s}$ This is a **second-order linear ordinary differential equation (ODE)**.

The Transfer Function $H(s)$:

The transfer function is defined as the ratio of the output to the input in the **s-domain** (Laplace domain), assuming zero initial conditions: $H(s) = \frac{V_c(s)}{V_s(s)}$.

Taking the Laplace Transform of the Differential Equation:

- $\mathcal{L}\{f(t)\} = F(s)$
- $\mathcal{L}\{f'(t)\} = sF(s)$
- $\mathcal{L}\{f''(t)\} = s^2F(s)$

Applying the Laplace transform to our differential equation: $\mathcal{L}\{LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c\} = \mathcal{L}\{v_s\}$ $LC \cdot s^2V_c(s) + RC \cdot sV_c(s) + V_c(s) = V_s(s)$

Factoring out $V_c(s)$ and Solve for $H(s)$ $V_c(s)(LCs^2 + RCs + 1) = V_s(s)$

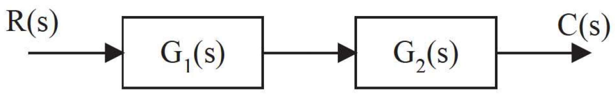
$$H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1}{LCs^2 + RCs + 1}$$

4. Block diagrams and simplification rules:

Because of their simplicity and versatility, block diagrams are commonly used by control engineers to represent many types of systems. A block diagram can be used simply to show the composition and connections of a system. Also, it can be used, combined with transfer functions, to describe the cause-and-effect interactions throughout the system. Transfer Function is described as the relationship between an input signal and an output signal to a device.

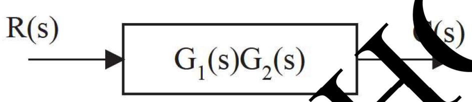
Three most basic simplifying rules are described in detail as follows.

Series Connection –



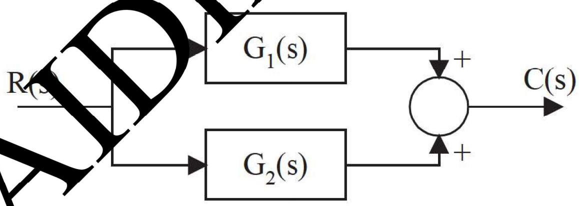
$$C(s) = G_1(s)G_2(s)R(s)$$

$$\Downarrow$$



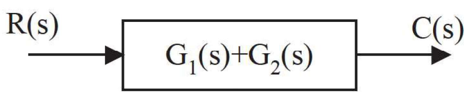
$$\frac{C(s)}{R(s)} = G_1(s)G_2(s)$$

Parallel Connection –



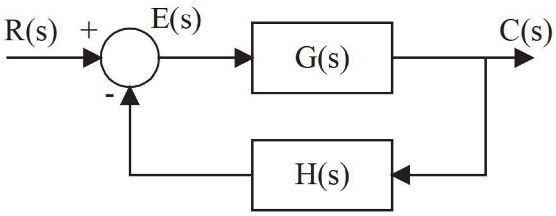
$$C(s) = [G_1(s) + G_2(s)]R(s)$$

$$\Downarrow$$



$$\frac{C(s)}{R(s)} = G_1(s) + G_2(s)$$

Feedback Control Systems –



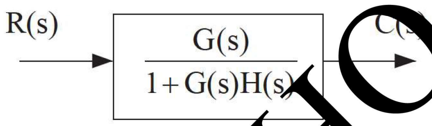
$$C(s) = G(s)E(s)$$

$$E(s) = R(s) - H(s)C(s) \qquad \text{(Error)}$$

$$C(s) = G(s)[R(s) - H(s)C(s)]$$

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

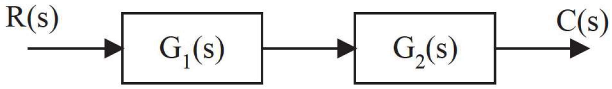
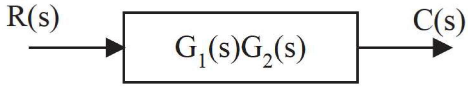
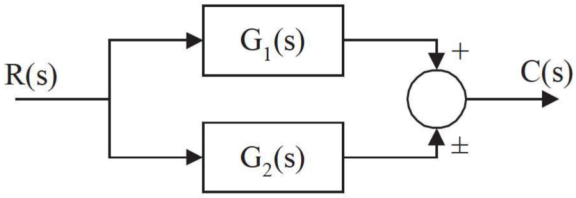
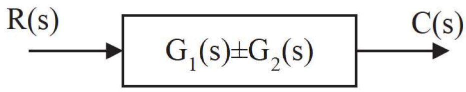
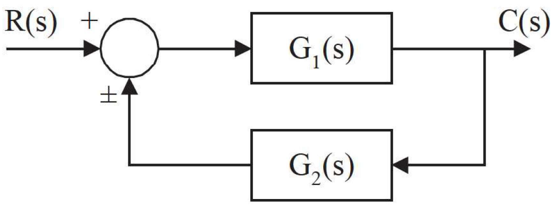
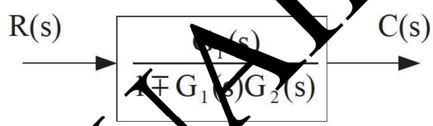
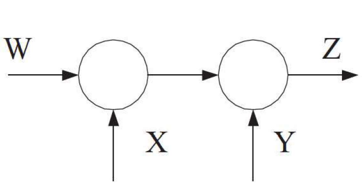
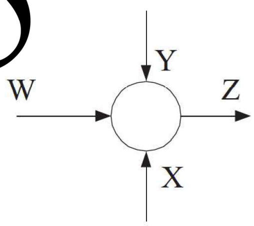
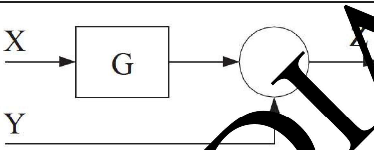
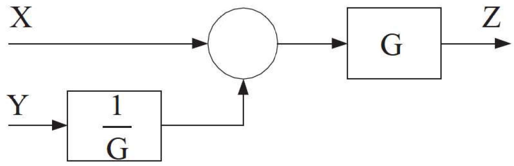
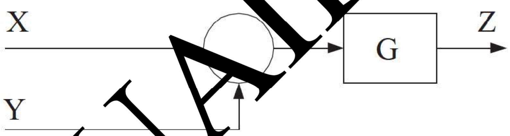
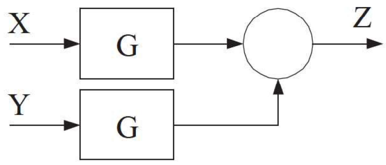
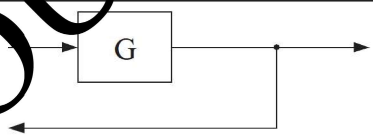
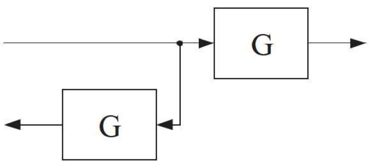
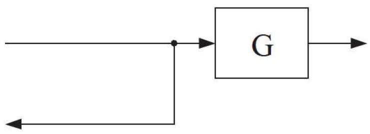
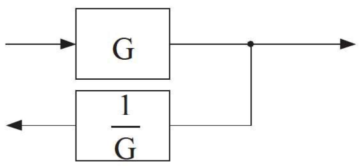
$$\Downarrow$$



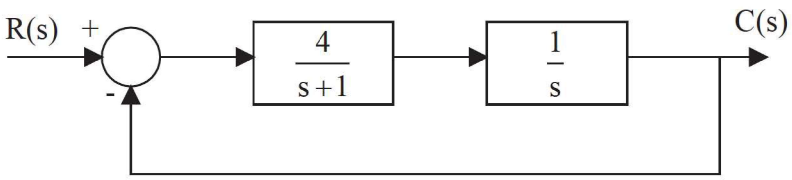
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Some basic rules of simplifying block diagrams are tabulated in Table 5.

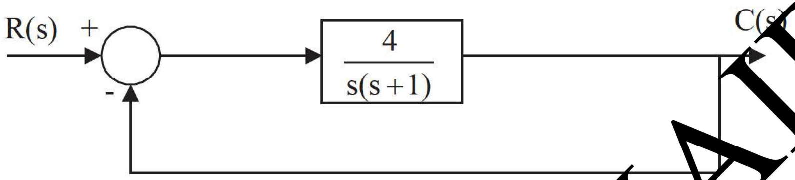
Table 5. Rules of simplifying block diagrams.

Case	Original Structure	Equivalent Structure
1		
2		
3		
4		
5		
6		
7		
8		

Example 1: Find the transfer function of the closed-loop system below.



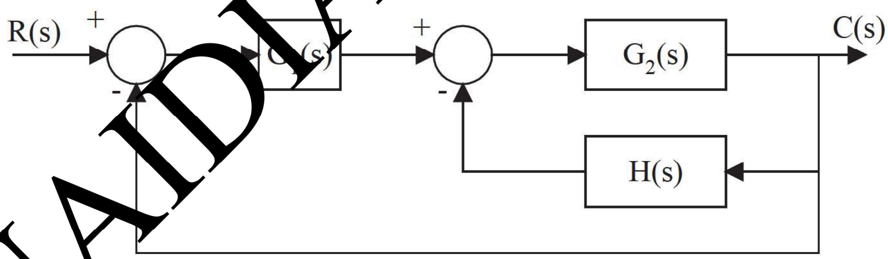
Solution:
Use case 1 to combine the two series blocks



Use case 3 to obtain the transfer function as follows

$$\frac{C(s)}{R(s)} = \frac{\frac{4}{s(s+1)}}{1+\frac{4}{s(s+1)}\cdot 1} = \frac{4}{s^2+s+4}$$

Example 2: Find the transfer function of the closed-loop system below.



Solution:
Use case 3 to simplify the inner feedback loop to obtain the following block diagram.