People’s Democratic Republic of Algeria
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Faculty of Exact Sciences
Physics Department

Field of study: *Physics* ***Level :*** *L3 Physics*

Course on
Electromagnetic Waves

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PREFACE

After four (4) years of experience teaching Electromagnetism Waves, I present this handout, which is considered an educational resource. It corresponds to the official curriculum taught to third-year (L3) students in Fundamental Physics. This document is intended to provide a deeper understanding of the concepts of electromagnetism and to help students address and solve problems related to this subject. It is presented in the form of a detailed course accompanied by solved exercises.

Propagation of electromagnetic waves is part of physics and engineering, which has led to the development of many technologies such as telecommunication, fiber optics, radar systems, microwave systems, etc. The interaction between electric and magnetic fields with different media helps to predict, control, and use the phenomenon of wave propagation.

The first chapter of the course reviews the basics of electromagnetism, which includes the fundamental equations by Maxwell, electric fields, magnetic fields, boundary conditions, etc. These are the fundamental concepts required to understand the phenomenon of wave propagation.

The second chapter is focused on the study of propagation in material media. The chapter includes the study of wave propagation in dielectric media, conductors, plasma media, etc. It also focuses on the influence of the medium's properties on the wave's propagation.

The third chapter deals with the propagation in anisotropic media. Anisotropic media are the media whose properties change with direction. The chapter helps to understand the phenomenon of wave propagation in anisotropic media.

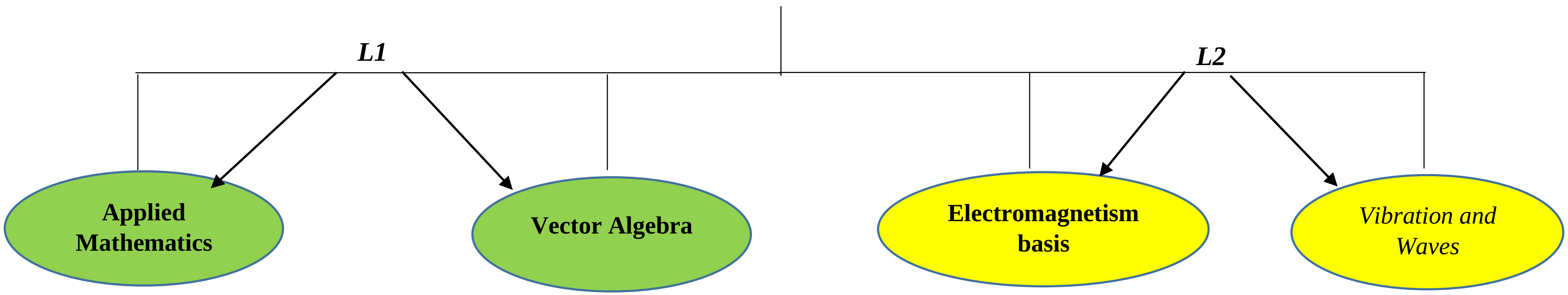
The fourth chapter covers nonlinear media and discusses various cases in which the properties of electromagnetic fields depend on their intensity. The effects of self-focusing, harmonic generation, and phase modulation are discussed in this chapter. All these effects are of fundamental importance in understanding laser devices and modern telecommunication systems.

In the fifth chapter on propagation in waveguides, various types of optical fibers, metallic waveguides, and dielectric waveguides are discussed. This chapter will help students understand various concepts related to practical applications.

At the end of this course, undergraduate students in physics, electronics, and telecommunications will have a thorough knowledge of basic concepts related to the propagation of electromagnetic waves and will be equipped with appropriate tools to analyze such propagation in various conditions.

PREREQUISITE FOR

ELECTROMAGNETIC WAVES



Applied Mathematics

- Differential and integral calculus
- Differential equations
- Trigonometric and exponential functions

Vector Algebra

- Dot product and cross product
- Operators: gradient, divergence, curl
- Manipulation of vector fields

Electromagnetism basis

- Coulomb, Gauss, Faraday and Ampère Laws
- Boundary conditions for electric and magnetic fields
- Permittivity, permeability and conductivity

Vibration and Waves

- Plane waves and characteristics (wavelength, phase velocity, group velocity)
- Reflection and transmission at interfaces
- Polarization concepts

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Chapter I
Reviews of Waves and Electromagnetism

INTRODUCTION

The purpose of this chapter is to review the fundamental concepts related to waves and electromagnetism. It aims to consolidate the theoretical foundations necessary for understanding the propagation of electromagnetic waves, highlighting their essential characteristics and the physical principles that govern them. This review will provide the essential framework for subsequently studying more complex structures.

I.1. Electromagnetic Waves

Electromagnetic wave, commonly known as light, is a type of wave that can propagate through various media, such as a vacuum or air at a speed close $3 \times 10^8 m/s$. These waves are produced, by accelerating electric charges and consist of mutually perpendicular oscillations of electric $\vec{E}$ and magnetic $\vec{B}$ fields, which themselves are perpendicular to the direction of wave propagation.

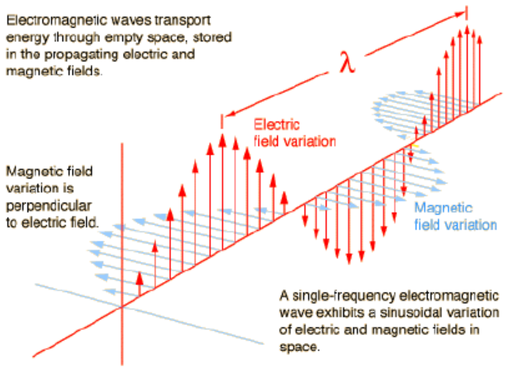


Figure I.1. : Propagation of $\vec{E}$ et $\vec{B}$ fields in the space.

In addition to transporting energy, electromagnetic waves are fundamental in carrying information over long distances makes them indispensable in fields such as telecommunications, remote sensing, radar technology, and medical imaging. Their ability of travelling into space without necessarily needing a material-carrying medium is the basis of many scientific and technological developments made in the modern world.

Electromagnetic waves are characterized by

- The period T, which represents the time of one oscillation.
- The frequency $f = {}^{1}/_{T}$, defined by the number of oscillations per second. It is measured in Hertz (Hz). Where 1Hz =1 oscillation per second.
- The wavelength , represents the distance between two consecutives oscillations of the wave. It is represented by λ and measured in meters. For example, the blue light has shorter wavelength than red light

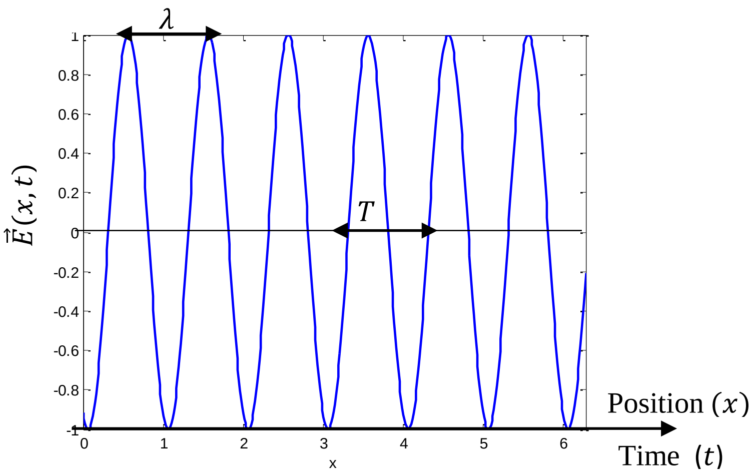


Figure I.2: Spatial and temporal variation of the electric field

An electromagnetic spectrum is obtained as a function of frequencies (or wavelengths) of electromagnetic waves ranged from radio waves to gamma rays (Figure I.3). Each region of this spectrum has distinct properties and applications. Radio and microwave frequencies find their applications in communication systems including radio, television, mobile phones, and satellite transmission. Infrared and visible light are the basis of optical communication and imaging technologies and X-rays and gamma rays are invaluable in medical diagnostics and treatments.

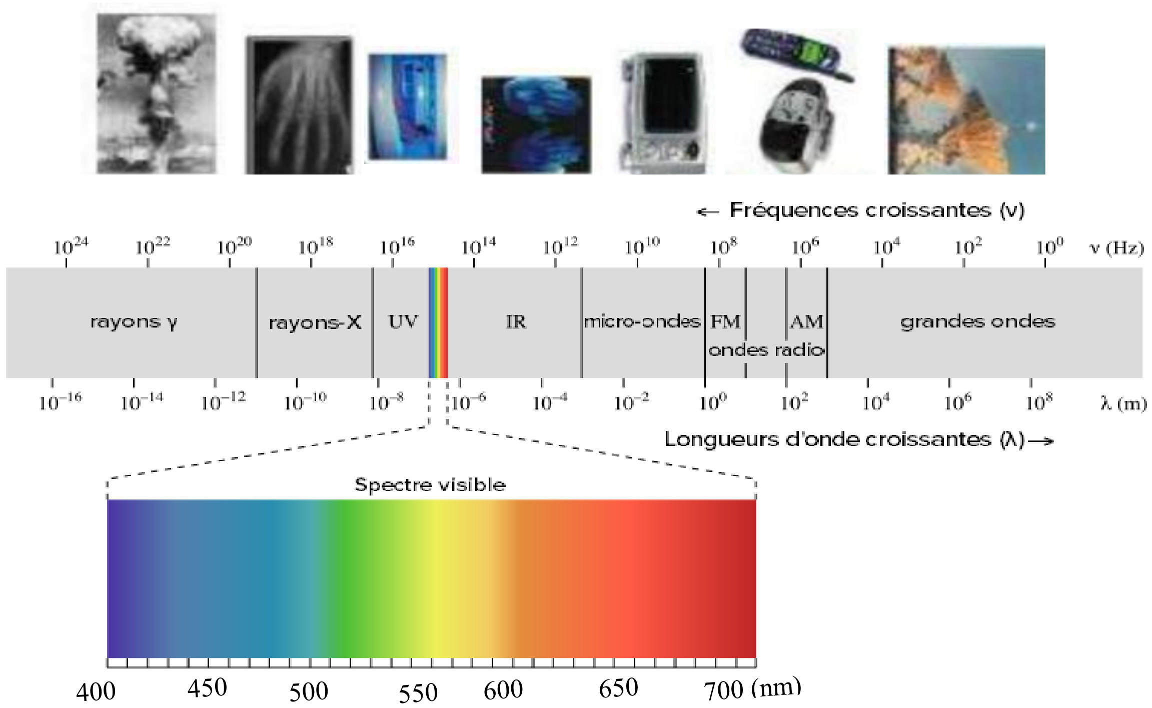


Figure I.3: Electromagnetic spectrum

I.2. Study of Electromagnetic Wave Propagation

I.2.1. Wave equation

The study of electromagnetic wave propagation requires solving D'Alembert's equation. Mathematically, a wave can be represented by a scalar or vector field denoted by ψ (can be the electric field or magnetic field) , depending on time and space variables, which satisfies D'Alembert's equation

$$\nabla^2\psi(\vec{r},t) - \frac{1}{v^2}\frac{\partial^2\psi(\vec{r},t)}{\partial t^2} = 0 \qquad (I.1)$$

Where $\vec{r} = \overrightarrow{OM}$ is the position vector and v is the wave propagation velocity. Accordingly, $\psi(\vec{r},t)$ is the amplitude of wave at point M at time t.

The Laplacian operator ∇^2 given by the sum of the second spatial derivatives

$$\nabla^2\psi(x,t) = \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{\partial^2\psi(x,t)}{\partial y^2} + \frac{\partial^2\psi(x,t)}{\partial z^2} \qquad (I.2)$$

It can be simplified given the nature of the problem (symmetries). The most important property of D'Alembert's equation is its linearity. This allows us to apply the superposition theorem to solutions.

In one dimensions, the wave equation is given by

$$\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi(x,t)}{\partial t^2} = 0 \qquad (I.3)$$

In the next sections, the principal types of wave equation solutions will be introduced. These include progressive plane waves, harmonic plane waves, and spherical waves.

I.2.2. Progressive Plane Waves

- A wave is said to be plane if, at a given instant t, the quantity that characterizes the propagating wave is the same at all points in a plane perpendicular to the direction of propagation.
- A progressive wave is the propagation of a disturbance from point A to point B, transporting energy and momentum but no matter.
- As it propagates, a progressive wave produces a reversible variation in the local physical properties of the medium (water height for waves, pressure for sound, electric and magnetic field for light).

Figure I.4 Example of a propagating wave (wave on the sea surface)

The solution to D'Alembert's equation (I.2) can be written as the superposition of two progressive plane wave (PPW) propagating in two opposite directions, which results in a plane wave of the type

$$\psi(x,t) = f(x - vt) + g(x + vt) \qquad (I.4)$$

where f is a plane wave propagating in the increasing x direction and g in the decreasing x direction.

In the expression for ψ, we can eliminate the choice of coordinate axis, $\vec{u}$ being the unit vector in the direction of propagation and $\vec{r}$ the position vector

$$\psi(\vec{r},t) = f(\vec{u}.\vec{r} - vt) + g(\vec{u}.\vec{r} + vt) \qquad (I.5)$$

I.2.3. Harmonic Progressive Wave

In what follows, we focus on a specific class of waves that satisfy the wave equation, namely harmonic progressive plane waves (HPPWs). Their distinctive feature lies in the functional form of $f(x - vt)$.

A harmonic progressive wave is a type of progressive wave whose displacement or amplitude varies sinusoidally in both space and time. It represents a periodic, continuous propagation of a disturbance through a medium.

For a wave propagating along a unit vector $\vec{u}$, the harmonic progressive wave can be written as

$$\psi(x,t) = \psi_0 \cos\left(\omega t - \vec{k}.\vec{r} + \varphi_0\right) \qquad (I.6)$$

By introducing the temporal period of the wave ($T = \frac{2\pi}{\omega}$) and the spatial period ($\lambda = v\,T$) the magnitude of the wave vector becomes

$$k = \frac{\omega}{v} = \frac{2\pi}{\lambda} \qquad (I.7)$$

The quantity $\varphi(\vec{r},t) = \omega t - \vec{k}.\vec{r} + \varphi_0$ represents the spatio-temporal phase

The function ψ thus exhibits a double periodicity, both in time (temporal period T) and in space (spatial period λ).

I.2.4. Wave surface

The wavefront is defined as the set of values of x such that: $\psi(M,t) = \psi_0 \cos(\varphi) = cste$ at a fixed t.

By differentiating φ at a fixed t, we can show that this condition implies

$$\varphi = cste \Rightarrow \frac{d\varphi}{dt} = 0 \Rightarrow -\vec{k}.\overrightarrow{dr} = 0 \qquad (I.8)$$

Therefore, for $\overrightarrow{dr}$ belonging to the wavefront, $\vec{k}$ is perpendicular to $\overrightarrow{dr}$. We can deduce that $\vec{k}$ is perpendicular to the wavefront (Figure I.5). On this surface, all the points have the same phase φ.

For the HPPW propagating along (Ox), the wave planes are parallel to the plane (Oyz) (Figure I.5).

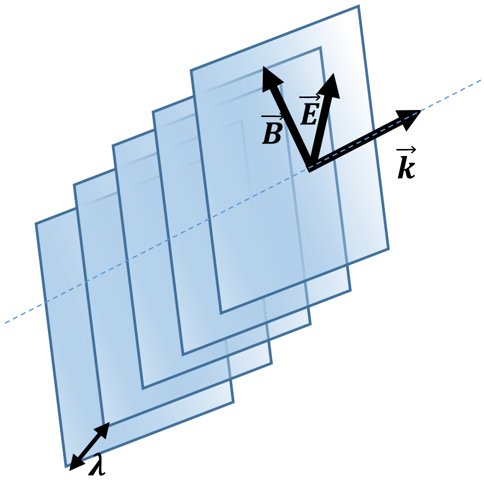


Figure I.5: Wavefront of a progressive plane wave (electromagnetic wave).

- ***Phase velocity***

The phase velocity v_φ is defined as the speed at which wavefront travels. This velocity can be found by differentiating φ.

Assuming that the HPPW propagates in the direction (Ox), this gives

$$d\varphi = \frac{\partial\varphi}{\partial t}dt + \frac{\partial\varphi}{\partial x}dx = \omega dt - kdx = 0 \qquad (I.9)$$

The phase velocity is given by

$$v_\varphi = \frac{dx}{dt} = \frac{\omega}{k} \qquad (I.10)$$

The wavelength corresponds to the distance between two successive wave planes

$$\lambda = v_\varphi T$$

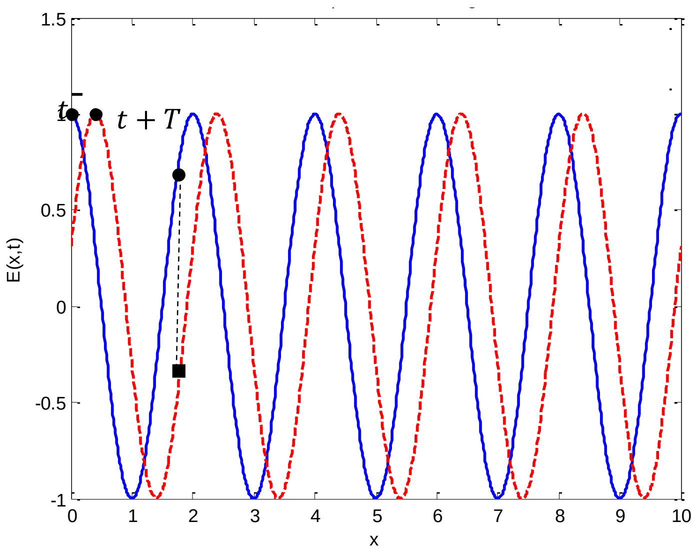


Figure I.6: Sinusoidal travelling wave. Trajectory of two points ● and ■ between t et $t+T$

For an electromagnetic wave in vacuum with a propagation speed $v = c$, in according with $(I.10)$ and $(I.7)$The phase velocity is given by $v_\varphi = c$

However, if propagation takes place in a material medium, then it is written as

$$v_\varphi = \frac{c}{n}$$

where it is inversely proportional to the refractive index n of the medium

- ***Group velocity***

Group velocity is an important concept in wave propagation. It represents the speed at which the overall shape of a wave packet, or the envelope of a group of waves, travels through a medium. Unlike phase velocity, which describes the motion of individual wave crests, the group velocity is associated with transmission of energy and information.

If we have two waves of different frequencies and equal amplitude, characterized by $\big((k-\Delta k),(\omega-\Delta\omega)\big)$ and $\big((k+\Delta k),(\omega+\Delta\omega)\big)$, then the total wave is given by

$$\psi(x,t) = \psi_1(x,t) + \psi_2(x,t)$$

$$\psi(x,t) = A\cos\big((k-\Delta k)x - (\omega-\Delta\omega)t\big) + Acos\big((k+\Delta k)x - (\omega+\Delta\omega)t\big)$$

Using a complex notation

$$\underline{\psi_1}(x,t) = Ae^{i[(k-\Delta k)x-(\omega-\Delta\omega)t]}$$

$$\underline{\psi_2}(x,t) = Ae^{i[(k+\Delta k)x-(\omega+\Delta\omega)t]}$$

Then, the wave function is expressed by

$$\underline{\psi}(x,t) = \underline{\psi_1}(x,t) + \underline{\psi_2}(x,t) = 2Ae^{i(kx-\omega t)}\big(e^{i(x\Delta k - t\Delta\omega)} + e^{i(x\Delta k + t\Delta\omega}\big)$$

Finally, the real total wave is given by:

$$\psi(x,t) = 2Ae^{i(kz-\omega t)}cos(x\Delta k - t\Delta\omega) \qquad (I.11)$$

In this case, the plane of constant amplitude moves at velocity v_g called group velocity

$$v_g = \frac{\Delta\omega}{\Delta k}$$

As $\Delta\omega$ and Δk are assumed to be small, we can write

$$v_g = \frac{d\omega}{dk} \qquad (I.12)$$

This is illustrated in Figure I.7. It is important to note that v_g represents the velocity of propagation of the wave energy. Thus, the group velocity v_g must be equal to or smaller than the speed of light c

If the medium is non dispersive: $\omega = ck$ implies $v_\varphi = v_g = c$.

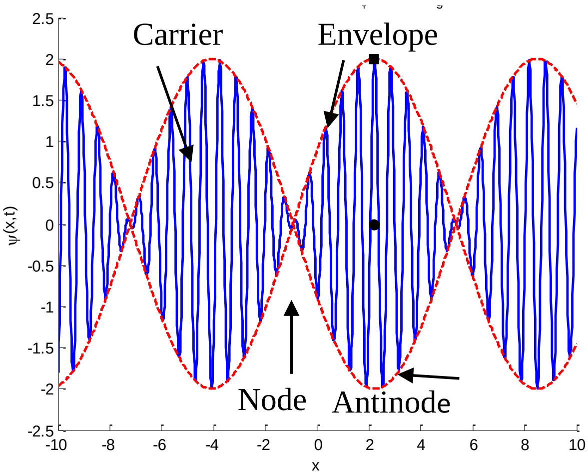


Figure I.7: Graphical illustration of group velocity

However, if the medium is dispersive (i.e., ω is a nonlinear function of k), the two velocities are different such as in the case of dielectrics, conductors and plasmas.

In this case, the wave vector must be considered as a function of the angular frequency ω. $\mathcal{D}(\vec{k},\omega) = 0$. This relationship is derived from the wave equation called the dispersion equation.

I.2.5. Spherical Waves

A spherical wave is a wave whose wavefronts (surfaces of constant phase) are spheres centered on a point source. In other words, the wave propagates equally in all directions from a single point (A small vibrating sphere in water, an isotropic antenna in radio communications…etc).

If the source is at the origin O , the wave depends only on the radial distance r and time t . A standard form of a spherical wave is:

$$\psi(r,t) = \psi_0(r)\, cos(\vec{k}.\vec{r} - \omega t) \qquad (I.13)$$

Starting from the 3D wave equation (I.1) and assuming spherical symmetry $\psi = \psi(r,t)$, it reduces to the radial wave equation:

$$\frac{1}{r}\frac{d^2}{dr^2}(r\psi) + k^2\psi = 0 \qquad (I.14)$$

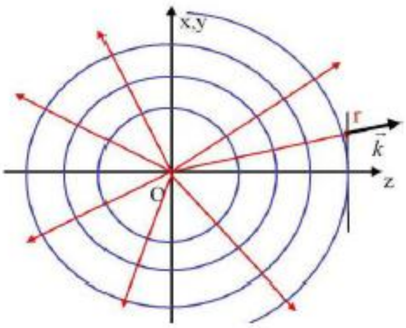


Figure I.8 : Representation of a spherical wave

Solutions describe outgoing and incoming spherical waves

$$\psi_{out}(r,t) = \frac{A}{r}cos(kr - \omega t), \psi_{in}(r,t) = \frac{B}{r}cos(kr + \omega t) \qquad (I.15)$$

- Spherical waves spread energy in all directions.
- The intensity decreases as $\frac{1}{r^2}$ with distance because energy is distributed over the growing surface of a sphere:

$$I(r) = \frac{P}{4\pi r^2} \qquad (I.16)$$

  Where P is the power of the source. This is known as the inverse square law.
- At large distances from the source $(r \gg \lambda)$, the curvature of the wavefront is negligible, so the wave locally behaves like a plane wave. This is called the "far-field approximation " radio and telecommunication, light from small point source…etc).

I.3. Electromagnetism

Electric and magnetic fields are closely related, as explained by James Clerk Maxwell (1860-1870). The relationships between these fields are described by a set of equations called Maxwell's equations, which predict the existence of electromagnetic wave.

Heinrich Hertz verified the existence of radio waves, which are a type of electromagnetic wave, by producing them experimentally based on Maxwell's equations. These experiments verified that radio waves travel at the speed of light, proving that they are part of the electromagnetic spectrum and can travel freely through space. In recognition of his contributions, the unit of frequency is named after him: the *Hertz*.

Before we examine the equations formulated by Maxwell, let's briefly review the concepts that will be frequently used throughout.

Field variables	***Names***	***Unit***
$\vec{E}$	Electric field	V/m
$\vec{D}$	Electric displacement field	C/m^2
$\vec{H}$	Magnetic Intensity	A/m
$\vec{B}$	Magnetic field	Tesla (T)
$\vec{J}$	Electric current density	A/m^2
ρ	Charge density	C/m^3

Constitutive Equations

$$\vec{E} = \varepsilon_0 \vec{D} \qquad \vec{B} = \mu_0 \vec{H} \qquad \vec{J} = \rho \vec{v}$$

Constants

- Vacuum permittivity ε_0

$$\varepsilon_0 = \frac{1}{36\,\pi \times \ 10^9} \qquad (S.I)$$

- Vacuum permeability μ_0

$$\mu_0 = 4\,\pi \ \times 10^{-7}\ (S.I)$$

- Speed of light *c*

$$c^2 = \frac{1}{\varepsilon_0 \mu_0} = 3 \times 10^8\ m/s$$

Basic vector operations

Before deriving the equations we will look at some needed vector analysis.

Let us quickly look it for electric field given by $\vec{E} = E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}$

- *Divergence of* $\vec{E}$

$$\vec{\nabla}.\vec{E} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).\left(E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}\right) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

- *Curl of* $\vec{E}$

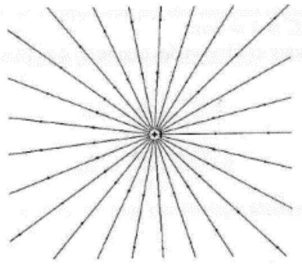
$$\vec{\nabla} \times \vec{E} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(E_x\vec{\imath} + E_y\vec{\jmath} + E_z\vec{k}\right)$$

- *Laplacian of* $\vec{E}$

$$\Delta\vec{E} = \vec{\nabla}^2\vec{E} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}$$

I.3.1. Maxwell’s Equations

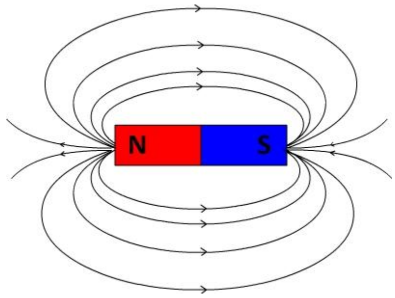
Maxwell’s equations describe the spatiotemporal behavior of the electromagnetic field ($\vec{E}, \vec{B}$) and how they are related to their sources. They are written as



- **Gauss’s Law** for Electricity

The electric field diverges from electric charges.
It describes how point charges act as sources of the electric field.

$$div\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (I.17)$$



- **Gauss’s Law** for Magnetism

Magnetic field lines $\vec{B}$ must be closed

$$div\,\vec{B} = 0 \qquad (I.18)$$

Magnetic field lines do not diverge (they do not extend to infinity) but rather run from one pole to the other.

- **Faraday’s Law of Induction**

$$\overrightarrow{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (I.19)$$

A variable magnetic field in time in the conductor causes a rotating electric field around the magnet. The sign (-) corresponds to Lenz's law, which states that the EMF (electromotive force) in the circuit opposes the cause that gives rise to it. This principle underlies electric generators and transformers.

- **Ampère-Maxwell Law**

$$\overrightarrow{rot}\vec{B} = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right) \qquad (I.20)$$

The magnetic field can be generated by electric currents (Ampère's theorem), or by the variation of an electric field (Maxwell's contribution). This law generalizes Ampère’s original law by including the displacement current term.

Maxwell's equations can be summarized in the following system of equations

$$\begin{cases} div\vec{E} = \dfrac{\rho}{\varepsilon_0} \qquad (MG) \\ div\,\vec{B} = 0 \\ \overrightarrow{rot}\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \qquad (MF) \\ \overrightarrow{rot}\vec{B} = \mu_0\left(\vec{j} + \varepsilon_0\dfrac{\partial \vec{E}}{\partial t}\right) \qquad (MA) \end{cases} \qquad (I.21)$$

It can be observed that this coupled system of equations links the spatial and temporal derivatives of the fields $\vec{E}$ and $\vec{B}$ to the sources $(\rho, \vec{j})$ that generate them.

Maxwell's equations are obtained in a medium similar to a vacuum ($\varepsilon = \varepsilon_0\varepsilon_r = \varepsilon_0$ and $\mu = \mu_0\mu_r = \mu_0$), but in which free charges and electric currents exist. It is therefore not a "true vacuum " where ρ and $\vec{j}$ must be equal to zero.

I.3.2. Energy and power of an electromagnetic wave

I.3.2.1. Electromagnetic energy

The energy carried by an electromagnetic wave, is stored in the electric field and magnetic field, and it can be transported through space.

To determine we will calculate the divergence of $\vec{E} \times \vec{B}$:

We have the following mathematical identity

$$\vec{\nabla}.\left(\vec{E} \times \vec{B}\right) = \vec{B}\,.\left(\vec{\nabla} \times \vec{E}\right) - \vec{E}\,.\left(\vec{\nabla} \times \vec{B}\right) \qquad (I.22)$$

Using the MF and MA relations, one obtains

$$\vec{\nabla}.\left(\vec{E} \times \vec{B}\right) = \vec{B}.\left(\frac{\partial \vec{B}}{\partial t}\right) - \vec{E}\,.\mu_0\left(\vec{j} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla}.\left(\vec{E} \times \vec{B}\right) = -\frac{1}{2}\frac{\partial \vec{B}^2}{\partial t} - \mu_0\left(\vec{j}.\vec{E}\right) - \frac{1}{2}\mu_0\varepsilon_0\frac{\partial \vec{E}^2}{\partial t}$$

By grouping all the terms together, we obtain

$$\frac{\vec{\nabla}.\left(\vec{E} \times \vec{B}\right)}{\mu_0} + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0}\right) + \vec{j}.\vec{E} = 0 \quad (I.23)$$

Let us define

$$\vec{\Pi} = \frac{\vec{E} \times \vec{B}}{\mu_0} \qquad (I.24)$$

$$w = \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0} \qquad (I.25)$$

$$\boxed{\vec{\nabla}.\vec{\Pi} + \frac{\partial w}{\partial t} + \vec{j}.\vec{E} = 0} \qquad (I.26)$$

This equation is referred to as Poynting's theorem expressing local energy balance describing how electromagnetic energy changes within a small region of space. It is a fundamental statement of energy conservation for electromagnetic fields.

It relates:

- Stored electromagnetic energy w (energy density)
- Energy flow out of region represented by the Poynting vector $\vec{\Pi}$. The divergence defines the electromagnetic energy radiated by the fields $\vec{E}$ and $\vec{B}$ per unit of time
- Energy delivered by the wave to matter which is dissipated through Joule heating ($\vec{j}.\vec{E}$)

I.3.2.2. Power of an electromagnetic wave

The total power of an electromagnetic wave is given by the flux of the Poynting vector through a surface area S by

$$\mathcal{P} = \iint \vec{\Pi}\,.\overrightarrow{dS} \qquad (I.26)$$

The energy study in a material depends on its nature (dielectric, conductor, etc.) and the shape of the surface (cylindrical, spherical, etc.) traversed by the Poynting vector.

It should be noted that the quantities involved in calculating of the energy and power carried by an electromagnetic wave are real-valued. Consequently, it is necessary to convert expressions from complex notation to their real form.

I.4. Propagation of electromagnetic waves in vacuum

I.4.1. Wave equation

In a vacuum (no free charges $\rho = 0$, and no electric currents $\vec{j} = \vec{0}$), Maxwell's equations take the following form

$$\begin{cases} \vec{\nabla}.\vec{E} = 0 & (MG) \\ \vec{\nabla}.\vec{B} = 0 & \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (MF) \\ \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} & (MA) \end{cases} \qquad (I.27)$$

Taking the curl of the Maxwell-Faraday equation , we obtain

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad (I.28)$$

Using the vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \Delta\vec{E} \qquad (I.29)$$

And taking account that $\vec{\nabla}.\vec{E} = 0$, this reduces to

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\Delta\vec{E} = -\mu_0\varepsilon_0 \frac{\partial^2\vec{E}}{\partial t^2}$$

Therefore, the Maxwell–Faraday and Maxwell–Ampère equations yield the wave equations

$$\begin{cases} \Delta\vec{E} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{E}}{\partial t^2} = \vec{0} & (I.30) \\ \Delta\vec{B} - \mu_0\varepsilon_0 \dfrac{\partial^2\vec{B}}{\partial t^2} = \vec{0} & (I.31) \end{cases}$$

These are d'Alembert equations, which describe the propagation of electromagnetic waves in vacuum with velocity $c = 1/\sqrt{\mu_0\varepsilon_0}$

We assume that the electromagnetic wave is a harmonic progressive plane wave. The electric $\vec{E}$ and magnetic $\vec{B}$ fields are expressed as

$$\vec{E} = \vec{E}_0\, cos(\omega t - \vec{k}.\vec{r}) \;\; , \vec{B} = \vec{B}_0 cos(\omega t - \vec{k}.\vec{r}) \quad (I.32)$$

In complex notation, the fields are given by

$$\underline{\vec{E}}(\vec{r},t) = \underline{\vec{E}_0} e^{i(\omega t - \vec{k}.\vec{r})} \quad , \quad \underline{\vec{B}}(\vec{r},t) = \underline{\vec{B}_0} e^{i(\omega t - \vec{k}.\vec{r})} \qquad (II.33)$$

With this representation, the differential operators simplify according to the substitutions:

$$\frac{\partial}{\partial t} \leftrightarrow i\omega, \qquad \vec{\nabla} \leftrightarrow -i\vec{k}.$$

I.4.2. Wave structure

In the complex notation, the vector operators act as follows

$$\frac{\partial\underline{\vec{E}}}{\partial t} = i\omega\underline{\vec{E}} \quad , \vec{\nabla}.\underline{\vec{E}} = -i\vec{k}.\underline{\vec{E}}\,, \;\; \vec{\nabla} \times \underline{\vec{E}} = -\, i\vec{k} \times \underline{\vec{E}} \qquad (II.34)$$

The Maxwell equations then reduce to

$$\begin{cases} \vec{\nabla}.\underline{\vec{E}} = i\,\vec{k}.\underline{\vec{E}} = 0 \\ \vec{\nabla}.\underline{\vec{B}} = i\,\vec{k}.\underline{\vec{B}} = 0 \\ \vec{\nabla} \times \underline{\vec{E}} = -\, i\,\vec{k} \wedge \underline{\vec{E}} = -\, i\,\omega\underline{\vec{B}} \\ \vec{\nabla} \times \underline{\vec{B}} = -\, i\,\vec{k} \wedge \underline{\vec{B}} = \dfrac{1}{c^2}\,(i\,\omega\underline{\vec{E}}\,) \end{cases} \qquad (I.35)$$

These relations confirm the transverse nature of plane electromagnetic waves

- From Gauss's law, $\vec{k}.\underline{\vec{E}} = 0 \Rightarrow \vec{k} \perp \underline{\vec{E}}$
- From Gauss's law for magnetism, $\vec{k}.\underline{\vec{B}} = 0 \Rightarrow \vec{k} \perp \underline{\vec{B}}$

- From Maxwell-Faraday's law:

$$\underline{\vec{B}} = \frac{\vec{k} \wedge \underline{\vec{E}}}{\omega} \qquad (I.36)$$

$\vec{E}$ and $\vec{B}$ vectors are orthogonal to the direction of propagation.

I.4.3. Dispersion equation

The wave propagation is rewritten as

$$\Delta\underline{\vec{E}} + \frac{\omega^2}{c^2}\underline{\vec{E}} = \vec{0} \qquad (I.37)$$

This equation is known as *Helmholtz's* equation.

The laplacian in the complex notation is given by

$$\Delta\underline{\vec{E}} = \frac{\partial^2\underline{\vec{E}}}{\partial x^2} + \frac{\partial^2\underline{\vec{E}}}{\partial y^2} + \frac{\partial^2\underline{\vec{E}}}{\partial z^2} = -k^2\underline{\vec{E}} \quad (I.38)$$

The dispersion equation is obtained

$$k^2 = \frac{\omega^2}{c^2} \qquad (I.39)$$

The propagation of electromagnetic waves in free space is non-dispersive: all frequencies travel at the same phase velocity ($v_\varphi = c$). This property distinguishes free-space propagation from dispersive media, where the relation between k and ω is more complex and leads to frequency-dependent group velocities, which will be the subject of the second chapter

I.4.4. Electromagnetic Energy Density

The electromagnetic energy density w_{EM} at any point in the medium traversed by an electromagnetic wave is given at each instant by the sum of the contributions of the electric and magnetic fields

$$w_{EM} = w_E + w_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2 \qquad (I.40)$$

For a plane wave, the magnetic field can be expressed as

$$\vec{B} = \frac{\vec{k} \wedge \vec{E}}{\omega} = \frac{k\vec{u} \times \vec{E}}{\omega} = \frac{\vec{u} \times \vec{E}}{c} \qquad (I.41)$$

where $\vec{u}$ denotes the unit vector along the direction of propagation.
In magnitude, this relation reduces to

$$B = \frac{E}{c} \qquad (I.42)$$

Substituting into the expression for the energy density gives

$$w_{EM} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}\frac{E^2}{c^2} = \varepsilon_0 E^2 \quad (I.43)$$

Thus, the total electromagnetic energy density is proportional to the square of the electric field amplitude.

I.4.5. Poynting Vector and energy transport

In free space, using the relation for B, one obtains

$$\vec{\Pi} = \frac{\vec{E} \times \vec{u} \times \vec{E}}{\mu_0 c} = \frac{E^2}{\mu_0 c}\vec{u} = \frac{cB^2}{\mu_0}\vec{u} \qquad (I.44)$$

As $\vec{u}.\vec{E} = 0$, the Poynting vector is oriented in the direction of wave propagation. It is measured in W/m^2

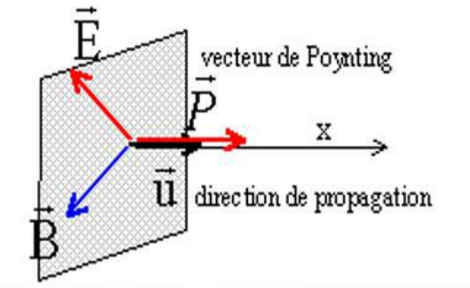


Figure I.9: Representation of Poynting Vector in the free space

Considering a surface S normal to the propagation direction, the flux of the Poynting vector through S is

$$\Phi = \Pi.S = \left(\frac{E^2}{\mu_0 c}\right)S \qquad (I.45)$$

The energy transported during a time interval dt is

$$dW = \varepsilon_0 E^2 Scdt = wScdt \qquad (I.46)$$

This expression shows that the energy contained in a cylindrical volume of base S and length cdt propagates at the speed of light c.

I.5. Electromagnetic waves Polarization

I.5.1. Harmonic plane waves polarization

We have seen that the vectors $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation. However, while remaining in the plane, they can move and describe different figures. The direction and trajectory described by the electric or magnetic fields during propagation are called the polarization of the wave (Figure I.10).

For a harmonic plane wave which propagates in the z- direction, the electric field vector components are given by

$$\vec{E}(z,t)\begin{cases}E_x = E_{0x}cos(\omega t - kz + \varphi_1)\\ E_y = E_{0y}cos(\omega t - kz + \varphi_2)\\ E_z = 0\end{cases} \qquad (I.46)$$

and the associated magnetic field is obtained from $\vec{B} = \frac{\vec{e}_z \times \vec{E}}{c}$

$$\vec{B}(\mathrm{z,t})\begin{cases}B_x = -\dfrac{E_{0y}}{c}cos(\omega t - kz + \varphi_1)\\ B_y = \dfrac{E_{0x}}{c}cos(\omega t - kz + \varphi_2)\\ B_z = 0\end{cases} \qquad (I.47)$$

I.5.2. Polarization States in the Wave Plane

For convenience, let us restrict to the plane $z = 0$. The electric field components reduce to

$$\begin{cases}E_x = E_{0x}cos(\omega t - \varphi_1)\\ E_y = E_{0y}cos(\omega t - \varphi_2)\end{cases} \qquad (I.48)$$

In the transverse plane (OXY), the end of the vector $\vec{E}$ traces a curve inscribed within a rectangle of sides $2E_{0x}$ and $2E_{0y}$. The nature of this curve depends on the value of phase shift $\Delta\varphi = \varphi_2 - \varphi_1$ between the x and y components.

a. Linear polarization

- For $\Delta\varphi = n\pi$ **,** (where $n \in Z$) the two components oscillate in phase (or in exact opposition). Thus, the electric vector remain constant. In fact, the ratio of the components is constant

$$\frac{E_x}{E_y} = \mp\frac{E_{0x}}{E_{0y}} = cste \qquad (I.49)$$

The electric field vector oscillates on the straight line in the plane wave. This corresponds to the linear polarization**.** The orientation of polarization is determined by the relative amplitudes E_{0x} and E_{0y}.

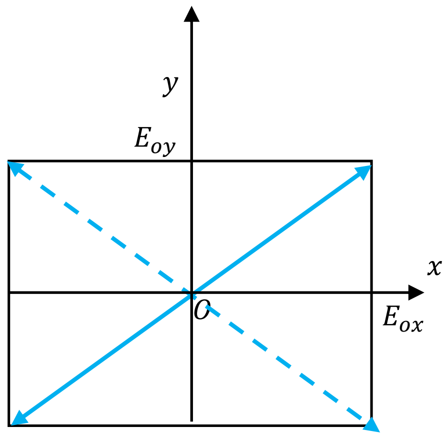


Figure I.10: Linear polarization

b. Elliptical polarization: $\Delta\varphi \neq n\pi$

In general, the components of the electric field are given by

$$E_x = E_{0x}cos(\omega t) \quad , \quad E_y = E_{0y}cos(\omega t - \varphi)$$

With the choice $\varphi_1 = 0$, this allows to write

$$\begin{cases} \dfrac{E_x}{E_{0x}} = cos(\omega t) \\ \dfrac{E_y}{E_{0y}} = cos(\omega t - \Delta\varphi) = cos(\omega t)cos(\Delta\varphi) + sin(\omega t)sin(\Delta\varphi) \end{cases}$$

By squaring the two equations and adding them, we obtain

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}}cos(\Delta\varphi) = sin^2(\Delta\varphi) \qquad (I.59)$$

This equation is that of an ellipse in the transverse plane. The end of the electric field vector describes an elliptical trajectory. The wave is said to be elliptically polarized.

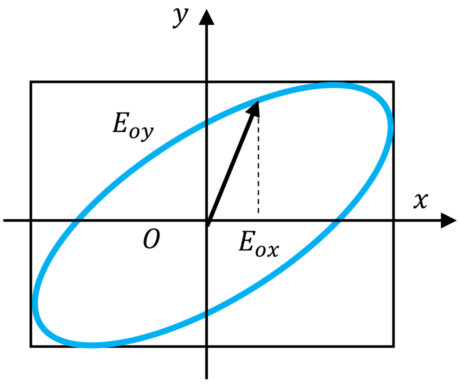


Figure I.11: Elliptic polarization

- **Circular polarization** $(\Delta\varphi = \mp(2n+1)\frac{\pi}{2})$

The amplitude of electric vector components are equal

$$E_{0x} = E_{0y}$$

The polarization is *circular*.

The sense of rotation of the $\vec{E}$ vector is determined by the sign of $\Delta\varphi$

Polarization is right-handed circular if the rotation is clockwise, otherwise, it is said to be left-handed circular.

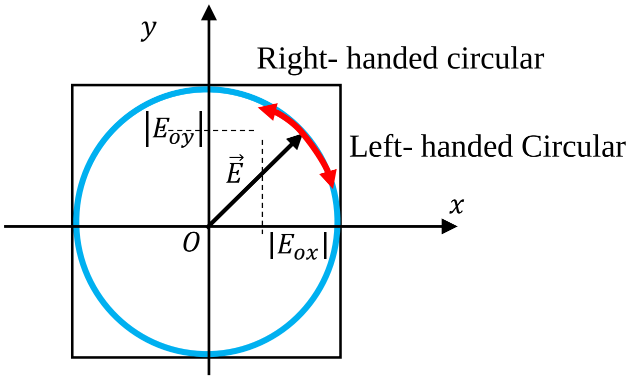


Figure I.12: Circular polarization

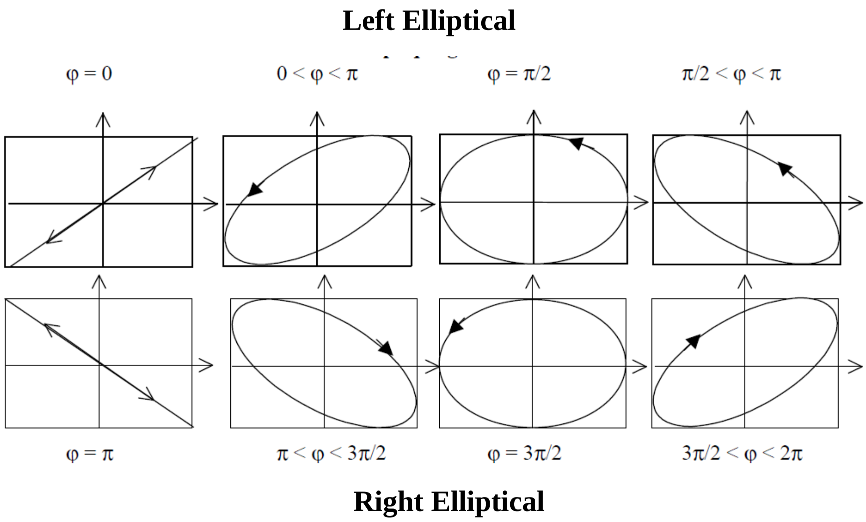


Figure I.13: Polarization states as a function of the phase shift $\Delta\varphi$

Exercise

Consider the electric field of an electromagnetic wave given by:

$$\vec{E} = 2\times\ 10^4\ cos\left(2.7\times\ 10^{14}\pi t - 4.5\ \times 10^5\pi\left(x+\sqrt{3}z\right)\right)\vec{e}_y$$

$\vec{e}_y$ is a unit vector along the (Oy) axis

1. Determine The

- Amplitude E_0
- Angular frequency $\omega$and the frequency f
- Wave vector $\vec{k}$
- Wavelength λ
- Phase velocity v
- Direction of propagation of the wave.

2. Show that $\vec{E}$ is a solution to the propagation equation.

3. Determine the expression for the corresponding magnetic field $\vec{B}$, calculate B_0.

4. Deduce the structural properties of the plane wave

Solution

1. The electric field is a harmonic plane wave with the general form

$$\vec{E} = E_0\ cos\left(\omega t - \vec{k}.\vec{r}\right)\vec{e}_y$$

The amplitude

$$E_0 = 2\times\ 10^4\ V/m$$

The angular frequency

$$\omega = 2.7\pi\times 10^{14} rad/s$$

The frequency

$$f = {}^{\omega}/_{2\pi} = {}^{2.7\times\pi\times 10^{14}}/_{2\pi} = 1.35\times 10^{14}\ Hz$$

The spatial phase:

$$\vec{k}.\vec{r} = (k_x x + k_z z) = 4.5\ \times 10^5\pi\left(x+\sqrt{3}z\right)$$

Thus the wave vector

$\vec{k} = 4.5\ \times 10^5\pi\left(\vec{e}_x+\sqrt{3}\vec{e}_z\right),\ k = 9\pi\times 10^5 m^{-1}$

The direction of propagation is therefore along

$$\vec{e}_k = {}^{1}/_{2}\left(\vec{e}_x+\sqrt{3}\vec{e}_z\right)$$

The phase velocity

$$v = {}^{\omega}/_{k} = {}^{2.7\pi \times 10^{14}}/_{9\pi \times 10^{5}} = 3 \times 10^{8} m/s$$

The wave propagates in vacuum (free space)

2.

In a homogeneous, source-free medium, the electric field satisfies the wave equation

$$\Delta\vec{E} - \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{0}$$

For a plane wave of the form

$$\vec{E} = \overrightarrow{E_0}\, cos\big(\omega t - \vec{k}.\vec{r}\big)$$

We have

$$\Delta\vec{E} = -k^2\vec{E} \quad , \frac{\partial^2\vec{E}}{\partial t^2} = -\,\omega^2\vec{E}$$

Substitution yields

$$\left(-k^2 + \frac{\omega^2}{v^2}\right)\vec{E} = \vec{0}$$

Since $v = {}^{\omega}/_{k}$, this equality is satisfied.

$\vec{E}$ is a solution of the wav equation

3. The magnetic field associated with the wave

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

We find

$$\vec{B} = \frac{E_0}{v}\ \ cos\big(\omega t - \vec{k}.\vec{r}\big)\big(-\sqrt{3}\vec{e}_x + \vec{e}_z\big)$$

Magnetic field amplitude is

$$B_0 = \frac{E_0}{v}$$

4. Wave structure

This wave is a plane electromagnetic wave, which is linearly polarized along y-direction

We have: $\vec{E} \perp \vec{B} \perp \vec{k}$ the wave is transverse

Chapter II

Electromagnetic wave propagation in isotropic media

II.1. INTRODUCTION

Electromagnetic waves propagate differently in materials than in a vacuum, and in many media their velocity depends on frequency, defining dispersive behavior. This property is fundamental for understanding signal transmission and energy transport. In this course, we focus on the propagation of electromagnetic waves in three fundamental types of media (dielectrics, conductors, and plasmas) with specific physical characteristics that govern their interaction with electromagnetic fields. The description of this propagation relies on Maxwell's equations combined with constitutive relations that incorporate material responses such as permittivity, permeability and electrical conductivity. These parameters, enable the modeling of both energy storage and dissipation, and provide the framework to explain key phenomena including attenuation and phase shifts.

II.2. Interaction of electromagnetic fields with matter

The interaction of an electromagnetic field with matter induces polarization in the medium by influencing its microscopic charges. This polarization creates an induced field that adds to the external field.

In dielectrics, the charges are bound and rearrange into electric dipoles, while in conductors the charges are free to move and produce currents. The essential distinction between the two lies in the behavior of the electric field, which in conductors, free charges redistribute to cancel the field, preventing it from penetrating the material, whereas in insulators the field penetrates and produces polarization effects.

II.3. Polarization of Matter

II.3.1. Types of Polarization in Dielectrics

As discussed earlier, when a dielectric medium is subjected to an external electric field, the bound charges react according to the microscopic structure of the material, leading to different types of polarization.

- Electronic polarization takes place when an electric field causes a disturbance in the distribution of the electrons in neutral atoms/molecules lacking a permanent dipole. The effect causes a shift in the center of the positive G^+ and negative G^-charges, leading to the creation of an induced dipole.
- Orientation polarization concerns polar molecules that already have a permanent dipole. Under the effect of the electric field, these dipoles tend to align themselves in the direction of the field, although thermal agitation limits this alignment.
- Ionic polarization occurs in ionic crystals, where the electric field causes a slight relative displacement of the positive and negative ions in the crystal lattice, thus generating an electric dipole.

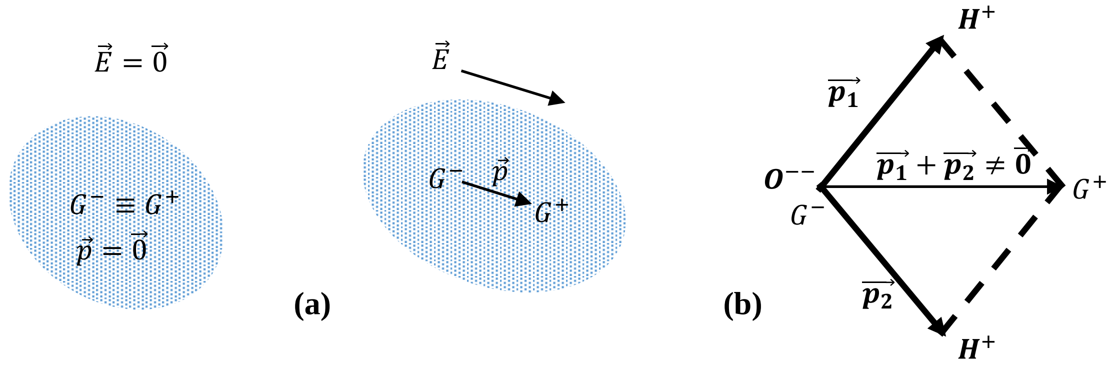


Figure II.1: (a) Electric dipole before and after excitation
(b) The water molecule as a permanent electric dipole

II.3.2. Polarization vector and bound charges

The polarization vector $\vec{P}(\vec{r}, t)$ represent the dipole moment per unit volume in the material

$$\vec{P}(\vec{r}, t) = \frac{d\vec{p}}{d\tau} \qquad (II.1)$$

where $d\vec{p}$ denotes the dipole moment contained within an infinitesimal volume element dτ. This vector can also be expressed as

$$\vec{P}(\vec{r}, t) = \frac{d\vec{p}}{d\tau} = n\vec{p} \qquad (II.2)$$

with n representing the number density of dipoles (i.e., the number of dipoles per unit volume). The dipole moment itself is defined as

$$\vec{p} = q\overrightarrow{G^-G^+} \qquad (II.3)$$

where q is the charge of the dipole, and $\overrightarrow{G^-G^+}$ is the vector connecting the barycenter of negative charges to the barycenter of positive charges.

By differentiating the polarization vector with respect to time, one obtains

$$\frac{\overrightarrow{dP}(\vec{r},t)}{dt} = n\frac{\overrightarrow{\partial p}}{\partial t} = nq\overrightarrow{v_p} = \vec{j_p} \qquad (II.4)$$

This quantity J_P is the current density associated with the motion or reorientation of bound charges.

To study the electric state of a polarized medium, we introduce the volume density of bound charges. In the absence of free charges and free currents, the principle of charge conservation leads to:

$$\frac{\partial \rho_p}{\partial t} + div\vec{j}_p = 0 \qquad (II.5)$$

Using equation (II.4), this relation simplifies to

$$\rho_p = -div\,\vec{P} \qquad (II.6)$$

II.3.3. Maxwell's Equations in dielectrics

By systematically exploring these cases, we will build a comprehensive understanding of how electromagnetic waves behave in real-world materials, bridging theory with practical applications in optics, telecommunications, and plasma physics.

- **Gauss's Law**

In free space

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

In a polarized medium, the total charge density is the sum of free charges ρ_f and bound charges ρ_P

$$\vec{\nabla}.\vec{E} = \frac{\rho_f + \rho_P}{\varepsilon_0} \qquad (II.8)$$

Using the relation (II.6), this becomes:

$$\vec{\nabla}.\vec{E} = \frac{\rho_f}{\varepsilon_0} - \frac{div\,\vec{P}}{\varepsilon_0} \qquad (II.9)$$

Using the electric displacement field

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \qquad (II.10)$$

we obtain the macroscopic form

$$\vec{\nabla}.\vec{D} = \rho_f \qquad (II.11)$$

- **Ampère-Maxwell Law**

In free space

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (II.12)$$

In matter, the current density includes contributions from both free and bound states

$$\vec{J} = \overrightarrow{J_f} + \overrightarrow{J_P} \qquad (II.12)$$

With the equation (II.12), we have

$$\vec{\nabla} \times \vec{B} = \mu_0 \overrightarrow{J_f} + \mu_0 \left(\frac{d\vec{P}}{dt} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Grouping terms

$$\vec{\nabla} \times \vec{B} = \mu_0 \overrightarrow{J_f} + \mu_0 \frac{\partial \vec{D}}{\partial t} \qquad (II.13)$$

The macroscopic Maxwell's equations in dielectrics become

$$\begin{cases} \vec{\nabla} . \vec{D} = \rho_f \\ \vec{\nabla} . \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \mu_0 \overrightarrow{J_f} + \mu_0 \varepsilon \dfrac{\partial \vec{D}}{\partial t} \end{cases} \qquad (II.14)$$

II.4. Linear Homogenous Isotropic (LHI) dielectric

We will consider linear, homogeneous, and isotropic (LHI) media

- Linear : there is a linear relationship in Fourier space between $\vec{P}$ and $\vec{E}$, called electrical susceptibility $[\chi_e]$

$$\vec{P} = \varepsilon_0 [\chi_e](\omega) \vec{E} \qquad (II.15)$$

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Isotropic: All directions are equivalent. the properties of the medium are independent of the orientation of the EM wave. $[\chi_e]$ is a scalar.

$$[\chi_e] = \begin{pmatrix} \chi_e & 0 & 0 \\ 0 & \chi_e & 0 \\ 0 & 0 & \chi_e \end{pmatrix}$$

- Homogeneous: the properties of the medium (permittivity, permeability and conductivity) are identical at all points of the EM wave displacement.

Under this assumption, the electric displacement vector is expressed as follows

$$\vec{D} = \varepsilon_0 \,(1 + \chi_e)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \qquad (II.16)$$

with the relative permittivity ε_r is defined as

$$\varepsilon_r = 1 + \chi_e$$

Electromagnetic waves in dielectric mediums are primarily transported via bound charge polarization and not through conduction currents. The important determining factor in this process is the value of electrical permittivity ε, which is defined as $\varepsilon = \varepsilon_0 \varepsilon_r$, which characterizes how the medium responds to an applied electric field.

II.4.1. Wave equation and dispersion

a. Perfect dielectric

We consider the case of a perfect, non-magnetic ($\mu_r = 1$) LHI medium ($\rho_f = 0, \vec{J_f} = \vec{0}$)

Under these conditions, Maxwell's equations are written as

$$\begin{cases} \vec{\nabla}.\vec{D} = 0 \\ \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \mu_0 \dfrac{\partial \vec{D}}{\partial t} \end{cases} \qquad (II.17)$$

By combining Maxwell’s equations and calculating the curl of MF equation, wave equation is given by

$$\Delta \vec{E} - \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \qquad (II.18)$$

Or, in the case of sinusoidal field ($\vec{E} = \vec{E}_0 \, e^{i(\omega t - \vec{k}.\vec{r})}$)

$$\Delta \vec{E} + \frac{\varepsilon_r}{c^2} \omega^2 \vec{E} = \vec{0} \qquad (II.19)$$

The Helmholtz equation (II.19) is similar to the corresponding equation (II.37) in a vacuum, except for a factor ε_r. The corresponding dispersion relation is then simply

$$k^2 = \frac{\varepsilon_r}{c^2} \omega^2 = \frac{n^2}{c^2} \omega^2 \qquad (II.20)$$

In perfect dielectrics, electromagnetic waves propagate as transverse plane waves with reduced velocity $v = \frac{c}{\sqrt{\varepsilon_r}}$.

In complex notation

$$\begin{cases} \vec{k}.\vec{D} = 0 \\ \vec{k}.\vec{B} = 0 \\ \vec{k}\times\ \vec{E} = \omega\vec{B} \\ \vec{k}\times\ \vec{B} = -\mu_0\omega\vec{D} \end{cases} \qquad (II.21)$$

The vectrors $\vec{k}$, $\vec{E}$ and $\vec{B}$ are mutually perpendicular, forming a right-handed orthogonal triad. The wave is transverse, both $(\vec{D},\vec{B})$ are perpendicular to the propagation direction.

b. Real dielectric

A real (lossy) dielectric is an imperfect insulating medium characterized by a finite electrical conductivity $\sigma \neq 0$, in contrast to an ideal lossless dielectric for which $\sigma = 0$. Electromagnetic energy propagating through a lossy dielectric is partially dissipated due to conduction losses.

Consider a linear, homogeneous, isotropic, and non-magnetic dielectric medium of permittivity$\varepsilon = \varepsilon_0\varepsilon_r$ with no free electric charges $\rho_f = 0$ and and a conduction current density $\vec{j}$ governed by Ohm's law ($\vec{j} = \sigma\overrightarrow{E}$).

The propagation of electromagnetic waves is described by Maxwell's equations with both displacement and conduction current contributions:

$$\begin{cases} \vec{\nabla}.\vec{E} = 0 \\ \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla}\times\ \vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\ \vec{B} = \mu_0(\sigma\vec{E} + \varepsilon_0\varepsilon_r\dfrac{\partial\vec{E}}{\partial t}) \end{cases} \qquad (II.22)$$

The wave equations are

$$\Delta\vec{E} - \mu_0\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\sigma\frac{\partial\vec{E}}{\partial t} \qquad (II.23)$$

$$\Delta\vec{B} - \mu_0\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} = \mu_0\sigma\frac{\partial\vec{B}}{\partial t} \qquad (II.24)$$

- *Dispersion equation*

Assuming the harmonic plane waves for electric field:

$$\vec{E} = \vec{E}_0 exp\left(i\left(\omega t - \vec{k}.\vec{r}\right)\right) \quad , \quad \vec{B} = \vec{B}_0 exp\left(i\left(\omega t - \vec{k}.\vec{r}\right)\right)$$

Maxwell’s equations become

$$\begin{cases} \vec{\nabla}.\vec{E} = i\,\overrightarrow{k}.\vec{E} = 0 \\ \vec{\nabla}.\vec{B} = i\,\overrightarrow{k}.\vec{B} = 0 \\ \vec{\nabla}\times\ \vec{E} = -i\vec{k}\wedge\underline{\vec{E}} = -\,i\omega\,\vec{B} \\ \vec{\nabla}\times\ \underline{\vec{B}} = -i\vec{k}\wedge\underline{\vec{B}} = \ i\mu_0\varepsilon\,\omega\vec{E}\ +\ \mu_0\sigma\vec{E} \end{cases} \qquad (II.25)$$

Calculating the curl of Maxwell-Faraday equation, we have

$$\vec{k} \wedge (\vec{k} \wedge \vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - k^2\vec{E} = (-\mu_0\varepsilon\omega^2 + i\omega\sigma)\vec{E}$$

The dispersion equation is given by

$$k^2 = (\mu_0\varepsilon\omega^2 - i\omega\sigma\mu_0) \qquad (II.26)$$

It can be observed that k^2 is generally complex and can be expressed in the form

$$k = k' - ik''$$

where k' represents the phase propagation of the wave, determining how the wave oscillates in space, while k'' accounts for attenuation, describing the exponential decay of the wave amplitude as it travels through the medium. In dielectric materials, it is often more convenient to work with the propagation constant, which directly incorporates both the phase shift and the damping effects, rather than the wave number itself.

In perfect dielectrics, electromagnetic waves propagate as transverse plane waves with lower velocity $v = \frac{c}{\sqrt{\varepsilon_r}} = \frac{c}{n}$. The refractive index n encapsulates how microscopic polarization slows the wave, while the propagation constant $\gamma = i\beta$ reflects purely oscillatory behavior without attenuation.

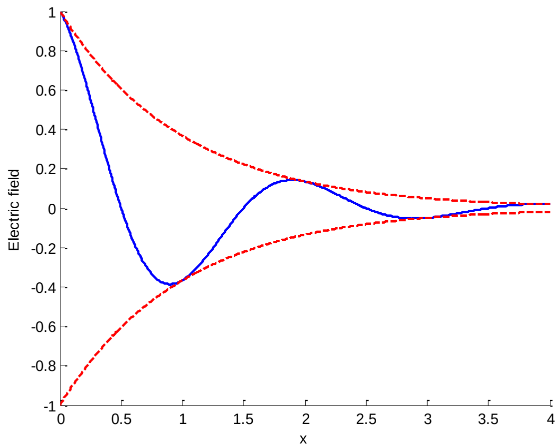


Figure II.2: Electric field in real dielectric

II.4.2. Complex refractive index

a. Bound Electron Model

We consider the model of an elastically bound electron, where the nucleus is assumed to remain fixed. When an external electric field oscillates, the electron is subject to the following forces:

- Electric force : $\vec{F_e} = -e\vec{E}$
- Elastic restoring force : $\vec{F_r} = -m\omega_0^2\vec{r}$. ($\omega_0$ is resonance frequency)

- Damping (frictionanl) force : $\vec{F}_{fr} = -\frac{m}{\tau}\dot{\vec{r}}$. (the energy loss due to radiation emitted by the accelerated electron).

Let $\vec{r}$ denote the position of the electron relative to the fixed barycenter of the positive charges (the nucleus). Applying Newton's second law yields:

$$-e\vec{E} - -m\omega_0^2\vec{r} - -\frac{m}{\tau}\dot{\vec{r}} = m\ddot{\vec{r}} \qquad (II.27)$$

Assuming a solution of the form of a harmonic oscillation: $\vec{r} = \vec{r_0}e^{-i\omega t}$ and $\vec{E} = \vec{E_0}e^{-i(\omega t - \vec{k}.\vec{r})}$, we obtain:

$$\vec{r} = -\frac{e}{m}\frac{1}{(\omega_0^2 - \omega^2) + i\frac{\omega}{\tau}}\vec{E} \qquad (II.28)$$

In this case, the dipole moment given by $\vec{p} = -e\vec{r}$ allows us to express the polarization vector in the following form:

$$\vec{P} = n\vec{p} = \frac{ne^2}{m}\frac{1}{(\omega_0^2 - \omega^2) - i\frac{\omega}{\tau}}\vec{E} \qquad (II.29)$$

Using the constitutive relation ($\vec{P} = \varepsilon_0\chi_e\vec{E}$), the electric susceptibility is obtained as:

$$\chi_e = \frac{ne^2}{\varepsilon_0 m}\frac{1}{(\omega_0^2 - \omega^2) - i\frac{\omega}{\tau}} \qquad (II.30)$$

By introducing the plasma frequency

$$\omega_p^2 = \frac{ne^2}{\varepsilon_0 m}$$

We can rewrite the susceptibility in a more compact form:

$$\chi_e(\omega) = \frac{\omega_p^2}{(\omega_0^2 - \omega^2) - i\frac{\omega}{\tau}}$$

$$\chi_{e(\omega)} = \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} - \frac{i\omega}{\tau}\frac{\omega_p^2}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} \qquad (II.31)$$

the relative permittivity is complex and can be written as:

$$\varepsilon_r = 1 + \chi_e(\omega) = 1 + \omega_p^2\left[\frac{(\omega_0^2 - \omega^2) + \frac{i\omega}{\tau}}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}}\right] \qquad (II.32)$$

where both the real and imaginary parts depend explicitly on the frequency ω.

b. Refractive index

The refractive index follows from the relation:

$$n^2(\omega) = \varepsilon_r = 1 + \chi_e(\omega)$$

Which, using the Lorentz form of χ_e, gives :

$$n^2(\omega) = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} + \frac{i\omega}{\tau}\frac{\omega_p^2}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} \qquad (II.33)$$

Thus, the refractive index is itself complex:

$$n(\omega) = n'(\omega) + in''(\omega)$$

In dilute media, where $\chi_e << 1$, one can approximate:

$$n(\omega) = \sqrt{1 + \chi_e} \approx 1 + \frac{1}{2}\chi_e$$

This yields explicit expressions for the real and imaginary parts:

$$n'(\omega) \approx 1 + \frac{1}{2}\frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} \qquad (II.34)$$

$$n''(\omega) \approx \frac{\omega}{2\tau}\frac{\omega_p^2}{(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2}} \qquad (II.35)$$

Since the wavevector is related to the refractive index by : $k = {}^{n\omega}/_{c}$, it is also complex

$$\vec{k}(\omega) = \overrightarrow{k'}(\omega) - i\overrightarrow{k''}(\omega)$$

With magnitudes

$$\begin{cases} \left\|\overrightarrow{k'}(\omega)\right\| = \dfrac{n'(\omega)}{c}\omega \\ \left\|\overrightarrow{k''}(\omega)\right\| = \dfrac{n''(\omega)}{c}\omega \end{cases} \qquad (II.36)$$

- The real part n' determines the phase velocity

$$v_\varphi = \frac{\omega}{|Re(k)|} = \frac{\omega}{k'} = \frac{c}{n'} \qquad (II.37)$$

If n' depends on ω, the medium exhibits dispersion. Far from resonance frequency ω_0, n' is real and varies weakly with frequency, so dispersion is small.

- The imaginary part k'' characterizes absorption of the wave by the medium. It is also called *the extinction index*

$$n'' = \frac{c}{\omega}|Im\,(k)| = \frac{c}{\omega}k'' \qquad (II.38)$$

- The penetration depth is given by

$$\delta = \frac{1}{k''} \qquad (II.39)$$

which provides the order of magnitude of the distance over which the wave amplitude decays significantly. For absorption to occur, the frequency must be within a range close to the medium's natural frequency. For example, glass absorbs UV radiation with a wavelength of 320 nm.

II.5. Electromagnetic waves in an Ohmic conductor

A metal can be described as a system of free electrons that move collectively, carrying their associated electron clouds as they drift through the lattice. These electrons are subjected to two main forces:

- Electrical force exerted by the applied external electric: field. $\overrightarrow{F_e} = q\vec{E}$
- Frictional (scattering) for $\overrightarrow{F_{fr}} = -\frac{m}{\tau}\vec{v}$, arising from collisions between free electrons and the lattice ions or impurities.

The average time interval between two successive collisions is called the relaxation time τ Applying the fundamental principle of dynamics (Newton's second law) to a single electron, the equation of motion can be expressed as

$$m\frac{d\vec{v}}{dt} = q\vec{E} - \frac{m}{\tau}\vec{v} \qquad (II.40)$$

m is the electron mass, $-e$ is the electron charge, $\vec{v}$ is the electron velocity.

The differential equation for the velocity vector $\vec{v}$ is:

$$\frac{d\vec{v}}{dt} + \frac{1}{\tau}\vec{v} = \frac{q}{m}\vec{E} \qquad (II.41)$$

In the case of a sinusoidal excitation, the velocity $\vec{v}$ is assumed to vary as:

$$\vec{v} = \vec{v}_0\, e^{-i\omega t}, \qquad \frac{d\vec{v}}{dt} = -i\omega\vec{v}$$

Substituting into equation $(II.56)$ gives:

$$-i\omega\vec{v} + \frac{1}{\tau}\vec{v} = \frac{q}{m}\vec{E}$$

Solving for the velocity vector yields:

$$\vec{v} = \frac{q}{m}\frac{\tau}{1 - i\tau\omega}\vec{E} \qquad (II.42)$$

The electrical conductivity is then defined as:

$$\sigma = \frac{\sigma_0}{1 - i\omega\tau} \qquad (II.43)$$

Where $\sigma_0 = \frac{nq^2\tau}{m}$ is the static conductivity

n is the electron density, $q = -e$ the electron charge.

For very short relaxation times $\tau \ll 1$, the electrical conductivity σ can be treated as a real quantity. In this regime, the current density vector $\vec{j}$ is locally related to the electric field by Ohm's law:

$$\vec{j} = \sigma\vec{E} \qquad (II.44)$$

II.5.1. Good conductors approximation

The Maxwell–Ampère relation can be written as:

$$\overrightarrow{\nabla \times} \vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0(\vec{j} + \vec{j}_D) \qquad (II.45)$$

where the displacement current density is defined as:

$$\vec{j}_D = \varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad (II.46)$$

At high frequencies, this term becomes significant since $\partial\vec{E}/\partial t \sim i\omega\vec{E}$

The magnitudes of the two current densities are

$$\|\vec{j}\| = \sigma\, E\ , \quad \|\overrightarrow{j_D}\| = \varepsilon_0\omega E$$

For the displacement current to be negligible compared to the conduction current, the following condition must hold

$$\varepsilon_0\omega \ll \sigma \qquad (II.47)$$

This is the good conductor approximation**.**

Example

For copper, the conductivity is approximately: $\sigma = 6\ 10^7 S.m^{-1}$

The condition $\varepsilon_0\omega \ll \sigma$ is satisfied when the wavelength: $\lambda \gg 28\ pm$

This corresponds to the ultraviolet, visible, and infrared regions of the spectrum

A material behaves as a good conductor when

- The conductivity σ is real, meaning that losses due to relaxation can be neglected
- The displacement current density $\overrightarrow{J_D}$ is much smaller than the conduction current density $\vec{j}$

In this regime, conduction dominates over displacement, and the electromagnetic response is governed primarily by the free electron current.

II.5.2.Wave equation

From the generalized wave equation, under the good conductor condition, we obtain

$$\Delta\vec{E} - \mu_0\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\sigma\frac{\partial\vec{E}}{\partial t} \qquad (II.48)$$

Unlike the previously derived d'Alembert wave equation**,** this is a diffusion equation**.** It introduces the diffusivity (diffusion coefficient)

$$D = \frac{1}{\mu_0\sigma} \qquad (II.49)$$

In this case, the dispersion relation becomes

$$k^2 = i\omega\mu_0\sigma \qquad (II.50)$$

The wave vector is therefore

$$k = \mp(1+i)\sqrt{\frac{\omega\mu_0\sigma}{2}} = \mp\frac{(1+i)}{\delta} \qquad (II.51)$$

where δ is the skin depth.

Note that the wave number k is complex. The real part is responsible for wave propagation and the imaginary part reflects wave attenuation. This attenuation leads to the skin effect, a phenomenon characteristic of electromagnetic wave propagation in metals (good conductors). The skin depth is defined as:

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} \qquad (II.52)$$

Thus, for a metal of finite thickness, the electromagnetic wave penetrates only over a depth δ, within which its amplitude decays exponentially. Beyond this depth, the wave becomes evanescent**.**

In good conductors, electromagnetic waves do not propagate freely as in dielectrics. Instead, they diffuse into the material with limited penetration. The skin depth δ quantifies this penetration, showing how conductivity and frequency control the confinement of electromagnetic fields near the surface of metals.

Examples

- The thickness of human skin is of the order of 10^{-3}m (a few millimeters).

- A sheet of paper has a thickness on the order of 10^{-4}m.
- The thickness of aluminum foil is typically on the order of 10^{-5}m.
- Structural walls have thicknesses on the order of 10^{-1}m

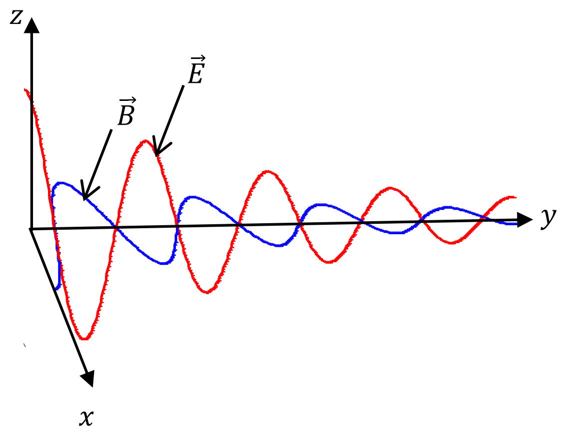


Figure II.3: Electromagnetic wave propagation in a good conductor

The absorption of an electromagnetic wave in a conductor depends on the relation between the skin depth δ and the material thickness e

Example

Consider an electromagnetic wave in a copper wire with diameter $a = 0.5\,mm$, $\sigma = 10^7 Sm^{-1}$ The electric current only flows in the region where the electric field is non-zero, and this region depends strongly on frequency through the skin depth.

- At low frequency $f = 50Hz$, the skin depth is much larger than the wire diameter ($\delta = 9.2\,mm \gg a$). The skin effect is negligible, the wave penetrates the entire conductor, and current is distributed throughout the cross-section.
- At intermediate frequency $f = 100\,KHz$, the skin depth ($\delta = 0.2\,mm$)becomes to wire diameter marking the onset of the skin effect.
- At high frequency $f = 100\,MHz$, the skin depth is extremely small ($\delta = 6.5\,10^{-6}\,mm$) The current is confined to a very thin surface layer, and the electromagnetic wave is fully absorbed before reaching the opposite side of the wire.

For industrial power frequencies around 50 Hz, relatively thick cables can be used since the skin effect is weak. At higher frequencies, however, thin conductors are preferred. When $\delta << e$, the electromagnetic wave is completely absorbed within the conductor before reaching the opposite boundary.

The case of a real conductor will be discussed as special case in the study of plasma.

II.6. Propagation in the plasma

II.6.1. Description of plasma

A plasma is a medium composed of atoms or molecules that are partially or fully ionized, while remaining globally electrically neutral. The charged particles (ions and electrons) that constitute the plasma interact through electromagnetic forces. The long-range nature of these forces allows a single particle to interact with many others, imparting a collective character to the interactions.

The creation of a plasma requires a significant energy input. This can be achieved through heating, bombardment with an intense laser beam, or by electrical discharge in a gas subjected to a large potential difference. Traditionally, plasmas are classified into two categories: *cold plasmas*, with temperatures below approximately $10^5 K$, and *hot plasmas*, with temperatures exceeding approximately $10^6 K$

Examples of plasmas in nature are numerous, including, ionosphere, stellar cores, which are hot and extremely dense plasmas.

II.6.2. Harmonic plane wave propagation in the plasma

a. ***Motion of ions and electrons***

In the following, we consider a plasma model consisting of n_i ions (with mass M and charge $+e$) and ne electrons (with mass m and charge $-e$) per unit volume. All interactions between ions are neglected—neither electrostatic attraction or repulsion nor collisions are taken into account. This assumption is valid for a low-density plasma. At thermodynamic equilibrium, the densities of electrons and ions are equal

$$n_e = n_i = n$$

In the regime of forced oscillations, the equation of motion for a charged particle in an electric field is

$$m\frac{d\vec{v}}{dt} = q\vec{E} \qquad (II.53)$$

Where $\vec{v}$ is the instantaneous velocity of the electron. Here we have neglected the force due to magnetic field which is only $({}^{v}/_{c})$ times the force due to electric field.

Assuming a harmonic electric field of the form: $\vec{E} = \underline{\vec{E}_0} e^{i(\vec{k}.\vec{r}-\omega t)}$

Integration yields

$$-i\, m\omega\underline{\vec{v}} = -e\,\underline{\vec{E}} \qquad (II.54)$$

Thus, the velocity of the electrons under the action of the electric field $\underline{\vec{E}}$ is:

$$\underline{\overrightarrow{v_e}} = -i\frac{e}{m\omega}\underline{\vec{E}} \qquad (II.55)$$

Similarly, the velocity of the ions is

$$\underline{\overrightarrow{v_i}} = i\frac{e}{M\omega}\underline{\vec{E}}$$

Comparing the two velocities gives

$$\frac{v_i}{v_e} = \frac{m}{M} \ll 1 \qquad (II.56)$$

Since the ion mass M is much larger than the electron mass m, the motion of the ions can be neglected relative to that of the electrons.

c. Conductivity of the medium

Let us denote the number of free electrons per unit volume. The complex current density is then:

$$\vec{j} = n_e(-e)\overrightarrow{v_e} + n_i(+e)\overrightarrow{v_i} \approx -en_e\overrightarrow{v_e} \qquad (II.57)$$

since the ion contribution is negligible compared to that of the electrons. Assuming that the electron density n_e is essentially equal to the equilibrium density n, we obtain:

$$\underline{\vec{j}} \approx i\frac{n\,e^2}{m\omega}\underline{\vec{E}} \qquad (II.58)$$

Comparing this expression with Ohm's law $\vec{j} = \sigma\vec{E}$, the complex conductivity of the medium is found to be

$$\sigma = i\frac{n\,e^2}{m\omega} \qquad (II.59)$$

It follows that the current density $\underline{\vec{j}}$ and the electric field $\underline{\vec{E}}$ are phase-shifted by π/2, i.e. they are orthogonal. As a consequence, the average radiated power vanishes

$$P = \frac{1}{2}\,Re\left(\vec{j}.\overrightarrow{E^*}\right) = 0 \qquad (II.60)$$

indicating the absence of energy loss. In general, energy dissipation in a conductor arises from the real component of the conductivity.

Remark

By introducing a friction term, one recovers the equations obtained in the study of a conductor (II.46). Without repeating the full derivation, the electrical conductivity is expressed as

$$\sigma = \frac{\sigma_0}{1 - i\omega\tau}$$

where τ is the mean collision time.

Two limiting cases arise

- Plasma regime: When the conductivity is purely imaginary ($\sigma \epsilon iR$), corresponding to collisionless oscillations of electrons.
- Good conductor regime: When τ is very small (frequent collisions), the conductivity becomes real ($\sigma = \sigma_0 \epsilon R$), which is the case for metals and good conductors.

II.6.3. Dispersion and absorption in the plasma

In the case of dilute ionized medium, we can assume $\rho = 0$, $\varepsilon \cong \varepsilon_0$ and $\mu = \mu_0$.

Maxwell's equations take the following form:

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} = 0 \\ \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \mu_0 \sigma \vec{E} + \mu_0 \varepsilon \dfrac{\partial \vec{E}}{\partial t} \end{cases} \qquad (II.61)$$

From these relations, the wave equation for the electric field $\vec{E}$ is obtained

$$\Delta \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0} \qquad (II.62)$$

A similar equation holds for magnetic field $\vec{B}$.

Assuming plane wave: $\underline{\vec{E}}(\vec{r}, t) = \underline{\vec{E}_0} e^{i(\omega t - \vec{k}.\vec{r})}$, the dispersion relation (II.26) is expressed as

$$k^2 = (\mu_0 \varepsilon \omega^2 - i\mu_0 \omega \sigma)$$

or equivalently

$$k^2 = \mu_0 \varepsilon_0 \omega^2 \left(1 - i \frac{\sigma}{\varepsilon_0 \omega}\right)$$

Using the conductivity expression (II.60) and the identity $\mu_0 \varepsilon_0 = \frac{1}{c^2}$, we obtain

$$k^2 = \left(\frac{\omega}{c}\right)^2 - \frac{\mu_0^2 n e^2}{m}$$

Which can be rewritten as

$$k^2 = \frac{\omega^2 - \omega_p^2}{c^2} \qquad (II.63)$$

Where $\omega_p = \sqrt{\frac{ne^2}{m\varepsilon_0}}$ is the electron plasma frequency

Since $\left({}^{\omega}/_{k}\right)$ represents the phase velocity of the electromagnetic wave in the medium, the refractive index is given by

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \qquad (II.64)$$

Two regimes can be distinguished

- For $\omega > \omega_p$, k is real and waves can propagate freely. At high frequencies, the plasma behaves as a high-pass filter with cutoff frequency ω_p.

  For example, the typical electron number density of metals is $n_0 \approx 10^{28} m^{-3}$ and the corresponding $\omega_p \sim 10^{16} s^{-1}$. For ultraviolet light $\omega > \omega_p$, the light can generally propagate in metals
- For $\omega < \omega_p$, wave vector k is purely imaginary : $k = ik'$. The electric field takes the form

$$\vec{E} = \overrightarrow{E_0} \ \exp\left(-\overrightarrow{k'}.\vec{r}\right) exp(-i\omega t)$$

  This indicates that the field decays exponentially along the direction of propagation. In this regime, no true propagation occurs, and the waves are evanescent**.** The plasma's conductivity is purely imaginary, meaning there is no energy absorption, and the incident wave is therefore completely reflected at the plasma boundary.

Example

The ionosphere is a dilute cold plasma with electron densities ranging from 10^{10} to $10^{12} m^{-3}$. Its plasma frequency is approximately $f_p = 10^7 Hz$

- For $f = 10^5 < f_p$: the ionosphere acts as a reflector. This explains the first transatlantic radio transmission achieved by Marconi in 1901. Amplitude-modulated radio waves can thus reach very distant points on Earth.
- For $f = 10^8 > f_p$: the ionosphere becomes transparent**.** These higher frequencies are used for satellite communications.

Exercise

Consider an isotropic, homogeneous, globally neutral ohmic conductor characterized by Ohm's law: $\vec{j} = \sigma\vec{E}$

When the conductor is subjected to a monochromatic electromagnetic wave of angular frequency ω, its conductivity becomes frequency-dependent and is given by $\sigma = {}^{\sigma_0}/_{(1 + i\omega\tau)}$ where:

- $\sigma_0 = {}^{ne^2\tau}/_{m} \approx 10^7 Sm^{-1}$ is the static conductivity
- τ is the relaxation time between two successive collisions,
- $\omega_c = {}^{1}/_{\tau} = 10^{14} rad/s$

1. Write Maxwell's equations in the conductor

2. Deduce the wave dispersion relation linking k^2 to ω

3. For low frequencies $\omega \ll {}^{1}/_{\tau}$,

a. determine the corresponding wavelength λ range and identify the associated region of the electromagnetic spectrum.

b. Same question for high frequencies $\gg {}^{1}/_{\tau}$.

4. In the low- frequency regime

a. Show that $k^2 = -i\,\mu_0\sigma_0\omega$

b. Deduce the expression of k as a function of $\mu_0\sigma_0\omega$. Knowing that: $-i = {}^{(1-i)^2}/_{2}$

c. Express k in terms of the skin depth $\delta = \left({}^{2}/_{\mu_0\sigma_0\omega}\right)^{1/2}$ characteristic damping length of the wave called skin depth.

6. for a radio wave frequency of 200 MHz

a. calculate skin depth δ (in micrometers).

b. Compare δ to the wavelength.

c. deduce the nature of wave propagation in the conductor.

Solution

1. *Maxwell's equations*

$$\begin{cases} \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0} = 0 \\ \vec{\nabla}.\vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \mu_0 \sigma \vec{E} + \mu_0 \varepsilon \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

Take the curl of Faraday's law:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0 \sigma \frac{\partial \vec{E}}{\partial t} + \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Using the identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\Delta \vec{E} + \vec{\nabla}(\vec{\nabla}.\vec{E}) = -\Delta \vec{E}$$

Because $\vec{\nabla}.\vec{E} = 0$

For monochromatic wave

$$\vec{E}(\vec{r}, t) = \overrightarrow{E_0} e^{i(\omega t - \vec{k}.\vec{r})}$$

$$k^2 = (\mu_0 \varepsilon_0 \omega^2 - i \mu_0 \omega \sigma)$$

This is the dispersion relation in a conducting medium.

2. Frequency regimes

a. For low frequency $\omega \ll \omega_c = 1/\tau$, $\sigma = \sigma_0$

The conduction term dominates over displacement current (good conductor)

$$\mu_0 \omega \sigma \gg \mu_0 \varepsilon_0 \omega^2$$

Corresponding the wavelength

$$\lambda = \frac{2\pi c}{\omega} \Rightarrow \lambda \gg \frac{2\pi c}{\omega_c} \approx 20\ \mu m$$

Range of radio waves, microwaves and infrared

b. For high frequency $\omega \gg \omega_c = 1/\tau$, $\sigma = \sigma_0/i\omega\tau$

The displacement current dominates

$$k^2 = \mu_0 \varepsilon_0 \omega^2$$

$$\lambda \ll 20\ \mu m$$

Corresponding to optical, ultraviolet, X-rays.

The conductor behaves almost like a dielectric.

4.a. *Expression of k at low frequency*

Neglecting the displacement current

$$k^2 = -i \mu_0 \omega \sigma$$

b. Using $-i = {(1-i)^2}/{2}$

$$k^2 = \frac{(1-i)^2 \mu_0 \omega \sigma}{2}$$

$$k = (1-i)\sqrt{\frac{\mu_0 \omega \sigma}{2}}$$

Skin depth is the quantity

$$\delta = \sqrt{\frac{\mu_0 \omega \sigma}{2}}$$

6. *Radio wave* $f = 200\ MHz$

a. The skin depth

$$\omega = 2\pi f \cong 1.26\ rad/s$$

Skin depth is

$$\delta = \sqrt{\frac{\mu_0 \omega \sigma}{2}} = 11\ \mu m$$

b. The wavelength in the vaccum is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{2 \times 10^8} = 1.5\ m$$

we can see that $\delta \ll \lambda$.

c. Nature of propagation

The attenuation is extremely strong. The wave penetrates only at very thin surface layer

The energy is rapidly dissipated as Joule heating

The wave is evanescent inside the conductor (skin effect)

Chapter III
Propagation of electromagnetic waves in anisotropic media
Birefringence

III.1. INTRODUCTION

In isotropic media, all directions in space are locally equivalent, which greatly simplifies the analysis of electromagnetic wave propagation. Indeed, in such materials, the speed of an electromagnetic wave is independent of both the direction of propagation and the orientation of the electric field oscillations. This uniformity means that isotropic media with a single refractive index, leading to propagation that is the same in every direction.

$$\Delta E - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \qquad (III.1)$$

However, it is important to note that there are many situations in which this is not the case, such as propagation in most crystalline solids. The atomic arrangement creates an environment that varies with direction inside the crystal. Consequently, when an electromagnetic wave interacts with such a medium, the motion of the electric charges within atoms and molecules depends on the orientation of the field vector relative to the crystal axes. As a result, when an electromagnetic wave interacts with the medium, the motion of the electric charges within atoms and molecules depends on the orientation of the field vector relative to the crystal axes.

This directional dependence is the signature of anisotropy and leads to optical phenomena like birefringence, the splitting of light into two rays with different velocities and refractive indices, depending on its polarization. Similar loss of isotropy is found also in amorphous materials subject to external actions applied along a given direction-such as mechanical stress, hydrodynamic flow, or intense electric and magnetic fields-which tend to align the molecules that had been distributed at random. In both crystalline and externally oriented amorphous media the propagation of electromagnetic waves becomes sensitive to direction and polarization, in marked contrast to the uniform behavior of waves in isotropic media.

Figure III.1: Observation of the doubling of the image of an object viewed through a calcite crystal showing birefringence.

III.2. Susceptibility tensor

In an anisotropic, linear material medium, the application of an electric field $\vec{E}$ displaces charges within the material, thereby inducing dipoles and producing a macroscopic polarization

$$\vec{P} = \varepsilon_0[\chi]\,\vec{E} \qquad (III.1)$$

It means that the induced dipoles are not necessarily aligned with the electric field $\vec{E}$.

We have

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad (III.2)$$

It should be noted that this relation holds only for sinusoidal or static electric fields. For fields with arbitrary time dependence, the response must instead be expressed through convolution integrals. The above relation is local, relating the complex amplitudes of the polarization vector $\vec{P}$ and the electric field $\vec{E}$. In general, the susceptibility coefficients χ_{ij} are complex quantities; however, here we restrict our attention to transparent media, in which they are real. We also consider an homogeneous medium, for which the susceptibility is independent of position. With these assumptions, the susceptibility tensor is initially characterized by nine coefficients. We show below that this tensor must be symmetric χ_{ij}, reducing the number of independent coefficients to six.

The electric displacement vector $\vec{D}$ is defined , up to factor ε_0, as the superposition of the incident field and the induced polarization

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$$

By substituting $\vec{P}$ with equation (III.1), we obtain:

$$\vec{D} = \varepsilon_0([I] + [\chi])\vec{E} = \varepsilon_0[\varepsilon_r]\vec{E} \qquad (III.3)$$

The relative permittivity tensor is defined by :

$$[\varepsilon_r] = [I] + [\chi] \text{ with } [I] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (III.4)$$

$[\varepsilon_r]$ represents the dielectric permittivity tensor of the material. In the absence of absorption, the matrix is symmetric and has real coefficients.

In the principal axes basis, the relative permittivity tensor $[\varepsilon_r]$ is diagonal, and thus can be written as:

$$[\varepsilon_r] = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 1+\chi_x & 0 & 0 \\ 0 & 1+\chi_y & 0 \\ 0 & 0 & 1+\chi_z \end{bmatrix} \qquad (III.5)$$

For a non-magnetic and isotropic medium $(\mu_r \approx 1)$, the principal refractive indices are obtained directly from these eigenvalues according to:

$n_x = \sqrt{\varepsilon_x}$, $n_y = \sqrt{\varepsilon_y}$ et $n_z = \sqrt{\varepsilon_z}$

This allows the relative permittivity tensor to be rewritten in terms of the refractive indices:

$$[\varepsilon_r] = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} \qquad (III.6)$$

The case of an isotropic medium with index n occurs when $n_x = n_y = n_z = n$

The electric induction vector is written as:

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad (III.7)$$

This demonstrates that the displacement $\vec{D}$ is generally not parallel to the electric field $\vec{E}$, and that the Poynting vector is not parallel to the wave vector $\vec{k}$. Consequently, the direction of energy flow does not coincide with that of the wave vector. Nevertheless, there exist three specific directions for which $\vec{D}$ is parallel to $\vec{E}$. In these cases, the wave propagation velocity is determined by the orientation of $\vec{D}$, and therefore also by that of $\vec{k}$.

The quantities ε_x, ε_y and ε_z are the eigenvalues of the dielectric tensor and represent the dielectric response of the crystal along the three principal directions.

In the study of anisotropic media, it is essential to introduce the concept of the **optical axis**. The optical axis is a privileged direction in the crystal along which light does not experience birefringence. In other words, when a light wave propagates along this direction, it does not split into two separate waves, and both polarization states propagate with the same phase velocity. The optical axis is therefore directly linked to the symmetry of the dielectric tensor and plays a fundamental role in determining the optical behavior of the crystal.

Another key concept is the distinction between the ordinary and extraordinary refractive indices**.**

- The ordinary refractive index, denoted n_0, corresponds to propagation directions perpendicular to the optical axis and remains constant regardless of direction. It describes the ordinary wave, which behaves as if the medium were isotropic.
- The extraordinary refractive index, denotedn_E, depends on the angle between the direction of propagation and the optical axis. This directional dependence reflects the anisotropic nature of the crystal and leads to variations in the phase velocity of the extraordinary wave.

The difference between the ordinary and extraordinary refractive indices is responsible for the phenomenon of birefringence, whereby an incident light beam splits into two orthogonally polarized waves that travel at different velocities inside the crystal.

There are two main classes of anisotropic media: uniaxial and biaxial crystals.

a. Uniaxial medium

This represents the most widely used type of medium. In uniaxial crystals, the dielectric tensor possesses two equal eigenvalues$\varepsilon_x = \varepsilon_y$, This degeneracy defines a single privileged direction in the medium, called the **optical axis**, and leads to two principal refractive indices: the ordinary refractive index n_O and the **extraordinary refractive index** n_E . In uniaxial media, two of the dielectric eigenvalues are equal which results in a single optical axis and two principal refractive indices: one ordinary index and one extraordinary index .

In the principal coordinate system, the relative permittivity tensor (ε_r) can be written in diagonal form

$$(\varepsilon_r) = \begin{pmatrix} \varepsilon_{Ord} & 0 & 0 \\ 0 & \varepsilon_{Ord} & 0 \\ 0 & 0 & \varepsilon_E \end{pmatrix} = \begin{pmatrix} n_O^2 & 0 & 0 \\ 0 & n_O^2 & 0 \\ 0 & 0 & n_E^2 \end{pmatrix} \qquad (III.8)$$

This representation shows that the medium is isotropic in the transverse XOY plane, while exhibiting anisotropic optical properties along the optical axis OZ.

In the transverse plane, the constitutive relation between the electric displacement field $\vec{D}$ and the electric field $\vec{E}$ reduces to:

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_x & 0 \\ 0 & \varepsilon_x \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \qquad (III.9)$$

Which can be written in vectror form as

$$\vec{D}_\perp = \varepsilon_0 \varepsilon_x \vec{D}_\perp \qquad (III.10)$$

The subscript $\perp$ denotes components perpendicular to the optical axis. This isotropic transverse response reflects the rotational symmetry around the optical axis, which defines the OZ direction as the unique symmetry axis of the crystal.

Optical axis and wave propagation

When light propagates along the optical axis $\theta = 0$, the medium behaves effectively as isotropic, and no birefringence occurs. In this case, a single refractive index governs propagation, and the wave does not split into ordinary and extraordinary components.

For arbitrary propagation directions, the wave decomposes into two orthogonally polarized modes:

- the ordinary wave, characterized by a constant refractive index n_O
- the extraordinary wave, characterized by an angle-dependent refractive index $n_E(\theta)$

This anisotropic propagation is responsible for birefringence, where two waves travel with different phase velocities inside the crystal.

Uniaxial crystals are further classified according to the relative magnitude of the ordinary and extraordinary refractive indices:

- A positive uniaxial crystal $(n_E > n_O)$: The extraordinary wave propagates more slowly than the ordinary wave. Typical examples include quartz.
- A negative uniaxial crystal $(n_E < n_O)$: The extraordinary wave propagates faster than the ordinary wave. A common example is calcite.

This classification is fundamental in polarization optics, as it determines the shape of the refractive index surface (ellipsoid) and the behavior of polarized light in birefringent materials.

b. Biaxial medium

Biaxial crystals are the second major class of anisotropic optical media. Unlike uniaxial crystals, their dielectric tensor has three distinct eigenvalues, meaning that all principal optical directions are different $(\varepsilon_x \neq \varepsilon_y \neq \varepsilon_{z)}$.

Consequently, the relative permittivity tensor (ε_r) in its principal axes is written as:

$$(\varepsilon_r) = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} = \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix} \qquad (III.11)$$

This leads to three different principal refractive indices: $n_x \neq n_y \neq n_z$

Optical axis in biaxial crystals

A fundamental consequence of this lower symmetry is the existence of two optical axes, along which light propagates without birefringence. Along these specific directions, the wave normal surface becomes degenerate, and the usual distinction between ordinary and extraordinary waves is no longer globally defined as in uniaxial media. These optical axes are determined by the symmetry properties of the dielectric tensor and lie within particular crystallographic planes.

For arbitrary propagation directions, a biaxial crystal supports two electromagnetic eigenmodes, but unlike the uniaxial case, both modes exhibit a strong directional dependence. Neither mode is associated with a constant refractive index and the phase velocities depend on the full three-dimensional orientation of the wave vector. The corresponding refractive indices are obtained from the general Fresnel equation, which yields two solutions associated with the two wave surfaces. A more complete description of wave propagation in biaxial crystals requires the introduction of the refractive index ellipsoid**.**

III.3. Propagation of a harmonic plane wave

III.3.1. Wave structure

We consider a plane wave of frequency ω and wave vector $\vec{k}$ propagating in an anisotropic medium. The associated electromagnetic fields $(\vec{E}, \vec{B})$ can be expressed, in complex notation

$$\vec{E} = \overrightarrow{E_0} e^{i(\omega t - \vec{k}.\vec{r})} \quad , \quad \vec{B} = \overrightarrow{B_0} e^{i(\omega t - \vec{k}.\vec{r})} \qquad (III.12)$$

where $\overrightarrow{E_0}$ et $\overrightarrow{B_0}$ may be complex (in the case of circular polarization)

The displacement field $\vec{D}$ and the magnetic field $\vec{H}$ can be written in an analogous form.

Maxwell's equations for such a wave reduce to

$$\begin{cases} i\vec{k} \times \vec{E} = i\omega\vec{B} & , \quad \vec{k} \times \vec{D} = 0 \\ i\vec{k} \times \vec{B} = -i\mu_0\omega\vec{D} & , \quad \vec{k} \times \vec{B} = 0 \end{cases} \qquad (III.13)$$

From these relations, it follows that $\vec{k}$ is orthogonal to the plane defined by $(\vec{B},\vec{D})$, which is therefore referred to as the wave plane**.** The electric field $\vec{E}$ lies in the plane $(\vec{k},\vec{D})$. The plane formed by $(\vec{E},\vec{B})$ is called the plane of vibration.

The propagation direction of the electromagnetic wave is determined by the Poynting vector $\vec{\Pi} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$, which, in anisotropic media, does not coincide with the wave vector $\vec{k}$ as it does in isotropic media.

III.2.2. Wave equation

The wave equation is obtained by eliminating the magnetic field from the two curl equations

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times(-i\omega\,\vec{B}) = \mu_0\varepsilon_0\omega^2\ [\varepsilon_r]\vec{E} = \frac{\omega^2}{c^2}[\varepsilon_r]\vec{E} \qquad (III.14)$$

Using the fact that

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \Delta\vec{E} \qquad (III.15)$$

Which can be written in terms of the wave vector $\vec{k}$ as follows

$$\vec{\nabla}(\vec{\nabla}.\vec{E}) - \Delta\vec{E} = -i\vec{k}(-i\vec{k}.\vec{E}) - (-k^2)\vec{E} = -\vec{k}(\vec{k}.\vec{E}) + k^2\vec{E} \qquad (III.16)$$

It should be noted that $\vec{\nabla}.\vec{E} \neq 0$, unlike in the case of an isotropic medium. This can be readily verified by using the principal-axis basis.

$\vec{\nabla}.\vec{D} = \varepsilon_0\vec{\nabla}.([\varepsilon_r]\vec{E}) = 0$ does not implie $\vec{\nabla}.\vec{E} = 0$

By combining the two expressions, we obtain the wave equation for an anisotropic medium

$$\omega^2\mu_0\vec{D} - k^2\vec{E} + \vec{k}(\vec{k}.\vec{E}) = \vec{0} \qquad (III.17)$$

This equation shows that $\vec{D}$, $\vec{E}$ and $\vec{k}$ all lies in the same plane.

The direction of the wave vector is characterised by the unit vector $\vec{u}$

$$\vec{k} = n\frac{\omega}{c}\vec{u}$$

The equation above can be expressed as

$$\vec{D} = \frac{k^2}{\omega^2\mu_0}\left(\vec{E} - \vec{u}(\vec{u}.\vec{E})\right) \qquad (III.18)$$

The Poynting vector given by $\vec{\Pi} = \vec{E}\times\vec{H}$, show that $(\vec{E},\vec{H},\vec{\Pi})$ form a right-handed trihedron. Since the displacement vector $\vec{D}$ is not parallel to the electric field $\vec{E}$, the poynting vector $\vec{\Pi}$ is therefore not aligned with the propagation direction $\vec{k}$.

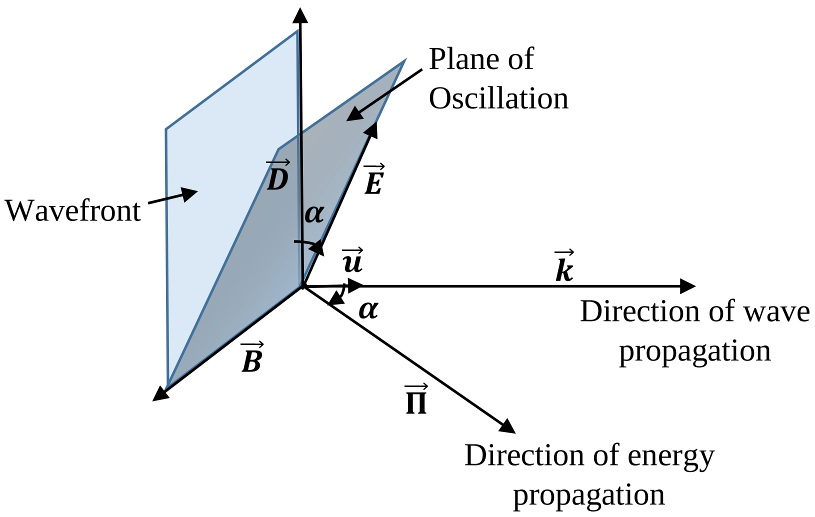


Figure III.2: Representation of wave and energy propagation directions in anisotropic media.

By combining the previous equations, we have

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \frac{k^2}{\omega^2\mu_0}\left(\vec{E} - \vec{u}(\vec{u}.\vec{E})\right) = \varepsilon_0 \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad (III.19)$$

III.3.3. Wave equation in principal axis

Let us consider the reference frame of the main axes of the medium. The components D_i of $\vec{D}$ simultaneously obey the relations:

$$D_i = \frac{k^2}{\omega^2\mu_0}\left(E_i - u_i(\vec{u}.\vec{E})\right) = \varepsilon_0 n_i^2 E_i \qquad (III.20)$$

We know that in vacuum : $\omega = k_0 c$ and $c^{-2} = \varepsilon_0\mu_0$. Taking into account the relation $k = nk_0$, the wave equation is expressed by

$$D_i = n^2\left(E_i - u_i(\vec{u}.\vec{E})\right) = \varepsilon_0 n_i^2 E_i \qquad (III.21)$$

From this equation, each component of the electric field vector $\vec{E}$ can be determined

$$E_i = \frac{n^2}{n^2 - n_i^2}\left(u_i(\vec{u}.\vec{E})\right) \qquad (III.22)$$

The unit vector $\vec{u}$ is characterized by its direction cosines with respect to the coordinate axes:

$$\vec{u} = \alpha\overrightarrow{u_X} + \beta\overrightarrow{u_Y} + \gamma\overrightarrow{u_Z}, \qquad \text{with} \quad \alpha^2 + \beta^2 + c^2 = 1$$

By expanding the wave equation along the three principal axes, we obtain:

$$n^2\left[\begin{pmatrix} E_X \\ E_Y \\ E_Z \end{pmatrix} - (\alpha E_X + \beta E_Y + \gamma E_Z)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}\right] = \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

This leads to the following system

$$n^2\begin{bmatrix}1-\alpha^2 & -\alpha\beta & -\alpha\gamma\\ -\alpha\beta & 1-\beta^2 & -\beta\gamma\\ -\alpha\gamma & -\beta\gamma & 1-\gamma^2\end{bmatrix}\begin{pmatrix}E_X\\ E_Y\\ E_Z\end{pmatrix}=\begin{pmatrix}n_x^2 & 0 & 0\\ 0 & n_y^2 & 0\\ 0 & 0 & n_z^2\end{pmatrix}\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}$$

Finally, we obtain

$$\begin{bmatrix}n_x^2-n^2(1-\alpha^2) & n^2\alpha\beta & n^2\alpha\gamma\\ n^2\alpha\beta & n_y^2-n^2(1-\beta^2) & n^2\beta\gamma\\ n^2\alpha\gamma & n^2\beta\gamma & n_z^2-n^2(1-\gamma^2)\end{bmatrix}\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\quad (III.23)$$

Thus, we obtain a homogeneous system of three equations in the unknowns E_X, E_Y and E_Z which can be written in matrix form: $(M)\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}=0$

The system admits non-trivial solutions $\left(E_x, E_y, E_z\right)\neq 0$, only if the determinant of the matrix is equal to zero

$$n_x^2\alpha^2\left(n^2-n_y^2\right)\left(n^2-n_z^2\right)+n_y^2\beta^2\left(n^2-n_x^2\right)\left(n^2-n_z^2\right)+n_z^2\gamma^2\left(n^2-n_x^2\right)\left(n^2-n_y^2\right)=0$$
$$(III.24)$$

Since the refractive index is positive, there are only distinct values, n' and n''. Each index can be associated with a plane wave denoted by $\overrightarrow{D'}$and $\overrightarrow{D''}$, respectively. These waves are linearly polarized in orthogonal directions. This behavior characterizes the phenomenon of birefringence.

Optical axis

In an anisotropic medium, the *optical axis* is the direction in space along which a wave can propagate with a single phase velocity (i.e. a single refractive index).

III.3.4. Fresnel equation and index surface

The Fresnel equation describes wave propagation in anisotropic media and leads to the concept of the *index surface*, which represents the dependence of the refractive index on the direction of propagation.

By dividing the equation by $(n^2-n_x^2)\left(n^2-n_y^2\right)(n^2-n_z^2)$, we obtain the Fresnel equation for the refractive indices, given by:

$$\frac{n_x^2\alpha^2}{(n^2-n_x^2)}+\frac{n_y^2\beta^2}{\left(n^2-n_y^2\right)}+\frac{n_z^2\gamma^2}{(n^2-n_z^2)}=0\qquad (III.25)$$

We define $\overrightarrow{OM} = n\vec{u}$, which characterizes the refractive-index surface. For reasons of symmetry, the analysis will be restricted to the positive quadrant where $x > 0$, $y > 0$ et $z > 0$.

Position Vector : $\overrightarrow{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Unit Vector $\vec{u}\begin{pmatrix} x/n \\ y/n \\ z/n \end{pmatrix}$

The refractive-index surface consists of two sheets, whose intersections with the planes $x = 0$, $y = 0$ et $z = 0$, may take the form of either elliptical arcs or circles.

$$\frac{n_x^2 {X_M}^2}{(n^2 - n_x^2)} + \frac{n_y^2 {Y_M}^2}{(n^2 - n_y^2)} + \frac{n_z^2 {Z_M}^2}{(n^2 - n_z^2)} = 0 \qquad (III.26)$$

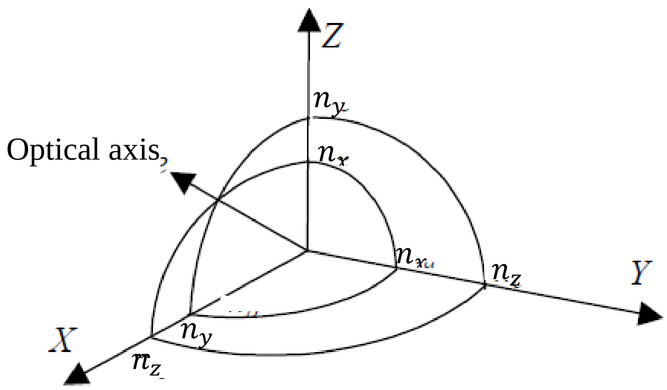


Figure III.3: Schematic representation of a cross sections of the refractive-index surfaces in the case of $n_z \geq n_y \geq n_x$.

This surface can be constructed as follows

- When the wave vector lies in the YOZ $(\alpha = 0)$, one solution of system (III.24) corresponds to a wave with electric field is oriented along the OX axis, with refractive index $n = n_x$. The intersection of the surface with the YOZ plane is therefore a circle of radius n_x
- The other solution, with the electric field perpendicular to OX, yields an intersection in the form of an ellipse with semi-axes n_y and n_z.

The shape of the surface is thus determined by its intersections with the three coordinate planes of the XYZ frame.

From the figure, it is evident that for almost all directions there exist two eigenmodes of polarization, each characterized by a distinct refractive index.

However, in two particular directions the two indices coincide. These directions define the optical axes, which explains the designation of the medium as *biaxial*. One of these axes is visible in the figure, while the other is its symmetric counterpart with respect to the OZ (or OZ) axis.

III.3.5. Index ellipsoid

It is possible to construct geometrically the two eigenvectors corresponding to linear polarizations, $\overrightarrow{D'}$ and $\overrightarrow{D''}$, with their associated refractive indices $n'$and n'', by introducing the index ellipsoid. We denote this surface by (Ɛ), defined as

$$(\mathcal{E}) \equiv \left(M, \overrightarrow{OM} = n\vec{d}\right)$$

Where, $\vec{d} = \frac{\vec{D}}{\|\vec{D}\|}$

Let us introduce the components of vectors

$$\overrightarrow{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad , \qquad \vec{d}\begin{pmatrix} D_x/D \\ D_y/D \\ D_z/D \end{pmatrix}$$

Consider the scalar product of equation (III.21) with vector $\vec{D}$

$$\vec{D}^2 = n^2\varepsilon_0\,\vec{E}.\vec{D}$$

Furthermore, in the principal axes system, $E_i = {D_i}/{n_i{}^2\varepsilon_0}$. We can then rewrite the scalar product $\vec{E}.\vec{D}$ with the components of $\overrightarrow{OM}$ as follows:

$$\frac{1}{n^2}\vec{D}^2 = \frac{1}{n^2}\vec{D}^2\left(\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2}\right) \qquad (III.27)$$

We find the equation of the surface (Ɛ)

$$\left(\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2}\right) = 1 \qquad (III.28)$$

This surface is an ellipsoid with semi-axes n_1, n_2 and n_3.

We consider the specific case of a uniaxial medium ($n_y = n_z$). The ellipsoid of indices is a revolute ellipsoid around the optical axis (oz). Consider the ellipse (Γ), which is the intersection of the wave plane (π), with a given normal direction $\vec{k}$, and the ellipsoid ($\mathcal{E}$). The vectors $\vec{D}'$ and $\vec{D}''$ belong to the wave plane (π).

$\vec{D}'$ and $\vec{D}''$ are orthogonal eigenstates. Their orientation is defined by the axes of the ellipse, whose semi-axes lengths are given by the extraordinary refractive index n_e along the optical axis and the ordinary refractive index n_o in the plane perpendicular to it.

It should be noted that the vibration $\vec{D}''$, which is perpendicular to the optical axis is here associated with the minor axis of the ellipse (Γ) and therefore with the ordinary refractive index. This is an ordinary vibration.

Thus, we have a simple way of identifying this direction of vibration: the ordinary displacement vector always vibrates in the direction perpendicular to the plane formed by the ordinary normal and the optical axis.

The extraordinary vibration can is then deduced: it corresponds to the direction perpendicular to the plane formed by the extraordinary wave normal and the of ordinary vibration direction.

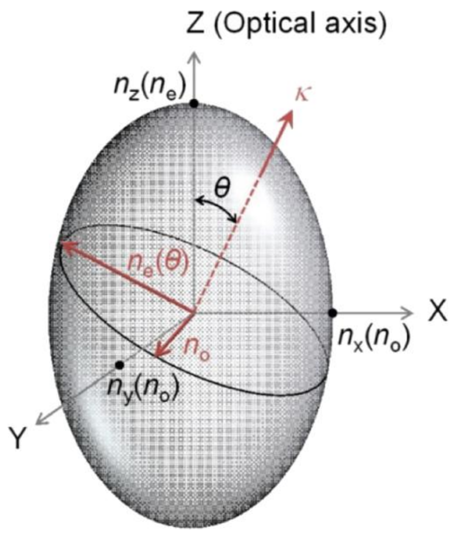


Figure III.4: Representation of the refractive-index ellipsoid for a uniaxial medium. The ellipsoid has an axis of rotational symmetry.

III.4. Wave Plates

Wave plates are plane-parallel plates cut from an anisotropic medium. In general, the medium is uniaxial (such as quartz or calcite), and the optical axis is parallel to the faces of the plate. These plates are used at normal incidence. Upon passing through the plate, the incident wave splits into an extraordinary wave, polarized parallel to the optical axis, and an ordinary wave, polarized perpendicular to the optical axis. These two waves propagate with different refractive indices, and therefore at different velocities, so that one emerges with a phase delay relative to the other. This is the origin of the term "wave plate".

Application: propagation of light in a birefringent plate

A monochromatic plane wave of angular frequency ω is incident on a birefringent plate of thickness e. The medium is a uniaxial crystal characterized by two principal refractive indices: the ordinary refractive index n_O and the extraordinary refractive index n_E.
We define a Cartesian coordinate system such that the unit vectors $\overrightarrow{e_x}$ and $\overrightarrow{e_y}$ correspond to the principal axes of the crystal. The x-direction is associated with the ordinary polarization state, while the y-direction is associated with the extraordinary polarization state.
The incident electric field $\vec{E}$ can be decomposed along these principal directions as:

$$\vec{E} = E_x\overrightarrow{e_x} + E_y\overrightarrow{e_y}$$

Inside the crystal, each polarization component propagates independently with a different refractive index. The wave number in the medium is given by:

$$k = \frac{n\omega}{c} = n\frac{2\pi}{\lambda}$$

where:

- n is the refractive index experienced by the wave.
- c is the speed of light in vacuum.
- λ is the wavelength in vacuum.
- e is the thickness of the plate.

After propagation through the plate, the two components acquire different phase shifts

$$\vec{E} = E_x e^{in_0\frac{2\pi}{\lambda}e}\overrightarrow{e_x} + E_y e^{in_e\frac{2\pi}{\lambda}e}\overrightarrow{e_y}$$

The output field can be rewritten by factoring out the ordinary phase term

$$\vec{E} = e^{in_0\frac{2\pi}{\lambda}e}\left(E_x\overrightarrow{e_x} + E_y e^{i\Delta\varphi}\overrightarrow{e_y}\right)$$

where the phase difference (or phase retardation) is defined as:

$$\Delta\varphi = \frac{2\pi}{\lambda}e(n_e - n_0) = \frac{2\pi}{\lambda}e\Delta n$$

- For $\Delta\varphi = 2m\pi$, or $e(n_e - n_0) = p\lambda$: λ blade
  The X and Y components are again in phase, and the blade behaves like a non-birefringent blade for this wavelength

- For $\Delta\varphi = 2m\pi + \pi$, ou $e(n_e - n_0) = p\lambda + \frac{\lambda}{2}$: half-wave plate $\frac{\lambda}{2}$

With a phase shift of π, the component along Y changes sign. The polarization at the output of the plate is therefore linear but forms an angle –θ with the OX axis. It is therefore symmetrical to the incident polarization with respect to one of the axes. Such a plate allows the polarization plane of a wave to be rotated continuously.

- For $\Delta\varphi = 2m\pi \pm {}^{\pi}/_{2}$ or $e(n_e - n_0) = p\lambda + \frac{\lambda}{4}$: quarter-wave blade $\frac{\lambda}{4}$

The two components in X and Y are in quadrature, we obtain an elliptical vibration with axes OX and OY. For the same blade, a right or left vibration can be obtained depending on whether the incident vibration is polarized along the first or second diagonal ($\theta = {}^{\pi}/_{4}$ ou $\theta = {}^{3\pi}/_{4}$)

Exercise 1

Quartz is a uniaxial birefringent material characterized by a difference between the extraordinary and ordinary refractive indices given by: $\Delta n = n_e - n_0 = 9.10^{-3}$.

1. For a wavelength $0.45\mu m$, determine the minimum thickness of the blade required to obtain a quarter-wave blade (${}^{\lambda}/_{4}$).

2. In practice, for mechanical reasons, a thickness of less than $100\ \mu m$, we therefore choose a thickness $112.5\ \mu m$

Show that the effective thickness $e(n_e - n_0)$ corresponds to $2\lambda + {}^{\lambda}/_{4}$ and produces the same effect at the wavelength considered.

3. Study the effect of this same blade for a neighboring wavelength $0.51\ \mu m$.

Verify that the effective thickness then corresponds to 2λ and conclude on the nature of the plate obtained.

4. Deduce why the characterization of a birefringent plate depends on the wavelength used.

Consider a birefringent quartz blade for which the difference in indices is $\Delta n = n_e - n_0 = 9.10^{-3}$.

Exercice 2

We consider a lithium niobate ($LiNbO_3$) crystal used as an electro-optic modulator Crystal length: We have the following characteristics

- $L = 2.0\ cm$
- Distance between electrodes : $d = 5.0\ mm$
- Light Wavelength : $\lambda = 633\ nm$
- Refractive Index: $n = 2{,}2$
- Coefficient électro-optique : $r = 30\ pm/V$

Uniform electric field, parallel to the electro-optic axis

1. Express the variation of the refractive index Δn as a function of the electric field $\vec{E}$.

2. Calculate the optical phase shift $\Delta\varphi$ as a function of the applied voltage.

3. Calculate the half-wave voltage V_π.

Solution

1. *Variation of the refractive index*

For the Pockels effect

$$\Delta n = -\frac{1}{2} n^3 r E$$

Since $E = {}^{V}/_{d}$, we have:

$$\Delta n = -\frac{1}{2} n^3 r \frac{V}{d}$$

2. *Optical phase shift*

The induced phase shift after propagation over a length L is

$$\Delta\varphi = \frac{2\pi}{\lambda} \Delta n L$$

By substituting Δn,

$$\Delta\varphi = -\frac{2\pi}{\lambda} \frac{1}{2} n^3 r \frac{V}{d} L$$

The phase shift is linear with respect to the applied voltage V.

3. Half-wave Voltage V_π

By definition, a half-wave plate corresponds to $\Delta\varphi = \pi$

$$\pi = -\frac{\pi}{\lambda} n^3 r \frac{V_\pi}{d} L$$

Thus

$$V_\pi = -\frac{\lambda d}{n^3 r L}$$

Numerical application

$$V_\pi = -\frac{633.\,10^{-9}\,5.\,10^{-3}}{(2.2)^3\,30.\,10^{-12}\,2.\,10^{-2}} \approx 500V$$

Chapitre IV

Propagation of electromagnetic waves in non-linear media

INTRODUCTION

In nonlinear media, polarization induced by an electromagnetic field is no longer proportional to the applied field, which gives rise to specific phenomena such as harmonic generation, frequency mixing, and soliton propagation. The objective of this chapter is to present the theoretical basis for describing this propagation using Maxwell's equations and appropriate constitutive relations, in order to understand the physical mechanisms responsible for these effects and their importance in applications such as optical fibers and optoelectronic devices.

IV.1. Statement of the problem

If an oscillating electric field $\vec{E}(t)$ propagates through a medium, it displaces the electronic clouds of atoms or molecules relative to its equilibrium position, thus inducing a dipole moment $\vec{p}$. Depending on the intensity of the electric field, we can have

- Linear regime: For a weak field, the medium responds proportionally to the excitation $\vec{E}(t)$. In this case, we can consider the dipole to be in a harmonic oscillator, thus describing the linear propagation of an electromagnetic wave (discussed in chapter II).
- Nonlinear regime: For strong field, the proportionality can no longer be assumed. Nonlinear effects appear in optics. The polarization must be expanded as a power series of the electric field

$$\vec{P}(t) = \varepsilon_0 \left[\chi^{(1)}\vec{E} + \chi^{(2)}\overrightarrow{E^2} + \chi^{(2)}\overrightarrow{E^3} + \cdots\right] \qquad (IV.1)$$

where

- $\chi^{(n)}$ are the rank-n electric susceptibility tensors,
- The linear term $\chi^{(1)}$ accounts for phenomena such as the refractive index and absorption.
- The higher-order terms ($\chi^{(2)}, \chi^{(3)}, \ldots$) are responsible for nonlinear optical effects, including second-harmonic generation, four-wave mixing, and self-phase modulation.

Nonlinear optics arises from the contribution of higher-order susceptibilities beyond the linear term. Its validity relies on a perturbative expansion of the material polarization, which is applicable only within a limited field-strength regime. The applied electric field must be strong enough for nonlinear effects to be observable, yet remain sufficiently weaker than the intra-atomic electric field so that higher-order terms remain progressively smaller. At extremely high field intensities, the perturbative description breaks down due to ionization of the medium, requiring a plasma-physics framework beyond standard nonlinear optics.

IV.2. Maxwell's Equations

The development of the polarization vector is naturally embedded in Maxwell's equations, which govern the dynamics of electromagnetic fields in matter

$$\vec{\nabla}\cdot\vec{D}(t)=\rho(t)$$
$$\vec{\nabla}\cdot\vec{B}(t)=0$$
$$\vec{\nabla}\times\vec{E}(t)=-\frac{\partial\vec{B}(t)}{\partial t} \qquad (IV.2)$$
$$\vec{\nabla}\times\vec{H}(t)=\vec{J}(t)+\frac{\partial\vec{D}(t)}{\partial t}$$

with the constitutive relation

$$\vec{D}(t)=\varepsilon_0\vec{E}(t)+\vec{P}(t)$$

This formulation highlights the important role of the polarization vector in relating the microscopic dipole dynamics to the macroscopic behavior of electromagnetic waves in matter. The polarization vector directly modifies the material's electromagnetic response. In linear optics, this simply alters the refractive index. In nonlinear optics, however, the higher-order terms couple different frequency components of the field, enabling frequency conversion and nonlinear wave mixing.

IV.3. Nonlinear Effects

These phenomena can be explained by frequency doubling of light, also known as second-harmonic generation, and the Pockels effect. In general, all second-order nonlinear optical effects only occur in anisotropic materials that lack inversion symmetry, i.e. non-centrosymmetric media.

Examples of nonlinear crystals

- **Lithium Niobate ($LiNbO_3$)** is the widely used non-linear crystal characterised by high second-order susceptibility χ^2, a wide transparency range and excellent electro-optical properties. It is used for frequency doubling, optical modulation via the Pockels effect and integrated devices.
- **Potassium Titanyl Phosphate (KTP)** is characterized by high conversion efficiency and good phase matching. Its robustness and stability make it particularly suitable for second harmonic generation in solid-state lasers.
- **Beta Barium Borate (BBO)** offers a very wide spectral transparency window (190–3500 nm) and a high damage threshold. It is commonly used for frequency conversion of ultrashort pulses, particularly in frequency doubling, amplification and parametric optical generation.
- **Lithium Triborate (LBO)** is an stable crystal combining broad optical transparency and a high damage threshold. It is mainly used for frequency doubling in high-power lasers and for parametric processes.

- **Gallium Arsenide (GaAs)** and other semiconductors
  These materials are distinguished by their strong optical nonlinearities and their compatibility with integration technologies. They are particularly used for generating terahertz radiation and for integrated photonics applications.These materials are distinguished by their strong optical non-linearities and their compatibility with integration technologies. They are used in particular for terahertz radiation generation and integrated photonics applications.

IV.3.1. Second-Harmonic Generation

Consider a monochromatic electric field at frequency ω

$$E(t) = A\cos(\omega t) \qquad (IV.3)$$

In linear optics, all processes (absorption, refraction, or transmission) preserve the frequency of the incident wave, so the output remains at the same frequency ω. In contrast, the second-order nonlinear polarization depends on the square of the electric field

$$P^{(2)}(t) = \varepsilon_0 \chi^{(2)} A^2 \cos^2(\omega t) = \varepsilon_0 \frac{\chi^{(2)}}{2} A^2\big(1 + \cos(2\omega t)\big) \qquad (IV.4)$$

This expression contains

- The constant term corresponds to a static polarization (optical rectification).

- The oscillatory term at frequency 2ω, which generates a new electromagnetic wave at twice the incident frequency.

Thus, a new frequency is created, something impossible in linear optics. What seems mathematically like a small correction therefore opens the way to entirely new optical phenomena.

The wave generated at 2ω can reach significant intensity and is readily exploited in experiments and optical devices. Second-harmonic generation was first observed experimentally by Franken in 1961, shortly after the invention of the laser. This pioneering experiment marks the birth of nonlinear optics on the experimental level.

Example

Second-harmonic generation (SHG) inside a frequency-doubled Nd:YAG laser cavity.

- The Nd:YAG laser emits infrared light at $\lambda_1 = 1064\, nm$, $\omega_1 = \frac{2\pi c}{\lambda_1}$.
- A nonlinear crystal (e.g., KTP, LBO, BBO) inside the cavity
    - The linear response allows the infrared wave at ω_1 to pass through.
    - The nonlinear response generates a polarisation term proportional to E^2, producing a new wave at frequency $2\omega_1$.
- Second-harmonic generation (green light): The new frequency corresponds to

$$\lambda_2 = \frac{\lambda_1}{2} = 532\, nm$$

which is visible green light.

IV.3.2. Wave equation

Starting from Maxwell's equations in a non-magnetic medium ($\mu = \mu_0$) and assuming the absence of free charges and currents ($\rho = 0, \vec{j} = \vec{0}$), the propagation equation governing a plane electromagnetic wave in a nonlinear dielectric medium can be expressed as follows:

$$\Delta\vec{E} - \frac{n^2}{c^2}\frac{\partial^2\vec{E}(\omega)}{\partial t^2} = \mu_0 \frac{\partial^2\overrightarrow{P^{NL}}(\omega)}{\partial t^2} \qquad (IV.5)$$

the nonlinear polarization is given by

$$\vec{P}^{NL} = \vec{P}^{(2)} = \varepsilon_0\chi^{(2)}\vec{E}^{(2)} \qquad (IV.6)$$

$\chi^{(2)}$ is the second-order nonlinear susceptibility tensor and ε_0 is the vacuum permittivity.

To solve the propagation equation, the following assumptions are made

- The nonlinear medium is homogeneous and isotropic, with a dielectric constant independent of position.

- All interacting waves propagate collinearly along the z −direction.
- The waves propagate in a single direction (forward propagation only).
- The waves are plane waves, linearly polarized, and oscillate in a plane perpendicular to the direction of propagation.

To derive the coupled differential equations from the wave equation, two filtering procedures are applied:

1. A physical approximation: the Slowly Varying Envelope Approximation (SVEA).
2. A mathematical projection based on the orthogonality of sinusoidal functions.

The second-harmonic electric field is written as

$$E_{2\omega}(z,t) = \frac{1}{2}A_{2\omega}(z)e^{i(k_{2\omega}z-2\omega t)} + \underbrace{\frac{1}{2}A_{2\omega}^{*}(z)e^{-i(k_{2\omega}z-2\omega t)}}_{\text{C.C}} \qquad (IV.6)$$

The spatial derivatives are

$$\frac{\partial E_{2\omega}}{\partial z} = \frac{1}{2}\left(\frac{dA_{2\omega}}{dz} + ik_{2\omega}A_{2\omega}\right)e^{i(k_{2\omega}z-2\omega t)} \qquad (IV.7)$$

$$\frac{\partial^2 E_{2\omega}}{\partial z^2} = \frac{1}{2}\left[\frac{\partial^2 A_{2\omega}}{\partial z^2} + 2ik_{2\omega}\frac{dA_{2\omega}}{dz} - k_{2\omega}^2 A_{2\omega}\right]e^{i(k_{2\omega}z-2\omega t)} \qquad (IV.8)$$

Under the SVEA, the field envelope $A_{2\omega}(z)$ varies much more slowly than the optical carrier, such that

$$\left|\frac{d^2 A_{2\omega}}{dz^2}\right| \ll k_{2\omega}\left|\frac{dA_{2\omega}}{dz}\right| \qquad (IV.9)$$

The second-order derivative of the envelope can therefore be neglected, yielding

$$\frac{\partial^2 E_{2\omega}}{\partial z^2} \approx \frac{1}{2}\left[2ik_{2\omega}\frac{dA_{2\omega}}{dz} - k_{2\omega}^2 A_{2\omega}\right]e^{i(k_{2\omega}z-2\omega t)} \qquad (IV.10)$$

- ***Nonlinear Polarization at the Second-Harmonic Frequency*** $\boldsymbol{P_{NL}^{(2)}(2\omega)}$

The second-order nonlinear polarization is given by

$$P_{NL}^{(2)}(2\omega) = \varepsilon_0\chi^{(2)}E_\omega^2 = \varepsilon_0\chi^{(2)}\left[\frac{1}{2}A_\omega(z)e^{i(k_\omega z-\omega t)} + \frac{1}{2}A_\omega^{*}(z)e^{-i(k_\omega z-\omega t)}\right]^2$$

$$= \varepsilon_0\chi^{(2)}\left[\frac{1}{4}A_\omega^2 e^{i(2k_\omega z-2\omega t)} + \frac{1}{4}A_\omega^{*2}(z)e^{-i(2k_\omega z-2\omega t)} + \frac{1}{4}A_\omega A_\omega^{*}\right]$$

Only the term oscillating at the frequency 2ω contributes to second-harmonic generation

$$P_{NL}^{(2)}(2\omega) = \frac{1}{4}\varepsilon_0\chi^{(2)}A_\omega^2 e^{i(2k_\omega z-2\omega t)} \qquad (IV.11)$$

- ***Derivation of the Coupled-Wave Equation***

Substituting the field and polarization expressions into the inhomogeneous wave equation and retaining only the terms oscillating at 2ω, one obtains

$$\frac{1}{2}\left[2ik_{2\omega}\frac{dA_{2\omega}}{dz}-k_{2\omega}^2A_{2\omega}\right]e^{i(k_{2\omega}z-2\omega t)}-\frac{1}{2}\left[-\frac{n^2(2\omega)^2}{c^2}A_{2\omega}(-2\omega)^2\right]e^{i(k_{2\omega}z-2\omega t)}$$
$$=\mu_0\frac{\partial^2}{\partial t^2}\left(\frac{1}{4}\varepsilon_0\chi^{(2)}A_\omega^2e^{i(2k_\omega z-2\omega t)}\right)$$

The terms proportional to $A_{2\omega}$ **c**ancel out because

$$k_{2\omega}=\frac{n(2\omega)}{c}$$

which satisfies the dispersion relation.

The resulting equation reduces to

$$ik_{2\omega}\frac{dA_{2\omega}}{dz}e^{ik_{2\omega}z}=\frac{1}{2}\mu_0\varepsilon_0\chi^{(2)}A_\omega^2(-2\omega)^2e^{2ik_\omega z}$$

The final expression is

$$\frac{dA_{2\omega}}{dz}=\frac{-4\omega^2}{2ik_{2\omega}c^2}\chi^{(2)}A_\omega^2e^{i(2k_\omega-k_{2\omega})z}$$

$$\frac{dA_{2\omega}}{dz}=i\frac{\omega}{n_{2\omega}c}\chi^{(2)}A_\omega^2e^{i\Delta kz}\qquad(IV.12)$$

where the phase mismatch is defined as $\Delta k=2k_\omega-k_{2\omega}$

The exponential factor $e^{i\Delta kz}$ accounts for the phase mismatch between the nonlinear polarization source and the generated second-harmonic field. When $2k_\omega\neq k_{2\omega}$, the two waves progressively drift out of phase, causing destructive interference and limiting the buildup of the second-harmonic amplitude $A_{2\omega}$

- ***Derivation of the fundamental-wave envelope equation*** $\frac{dA_\omega}{dz}$

The derivation of the evolution equation for $A_\omega$follows a similar procedure. In this case, the process corresponds to the reverse interaction: the generated second-harmonic field mixes with the fundamental field through the second-order nonlinearity, thereby transferring energy back to the frequency ω.

The nonlinear polarization $P_{NL}^{(2)}(\omega)$ is given by

$$P_{NL}^{(2)}(\omega)=\varepsilon_0\chi^{(2)}(E_{tot})^2\qquad(IV.13)$$

where $E_{tot}=E_\omega+E_{2\omega}$

The fundamental and second-harmonic fields are expressed as

$$E_\omega(z,t)=\frac{1}{2}A_\omega(z)e^{i(k_\omega z-\omega t)}+\frac{1}{2}A_\omega^*(z)e^{-i(k_\omega z-\omega t)}$$
$$E_{2\omega}(z,t)=\frac{1}{2}A_{2\omega}(z)e^{i(k_{2\omega}z-2\omega t)}+\frac{1}{2}A_{2\omega}^*(z)e^{-i(k_{2\omega}z-2\omega t)}$$

$$P_{NL}^{(2)}(\omega) = \varepsilon_0 \chi^{(2)} (E_\omega^2 + E_{2\omega}^2 + 2E_\omega E_{2\omega}) \qquad (IV.14)$$

To derive the contribution at the fundamental frequency ω, only those terms oscillating at $\mp\omega$ are retained. The terms E_ω^2 and $E_{2\omega}^2$ do not contain frequency components at ω; therefore, the only relevant contribution arises from the cross-product term $2E_\omega E_{2\omega}$.

$$2E_\omega E_{2\omega} = 2\left[\frac{1}{2}(E_{source}^{\omega} + E_{source}^{-\omega})\right]\left[\frac{1}{2}(E_{source}^{2\omega} + E_{source}^{-2\omega})\right]$$

$$2E_\omega E_{2\omega} = \frac{1}{2}(E_\omega E_{2\omega} + E_\omega E_{-2\omega} + E_{-\omega} E_{2\omega} + E_{-\omega} E_{-2\omega}) \qquad (IV.15)$$

Among the resulting products, the component oscillating at frequency $+\omega$ originates from the interaction between $E_{2\omega}$ and E_ω

$$P_{NL}^{(2)}(\omega) = \varepsilon_0 \chi^{(2)} 2\left(\frac{1}{2}A_{2\omega}(z)e^{i(k_{2\omega}z-2\omega t)}\right)\left(\frac{1}{2}A_\omega^*(z)e^{-i(k_\omega z-\omega t)}\right)$$

Substituting this expression into the wave equation and applying the Slowly Varying Envelope Approximation (SVEA) to the fundamental field leads to

$$P_{NL}^{(2)}(\omega) = \left[\frac{1}{2}\varepsilon_0 \chi^{(2)} A_{2\omega} A_\omega^* e^{i(k_{2\omega}-k_\omega)z}\right] e^{-i\omega t} \qquad (IV.16)$$

After evaluating the temporal derivatives and simplifying, one obtains

$$ik_\omega \frac{dA_\omega}{dz} e^{i(k_\omega z-\omega t)} = \mu_0 \frac{\partial^2 P_{NL}(\omega)}{\partial t^2} = -\mu_0(-\omega^2)\frac{1}{2}\varepsilon_0 \chi^{(2)} A_{2\omega} A_\omega^* e^{i(k_{2\omega}-k_\omega)z}\, e^{-i\omega t}$$

$$\frac{dA_\omega}{dz} = \frac{i\omega\chi^{(2)}}{n_\omega c} A_{2\omega} A_\omega^* e^{-i\Delta k z} \quad (IV.17)$$

With $\Delta k = k_{2\omega} - k_\omega$

Coupled wave equation

The evolution of the fundamental and second-harmonic fields is therefore governed by the coupled equations

$$\frac{dA_{2\omega}}{dz} = i\frac{\omega}{n_{2\omega}c}\chi^{(2)} A_\omega^2 e^{i\Delta k z}$$

$$\frac{dA_\omega}{dz} = i\frac{\omega\chi^{(2)}}{n_\omega c} A_{2\omega} A_\omega^* e^{-i\Delta k z}$$

These coupled equations describe the continuous exchange of energy between the fundamental and second-harmonic waves through the second-order nonlinear susceptibility $\chi^{(2)}$. The first equation represents the generation of the second harmonic from the fundamental field, whereas the second equation accounts for the reciprocal interaction, whereby the generated second-harmonic wave influences the evolution of the fundamental field. Consequently, neither wave can be treated as completely independent and the nonlinear medium acts as an active mediator of energy transfer between the two frequency components.

In general, the coupled-wave equations are nonlinear and difficult to solve analytically. A common simplifying assumption is perfect phase matching, corresponding to $\Delta k = 0$.

Under these conditions, the fundamental and second-harmonic waves remain in phase throughout the medium, leading to maximal energy transfer and efficient second-harmonic generation.

If the fundamental wave is assumed to be undepleted (its amplitude remains approximately constant along the propagation direction), the intensity of the generated second-harmonic wave is found to scale as

$$I_{2\omega}(z) \propto z^2 I_\omega^2 \qquad (IV.17)$$

This result shows that the second-harmonic intensity grows quadratically with the crystal length and is proportional to the square of the fundamental-wave intensity, which is a characteristic signature of second-order nonlinear optical processes.

IV.3.3. Sum- and difference-frequency generation

If the incident wave contains two different frequencies

$$\vec{E} = \overrightarrow{E_1} cos(\omega_1 t) + \overrightarrow{E_2} cos(\omega_2 t) \qquad (IV.18)$$

The second-order term of the response contains frequencies 0, $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$, $\omega_1 - \omega_2$, because

$$\left(cos(\omega_1 t) + cos(\omega_2 t)\right)^2 = cos^2(\omega_1 t) + cos^2(\omega_2 t) + 2cos(\omega_1 t)cos(\omega_2 t)$$

$$= 1 + \frac{1}{2}[cos(2\omega_1 t) + cos(2\omega_2 t) + \cos((\omega_1 + \omega_2)t) + \cos((\omega_1 - \omega_2)t)]$$

This effect is used to triple the frequency of an infrared YAG laser.

First, the infrared beam is passed through a doubling crystal, which produces two beams with frequencies $\omega_1 = \omega$ and $\omega_2 = 2\omega$. These two frequencies are then combined in another crystal. The frequency $\omega_1 + \omega_2 = 3\omega$ is selected by phase matching.

IV.3.4 Pockels Effect

The Pockels effect, known as the linear electro-optic effect, refers to the change in the optical properties of a material when subjected to an external electric field. The application of an electric field causes a variation in the crystal's refractive index, thereby altering the propagation of light within the medium. This variation is proportional to the intensity of the applied field.

The Pockels effect is observed in anisotropic non-centrosymmetric media, which is characterized by the change in the ordinary and extraordinary refractive indices under the influence of the static electric field. This is one of the phenomena of second-order nonlinear optics.

In the presence of the static electric field (DC field), denoted as $\overrightarrow{E_0}(\omega_1 = 0)$, and the optical electric field, denoted as $\overrightarrow{E_\omega}(\omega_2 = \omega)$, the total electric field in the medium is the sum of the two.

The second-order nonlinear polarization $P^{(2)}$ then contains a term oscillating at the optical ω, which is proportional to the product of the amplitudes of the static and optical fields, $E_0 E_\omega$. This term effectively modifies the linear optical response of the medium.

As a result, the ordinary and extraordinary refractive indices become dependent on the applied electric field. To first order in E_0, they can be written as

$$n_o = n_o + \Delta n_o \qquad , \quad n_e = n_e + \Delta n_e$$

where the refractive-index variations Δn_o and $+\Delta n_e$ are linearly proportional to the static electric field:

$$\Delta n_i = -\frac{1}{2} n_i^3 r_i E_0 \qquad , \qquad i \in \{o, e\} \qquad (IV.19)$$

Here, r_i denotes the appropriate electro-optic (Pockels) coefficient, which depends on the crystal symmetry and the orientation of the applied field.

Physically, the Pockels effect can be interpreted as an electric-field–induced birefringence, resulting from the coupling between the static field and the optical field through the second-order nonlinear susceptibility $\chi^{(2)}$. This linear dependence of the refractive index on the applied electric field distinguishes the Pockels effect from higher-order electro-optic effects, such as the Kerr effect, which scales quadratically with the electric field.

The ordinary and extraordinary indices verify

$$n^2 = n_{lin}^2 + {}^{\alpha(E_0 E_\omega)}/_{\varepsilon_0}\, E_\omega$$

$$n = n_{lin} + \frac{\alpha}{2 n_{lin} \varepsilon_0} E_0 \qquad (IV.20)$$

The quantity n_{lin} is the contribution to the linear response index.

IV.4. The third-order nonlinear effects

Nonlinear optical effects can occur in isotropic media. In the following, we focus on the Kerr effect and the optical Kerr effect, both of which originate from third-order nonlinearities.

IV.4.1. Form of the Third-Order Nonlinear Term

The third-order nonlinear polarization arises from the product of three electric-field components. If the corresponding angular frequencies are ω_1, ω_2 et ω_3, the product of the three oscillating fields can be written as:

$$cos(\omega_1 t)cos(\omega_2 t)cos(\omega_3 t) = \frac{1}{8}\left(e^{i\omega_1 t} + e^{-i\omega_1 t}\right)\left(e^{i\omega_2 t} + e^{-i\omega_2 t}\right)\left(e^{i\omega_3 t} + e^{-i\omega_3 t}\right)$$

From this expansion, it is clear that the nonlinear response contains frequency components at all possible combinations $\mp\omega_1 \mp \omega_2 \mp \omega_3$

If the incident field contains a single angular frequency ω the third-order nonlinear response generates frequency components at

$$3\omega = \omega + \omega + \omega$$

$$\omega = \omega + \omega - \omega$$

as well as their negative-frequency counterparts, $-\omega = \omega - \omega - \omega \quad , \quad -3\omega$

The component oscillating at the fundamental frequency ω is responsible for the optical Kerr effect, which manifests as an intensity-dependent modification of the refractive index, while the component at 3ω leads to third-harmonic generation (THG).

If the incident field contains two angular frequencies ω_1 and ω_2, the third-order nonlinear polarization gives rise to a rich set of frequency components, including

$$3\omega_1,\ 2\omega_1 + \omega_2, 2\omega_1 - \omega_2, \omega_1 + 2\omega_2, \omega_1 - 2\omega_2, 3\omega_2, \omega_1, \omega_2$$

together with their corresponding negative frequencies.

These terms describe various four-wave mixing processes, such as frequency conversion, self-phase modulation, and cross-phase modulation, all of which are governed by the third-order nonlinear susceptibility $\chi^{(3)}$.

IV.4.2. Static Kerr Effect

A static electric field $\overrightarrow{E_0}$ with angular frequency $\omega_1 = 0$ is applied to an isotropic medium traversed by an electromagnetic wave with electric field $\overrightarrow{E_\omega}$ and angular frequency $\omega_2 = \omega$. Since the medium is isotropic, there is no second-order effect.

The propagation of the wave with frequency ω is affected by a third-order nonlinear effect the response contains a term $\alpha E_0^2 E_\omega e^{i\omega t}$ which contributes to the index of this wave

$$n^2 = n_{lin}^2 + \frac{\alpha E_0^2 E_\omega}{\varepsilon_0 E_\omega} = n_{lin}^2 + \frac{\alpha}{\varepsilon_0} E_0^2 \qquad (IV.21)$$

And so the index

$$n = n_{lin} + \frac{\alpha}{2 n_{lin} \varepsilon_0} E_0^2 \qquad (IV.22)$$

Depends on the square of the static electric field.

IV.4.3. Optical Kerr Effect

The Kerr effect is a phenomenon whereby almost all transparent isotropic substances subjected to an electric field undergo a change in their optical properties and exhibit the phenomenon of birefringence. The behavior is like that of a uniaxial crystal with an optical axis parallel to the electric field.

An isotropic medium is illuminated with an electromagnetic wave $\overrightarrow{E_\omega}$ with a frequency ω and high intensity. The Kerr effect is caused not by a static field, but by the electric field of the wave itself.

The third-order response to the electromagnetic wave contains terms of the form $E_\omega^3 e^{i\omega t} e^{i\omega t} e^{-i\omega t} = I E_\omega e^{i\omega t}$, where I is the intensity of the wave, which describe a modification of the wave with frequency ω. They contribute to the refractive index of the medium by a term $\frac{\alpha}{2 n_{lin} \varepsilon_0} I = n_{NL} I$. And the electromagnetic wave sees an index

$$n = n_{lin} + n_{NL} I \qquad (IV.23)$$

Chapter V

Propagation of electromagnetic waves in waveguides

V.1. INTRODUCTION

A waveguide is a device used to guide high-frequency electromagnetic waves. These waves are commonly used in microwave and radar technology. Propagation of these waves in a waveguide is not free, but is instead restricted by the boundary conditions provided by the conductive walls of the waveguide. The study of waveguides has enabled us to propagate signals with low losses and high accuracy.

The purpose of this chapter is to present the main types of waveguides and explain the propagation conditions and associated cutoff frequencies. We then analyze the fundamental and higher modes derived from the solutions to Maxwell's equations in order to highlight their specific characteristics.

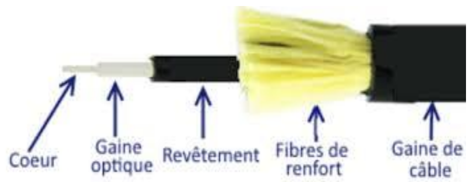
V.2. Guided Propagation and propagation modes

V.2.1. Guided propagation

Guided propagation refers to the transport of an electromagnetic wave in a medium bounded by conductive or dielectric walls, called a waveguide. This device allows electromagnetic energy to be channeled along the guide with a low attenuation rate, making it an essential tool for high frequencies.

For frequencies above 1 GHz, metal (rectangular, circular) or dielectric (optical fiber) waveguides are used (Figure V.1). These structures are advantageous replacements for conventional lines (such as coaxial cables), which become too dissipative at these frequencies.

(a)



(b)

Figure V.1: Different types of waveguides: (a) rectangular and circular waveguides; (b) optical fibers.

In a waveguide, the incident waves and those reflected from the walls superimpose to form a resulting wave that propagates on average parallel to the walls.

V.2.2. Propagation Modes

The distribution of fields in a waveguide differs from that of free waves in a vacuum. There are several modes of propagation:

1. TEM (Transverse Electromagnetic) modes: The electric and magnetic fields are entirely transverse (possible in coaxial lines, but not in hollow waveguides)

$E_z = 0$ and $B_z = 0$.

2. TE (Transverse Electric) modes : Only the electric field is purely transverse to the direction of propagation $E_z = 0$ and $B_z \neq 0$

3. TM (Transverse Magnetic) modes : the magnetic field has no component in the direction of propagation. $E_z \neq 0$ and $B_z = 0$

4. HE (Hybride) modes: The electric and magnetic fields are not transverse to the direction of propagation.

V.3. Rectangular Guide

V.3.1. Description of waveguide

Let us consider a hollow metallic waveguide with a rectangular cross section of width *a* in the *x*-direction and height *b* in the *y*-direction. The walls are assumed to be perfectly conducting.
We assume that $\varepsilon_r = 1$ and $\mu_r = 1$, and that the metal is perfect ($\sigma \to +\infty$). Thus, there is no energy dissipation.

- In the metal, $\vec{E} = \vec{0}$ (the conductor is perfect), $\vec{B} = \vec{0}$ (static in fact),
- In the immediate vicinity, $\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{n}$, $\vec{B} = \mu_0 \vec{j} \times \vec{n}$

We now consider an electromagnetic wave propagating in a vacuum along the Oz axis. It is possible to analyze one of its linearly polarized components, parallel to one of the sides of the rectangular section.
We assume that the electric field is linearly polarized along the y-axis

$$E_y = A(x, y)e^{i(\omega t - \beta z)} \quad (V.1)$$

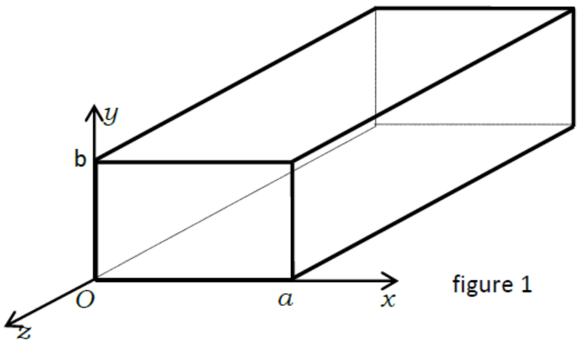


Figure V.2: Cross section of a rectangular waveguide $(a \times b)$.

V.3.2. Electromagnetic field

Assuming that propagation occurs along the (oz) axis, we then seek solutions of the form

$$\vec{E}(\vec{r},t) = Re\left(\underline{\vec{E}}(x,y)e^{i(\omega t-\beta z)}\right) \qquad \vec{B}(\vec{r},t) = Re\left(\underline{\vec{B}}(x,y)e^{i(\omega t-\beta z)}\right)$$

In this case, we obtain the following operators

$$\frac{\partial}{\partial t} \to i\omega \quad , \quad \vec{\nabla} \to i\beta\overrightarrow{u_z} + \frac{\partial}{\partial x}\overrightarrow{u_x} + \frac{\partial}{\partial y}\overrightarrow{u_y} = i\beta\overrightarrow{u_z} + \vec{\nabla}_T, \quad \vec{\nabla}^2 \to \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2$$

V.3.3. Maxwell's Equations

We are in a homogeneous medium, without charges or currents.

Maxwell-Gauss : $\frac{d\underline{E_x}}{dx} + \frac{d\underline{E_y}}{dy} + i\beta\underline{E_z} = 0 \qquad (V.2)$

Maxwell-flux : $\frac{d\underline{B_x}}{dx} + \frac{d\underline{B_y}}{dy} + i\beta\underline{B_z} = 0 \qquad (V.3)$

Maxwell-Faraday : $\begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ ik \end{pmatrix} \times \begin{pmatrix} \underline{E_x} \\ \underline{E_y} \\ \underline{E_z} \end{pmatrix} = i\omega \begin{pmatrix} \underline{B_x} \\ \underline{B_y} \\ \underline{B_z} \end{pmatrix} \Rightarrow \begin{cases} \frac{d}{dy}\underline{E_z} - i\beta\underline{E_y} = i\omega\underline{B_x} \\ i\beta\underline{E_x} - \frac{d}{dx}\underline{E_z} = i\omega\underline{B_y} \\ \frac{d}{dx}\underline{E_y} - \frac{d\underline{E_x}}{dy} = i\omega\underline{B_z} \end{cases} \qquad (V.4)$

Maxwell-Ampère : $\begin{cases} \frac{d}{dy}\underline{B_z} - i\beta\underline{B_y} = i\frac{\omega}{c^2}\underline{E_x} \\ i\beta\underline{B_x} - \frac{d}{dx}\underline{B_z} = -i\frac{\omega}{c^2}\underline{E_y} \\ \frac{d}{dx}\underline{B_y} - \frac{d\underline{B_x}}{dy} = -i\frac{\omega}{c^2}\underline{E_z} \end{cases} \qquad (V.5)$

The transverse electromagnetic field $\left(\overrightarrow{\underline{E_T}}, \overrightarrow{\underline{B_T}}\right)$ can be written as follows

$$\overrightarrow{\underline{E_T}} = \frac{-1}{\frac{i\omega}{c^2}\left(1-\frac{\beta^2 c^2}{\omega^2}\right)}\left(\vec{\nabla}_T \underline{B_z} \times \overrightarrow{u_z} + \frac{\beta}{\omega}\vec{\nabla}_T \underline{E_z}\right)$$

$$\overrightarrow{\underline{B_T}} = \frac{1}{i\omega\left(1-\frac{\beta^2 c^2}{\omega^2}\right)}\left(\vec{\nabla}_T \underline{E_z} \times \overrightarrow{u_z} + c^2\frac{\beta}{\omega}\vec{\nabla}_T \underline{B_z}\right)$$

All transverse field components are determined by the longitudinal components$\underline{E_z}$ and $\underline{B_z}$.

V.3.4. *Wave equation*

The propagation equation for the electromagnetic field is given by Helmholtz's equation:

$$\left(\vec{\nabla}^2 + k^2\right)\{\vec{E}, \vec{B}\} = \vec{0} \;\; (V.5)$$

with $k^2 = \frac{\omega^2}{c^2}$ (dispersion relation in a free space)

By replacing all operators, and taking into account that there is an invariance according to (Oz), we obtain

$$\frac{\partial^2 \underline{\vec{E}}}{\partial x^2} + \frac{\partial^2 \underline{\vec{E}}}{\partial y^2} + (k^2 - \beta^2)\underline{\vec{E}} = \vec{0} \qquad (V.6)$$

$$\frac{\partial^2 \underline{\vec{B}}}{\partial x^2} + \frac{\partial^2 \underline{\vec{B}}}{\partial y^2} + (k^2 - \beta^2)\underline{\vec{B}} = \vec{0} \qquad (V.7)$$

Let's define: $k_c^2 = k^2 - \beta^2$

The wave equation for rectangular guide is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)\{\overrightarrow{\underline{E}}, \overrightarrow{\underline{B}}\} = \vec{0} \qquad (V.8)$$

- ***Boundary conditions***

Electric field $\underline{\vec{E}}$

$$\underline{E}(0, y) = \underline{E}(a, y) = 0$$

$$\underline{E}(x, 0) = \underline{E}(x, b) = 0$$

Magnetic field $\underline{\vec{B}}$

$$\underline{B}(0, y) = \underline{B}(a, y) = 0$$

$$\underline{B}(x, 0) = \underline{B}(x, b) = 0$$

In a rectangular guide, it is impossible to have a TEM mode. We will focus on TE and TM modes.

V.3.5. Transverse Electric modes (TE) : $\underline{E_z} = 0$ et $\underline{B_z} \neq 0$

Using the method of separation of variables and taking into account the boundary conditions, the solution of equation (V.8) is:

$$\underline{B_z} = \underline{B_0} cos\left(\frac{m\pi}{a}x\right) cos\left(\frac{n\pi}{b}x\right) \qquad (V.9)$$

With $(m, n) \neq (0,0)$

This mode is called transverse electric TE_{mn}

V.3.6. Cut-off pulse

Mode propagation condition TE_{mn} is $Re(\beta) \neq 0$

Taking account that : $\beta^2 = k^2 - k_c^2$, $k^2 > k_c^2$

Then, we have

$$\omega^2 > c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \qquad (V.10)$$

The cut-off pulse corresponds to

$$\omega_c = \pi c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad (V.11)$$

If the dielectric inside the waveguide is different from a vacuum, simply replace $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$. In general, the cut-off frequency in a rectangular waveguide becomes

$$f_{c,m,n} = \frac{c}{2\pi\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad (V.12)$$

Assuming $a > b$, the lowest cut-off frequency is $f_{c,min} = f_{c,1,0}$

- For $f < f_{c,1,0}$, no propagation occurs $(\lambda_0 > 2a)$
- For $f > f_{c,1,0}$ only the TE_{10} mode can propagate.

The corresponding wave length is :

$$\lambda = \frac{c}{f_{c,m,n}} = \frac{2\pi\sqrt{\varepsilon_r}}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

In this case, the longitudinal magnetic field is given by

$$\overrightarrow{B_z} = Re\left(\underline{B_0} cos\left(\frac{\pi}{a}x\right) e^{i(\beta z - \omega t)}\right)\overrightarrow{u_z} \qquad (V.13)$$

- ***Phase velocity***

$$v_\varphi = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}} \qquad (V.14)$$

Which can rewritten in the following form

$$v_\varphi = \frac{c}{\left(1 - \frac{\omega_c^2}{\omega^2}\right)} \qquad (V.15)$$

V.3.7. Study of $TE_{1,0}$ mode

Longitudinal magnetic field

$$\underline{\overrightarrow{B_z}} = Re\left(\underline{B_0} cos\left(\frac{\pi}{a}x\right)\ e^{i(\beta z - \omega t)}\right)\overrightarrow{u_z}$$

In real notation

$$\overrightarrow{B_z} = B_0 cos\left(\frac{\pi}{a}x\right) cos(\beta z - \omega t)\overrightarrow{u_z} \quad (V.16)$$

The dispersion equation is given by

$$\beta^2 = \frac{\omega^2}{c^2} - \frac{\pi^2}{a^2} \qquad (V.17)$$

- ***Transversal magnetic field***

$$\underline{\overrightarrow{B_T}} = i\frac{ka}{\pi}\underline{B_0} sin\frac{\pi x}{a}\overrightarrow{u_x} \qquad (V.18)$$

Or, in real form

$$\overrightarrow{B_T} = \frac{ka}{\pi}B_0\ sin\frac{\pi x}{a}\ sin(\omega t - \beta z)\overrightarrow{u_x} \qquad (V.19)$$

- ***Transversal electric field***

According to the equation (V.16)

$$\underline{\overrightarrow{E_T}} = -ia\frac{\omega}{\pi}\underline{B_0} sin\frac{\pi x}{a}\overrightarrow{u_y} \qquad (V.20)$$

$$\overrightarrow{E_T} = -a\frac{\omega}{\pi}B_0\ sin\frac{\pi x}{a}\ sin(\omega t - \beta z)\overrightarrow{u_y} \qquad (V.21)$$

The transverse electric field takes the following form

$$\overrightarrow{E_T} = -E_0 sin\frac{\pi x}{a}\ sin(\omega t - \beta z)\overrightarrow{u_y} \qquad (V.22)$$

with $E_0 = a\frac{\omega}{\pi}B_0$

This means that vectors $\overrightarrow{E_T}$, $\overrightarrow{B_T}$ and $\vec{k}$ form a direct trihedron.

The TE and TM modes are symmetrical, so the same conclusions are applied to TM modes.

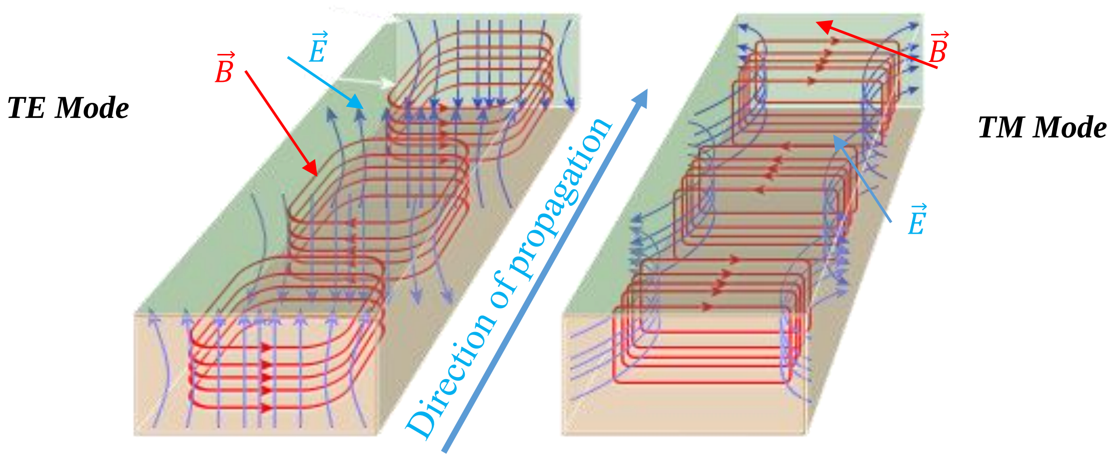


Figure V.3: Schematic representation of TE and TM modes in a rectangular waveguide.

The Figure V.3 highlights the fundamental difference between Transverse Electric TE and Transverse Magnetic TM modes. In the TE mode, the electric field has no component along the direction of propagation, whereas in the TM mode, the magnetic field lacks a longitudinal component. These modes constitute the fundamental solutions of Maxwell's equations in waveguides and play a crucial role in telecommunications systems, radar technology, and microwave components.

V.4. Cylindrical guides

In cylindrical coordinates $(\vec{e_r}, \vec{e_\theta}, \vec{e_z})$, the electromagnetic field $(\vec{E}, \vec{B})$ is written as

$$\vec{E}(\vec{r}, \theta, z) \rightarrow \begin{pmatrix} E_r \\ E_\theta \\ E_z \end{pmatrix} = \vec{E}_0(r, \theta) e^{i(\omega t - kz)}$$

$$\vec{B}(\vec{r}, \theta, z) \rightarrow \begin{pmatrix} B_r \\ B_\theta \\ B_z \end{pmatrix} = \vec{B}_0(r, \theta) e^{i(\omega t - kz)}$$

In the following, attention will be focused on optical fibers, which are the most widely used medium in telecommunications.

V.4.1. Optical fiber

Optical fiber is a glass wire capable of guiding light over long distances with very low attenuation. Current applications include the physical infrastructure for telecommunications. The smallest diameter ever, makes it highly appealing for many applications such as telecommunications, sensors, and medicine

Optical fibers have several advantages over other mediums in communication:

• Low attenuation for a broad frequency band that will support long-distance transmission without much attenuation.

• Its low cost of manufacturing makes it feasible for mass implementation.

• High Bandwidth (approximately 30THz), which can transmit huge data at the same time.

There are two kinds of fiber:

• Step index fiber, which is the common one, where the refractive index is constant in the core and changes abruptly when one moves from the core to the cladding regions. The abrupt change in refractive index helps confine the electromagnetic wave in the core through total internal reflection.

• Gradient-index fiber: This type of fiber has a gradual reduction in the refractive index from the center of the core to the cladding in the direction perpendicular to the core. This helps minimize modal dispersion through the constant deviation of the light ray from the core.

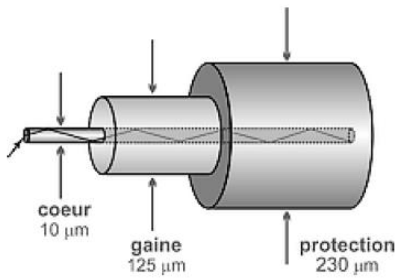


Figure V.4: Optical Fiber

V.4.2. Problem formulation

Consider an optical fiber consisting of a dielectric core with refractive index n_1 and thickness a surrounded by a dielectric cladding with refractive index n_2 $(n_2 < n_1)$. This difference in refractive index allows light to be confined. This can occur if total internal reflection occurs at the core-cladding interface.

Guidance condition in optical fiber

- *Total internal reflection*

Total internal reflection allows light to be guided in an optical fiber when it passes from the core with index n_1 to the cladding with index n_2, where $n_1 > n_2$.

It is described by Snell-Descartes' law:

$$n_1 \, sin\theta_1 = n_2 \, sin\theta_2 \qquad (V.23)$$

The critical angle is defined by

$$\sin\theta_c = {}^{n_2}/_{n_1} \qquad (V.24)$$

When $\theta_1 > \theta_c$, refraction becomes impossible and the light is entirely reflected in the core. This phenomenon ensures the confinement and guided propagation of light in the optical fiber.

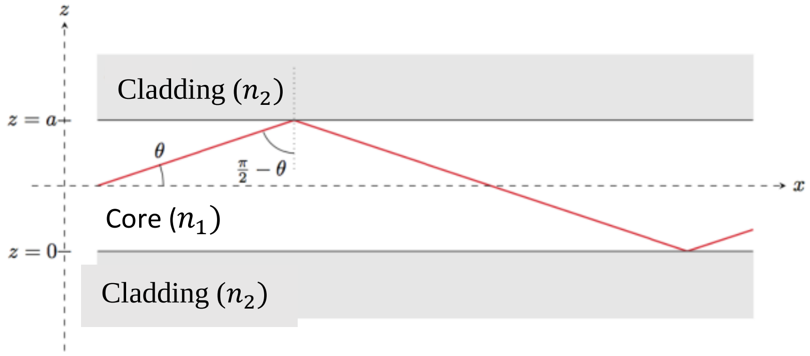


Figure V.5 : Diagram of a step-index optical fiber

Numerical aperture (NA)

The numerical aperture (NA) characterizes the ability of an optical fiber to accept incident light and guide it by total internal reflection. It corresponds to the maximum angle of acceptance of light rays entering the fiber. The numerical aperture given by

$$NA = \sqrt{n_1^2 - n_2^2} \qquad (V.25)$$

where n_1 and n_2 are the refractive indices of the core and cladding, respectively. The numerical aperture depends directly on the refractive index difference between these two regions. A larger NA improves coupling efficiency by allowing the fiber to collect light over a wider range of incident angles; however, it also increases the number of supported propagation modes, which may lead to higher modal dispersion in multimode fibers.

Normalized frequency (V-Number)

The normalized frequency determines the number of guided propagation modes in an optical fiber. It is expressed as

$$V = \frac{2\pi}{a}\sqrt{n_1^2 - n_2^2} \qquad (V.26)$$

This quantity distinguishes single-mode fibers from multi-mode fibers. This parameter directly influence transmission performance, particularly bandwidth and dispersion.
The V-number defines the propagation regime

- For $V < 2.405\ V$, the fiber operates in the single-mode regime
- For $V > 2.405\ V$, multiple modes are supported

V.4.3. Propagation equation

The optical fiber is composed of two dielectric media, namely the core and the cladding. Both materials are assumed to be linear, homogeneous, isotropic, non-conductive, and free of free electric charges and currents$\left(\rho_f = 0\right)$ and currents $\left(\vec{J_f} = \vec{0}\right)$.

Under these assumptions, Maxwell's equations in each region of the fiber reduce to the homogeneous wave equations for the electric and magnetic fields:

$$\left(\Delta - \frac{\varepsilon_r}{c^2}\frac{\partial^2}{\partial t^2}\right)\{\vec{E},\vec{B}\} = \vec{0} \qquad (V.27)$$

where $\varepsilon = \varepsilon_0\varepsilon_r$ is the permittivity of the medium and $\mu \approx \mu_0$ for non-magnetic dielectrics.

V.4.3.1. Relationship between transverse fields and longitudinal fields

It can be shown that all transverse components can be expressed as functions of the longitudinal components E_z and B_z. This simplifies the equations once the polarization has been chosen ($E_z = 0$ for TE, $B_z = 0$ for TM)

$$\vec{\nabla}\times \vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \begin{cases} \dfrac{1}{r}\dfrac{\partial E_z}{\partial\theta} - \dfrac{\partial E_\theta}{\partial z} = -i\omega B_r \\ \dfrac{\partial E_r}{\partial z} - \dfrac{\partial E_z}{\partial r} = -i\omega B_\theta \\ \dfrac{\partial E_r}{\partial\theta} - \dfrac{1}{r}\dfrac{\partial E_\theta}{\partial r} = -i\omega B_z \end{cases} \qquad (V.28)$$

$$\vec{\nabla}\times \vec{B} = \frac{\varepsilon_r}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \begin{cases} \dfrac{1}{r}\dfrac{\partial B_z}{\partial\theta} - \dfrac{\partial B_\theta}{\partial z} = i\omega\dfrac{\varepsilon_r}{c^2}E_r \\ \dfrac{\partial B_r}{\partial z} - \dfrac{\partial B_z}{\partial r} = i\omega\dfrac{\varepsilon_r}{c^2}E_\theta \\ \dfrac{\partial B_r}{\partial\theta} - \dfrac{1}{r}\dfrac{\partial B_\theta}{\partial r} = i\omega\dfrac{\varepsilon_r}{c^2}E_z \end{cases} \qquad (V.29)$$

Combining these two systems of equations allows all components to be expressed as functions of E_z and B_z

$$E_r = \frac{-i}{\dfrac{\omega^2}{c^2} - K^2}\left(K\frac{\partial E_z}{\partial r} + \frac{\omega}{r}\frac{\partial B_z}{\partial\theta}\right) \qquad (V.30)$$

$$E_\theta = \frac{-i}{\frac{\omega^2}{c^2}-K^2}\left(\frac{K}{r}\frac{\partial E_z}{\partial \theta} - \omega\frac{\partial B_z}{\partial r}\right) \qquad (V.31)$$

$$B_r = \frac{-i}{\frac{\omega^2}{c^2}-K^2}\left(K\frac{\partial B_z}{\partial r} - \frac{\mu_0\varepsilon_0\varepsilon_r\omega}{r}\frac{\partial E_z}{\partial \theta}\right) \qquad (V.32)$$

$$E_\theta = \frac{-i}{\frac{\omega^2}{c^2}-K^2}\left(\frac{K}{r}\frac{\partial B_z}{\partial \theta} - \mu_0\varepsilon_0\varepsilon_r\omega\frac{\partial E_z}{\partial r}\right) \qquad (V.33)$$

V.4.3.2.Propagation equation Solutions

The components E_z and B_z are scalar functions. The projection of the propagation equation of $\vec{E}$ onto the direction (Oz) gives

$$\Delta E_z(r,\theta,z) - \frac{\varepsilon_r}{c^2}\frac{\partial^2 E_z(r,\theta,z)}{\partial t^2} = 0 \qquad (V.34)$$

The Laplacian in cylindrical coordinates is expressed as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (V.35)$$

The Helmholtz's equation take the following form

$$(\Delta + k^2 - \beta^2)E_z(r,\theta) = 0 \qquad (V.36)$$

To solve this equation, we must use the method of separation of variables, which allows us to write:

$$E_z(r,\theta,z) = f(r)g(\theta)e^{i(\omega t - Kz)} \qquad (V.37)$$

The solutions to Helmholtz's equation are in the form of Bessel functions (J, K) as follows

$$E_z(r,\theta,z) = e^{i(\omega t - Kz)}\begin{cases} A_{core}e^{ic\theta}J_m\left(u_{core}\frac{r}{a}\right) & pour\ 0 \le r < a \\ A_{cladding}e^{im\theta}K_m\left(w_{cladding}\frac{r}{a}\right) & pour\ a < r < b \end{cases} \qquad (V.38)$$

$$B_z(r,\theta,z) = e^{i(\omega t - Kz)}\begin{cases} B_{core}e^{im\theta}J_m\left(u_{core}\frac{r}{a}\right) & pour\ 0 \le r < a \\ B_{cladding}e^{im\theta}K_m\left(w_{cladding}\frac{r}{a}\right) & pour\ a < r < b \end{cases} \qquad (V.39)$$

The quantities u and w are given by

$$u = a\sqrt{k^2n_1^2 - \beta^2} \qquad w = a\sqrt{\beta^2 - k^2n_2^2} \qquad (V.51)$$

A_{core} and B_{core} are the amplitudes of the electric and magnetic fields in the core of the fiber, respectively.

Similarly, $A_{cladding}$ et $B_{cladding}$ are the respective amplitudes of the electric and magnetic fields in the cladding.

The transverse components of the electromagnetic field are obtained by substituting the expressions of $E_z(r,\theta,z)$ and $B_z(r,\theta,z)$ into the expressions for E_r, E_θ, B_r et B_θ

A more detailed study could be carried out in more specialized courses; here, we will simply outline, in a simplified manner, the modes that may appear in optical fibers, namely the LP modes.

V.5. Linearly polarized (LP) Modes I optical fiber

In a step-index optical fiber, light does not propagate arbitrarily, but in guided modes, which are solutions to Maxwell's equations satisfying the boundary conditions at the core-cladding interface. For weakly guided fibers $(n_1 = n_2)$, the exact vector solutions can be approximated by linearly polarised (LP) modes denoted LP_{lm}. These modes describe the transverse distribution of the field in the fiber

LP modes are expressed as

$$LP_{lm}(r,\theta) = R_{lm}(r)cos(l\theta)$$

- l is the azimuthal order that defines the number of angular variations (number of petals /2)
- m is the radial order that represents the number of radial maxima (number of crowns)
- $R_{lm}(r)$ is the radial part

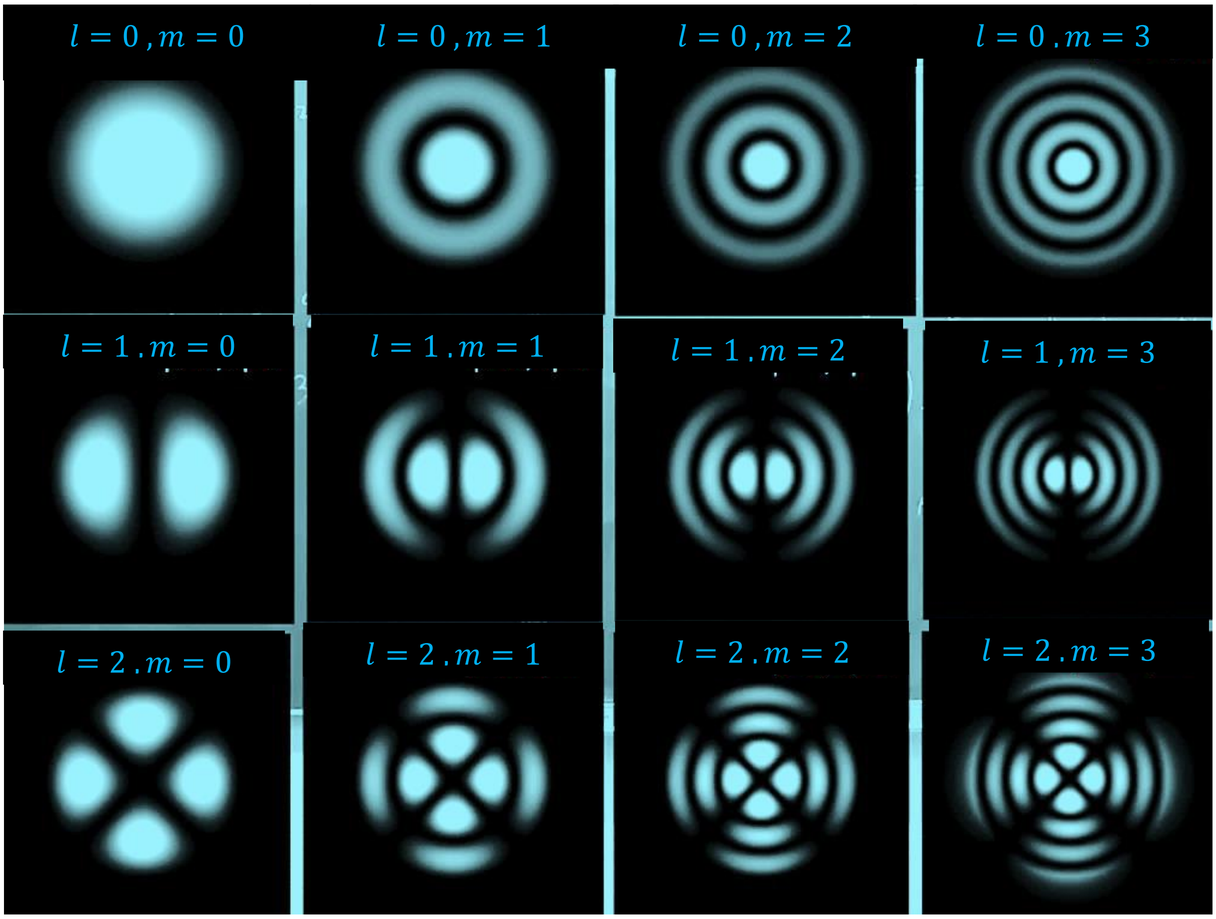


Figure V.6: Intensity distribution images of some LP modes.

In summary, Figure V.6 clearly illustrates the influence of the radial and azimuthal mode orders on the spatial distribution of optical power within the fiber. The increase of m generates additional radial rings, whereas the increase of l produces angular lobes. These modal intensity distributions constitute a fundamental characteristic of light propagation in multimode optical fibers and provide valuable insight into the guiding properties of the fiber

Exercises

Exercise 1

Let's consider a rectangular waveguide operating in the X-band (8.2–12.4 GHz) with cross-section dimensions:

$$a = 20\,mm, b = 10\,mm$$

The waveguide is filled with a dielectric of relative permittivity: $\varepsilon_r = 2.56$.

Calculate the propagating TE modes up to 15 GHz.

Solution

For a TE_{mn} mode in a rectangular waveguide:

$$f_{c,m,n} = \frac{c}{2\pi\sqrt{\varepsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$f_{c,m,n} = \frac{3 \times 10^8}{2\pi\sqrt{2.56}}\sqrt{\left(\frac{m\pi}{2 \times 10^{-2}}\right)^2 + \left(\frac{n\pi}{10^{-2}}\right)^2}$$

We find,

$$f_{c,m,n} = 9.375 \times 10^9\sqrt{(50m)^2 + (100n)^2}$$

Let's compute cutoff frequencies for the lowest modes:

- TE_{10} Mode: $m = 1, n = 0$

$$f_{c,10} = 9.375 \times 10^9 \times 50 \approx 4.69 \text{ GHz}$$

Propagation in X-band

- TE_{01} Mode: $m = 0, n = 1$

$$f_{c,01} = 9.375 \times 10^9 \times 100 \approx 9.38 \text{ GHz}$$

- TE_{11} Mode: $m = 1, n = 1$

$$f_{c,11} = 9.375 \times 10^9 \times \sqrt{(50)^2 + (100)^2} \approx 10.5 \text{ GHz}$$

- TE_{20} Mode: $m = 2, n = 0$

$$f_{c,20} = 9.375 \times 10^9 \times \sqrt{(50 \times 2)^2} \approx 9.38 \text{ GHz}$$

- TE_{21} Mode: $m = 2, n = 1$

$$f_{c,21} = 9.375 \times 10^9 \times \sqrt{(100)^2 + (100)^2} \approx 12.5 \text{ GHz}$$

Exercise 2

Consider a rectangular waveguide with dimensions a and b ($a \geq b$). This waveguide is excited simultaneously by two orthogonal polarizations: horizontal and vertical.

a) Determine the conditions that the dimensions a and b must satisfy so that a wave, in a frequency band between f_1 and f_2, can propagate only according to the two fundamental modes TE_{10} and TE_{01} , without any higher-order modes being excited.

b) Establish the necessary relation between f_1 and f_2 so that this situation is achievable.

c) Verify in this context, that the choice $a = 1.8\ cm$ and $b = 1.7\ cm$ is consistent for a band centered around $f_1 = 9GHz$ and $f_2 = 9GHz$.

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