الجمهورية الجزائرية الديمقراطية الشعبية
People's Democratic Republic of Algeria

Ministry of Higher Education
and Scientific Research
University Abderrahmane MIRA
Bejaia

جامعة بجاية
Tasdawit n Bgayet
Université de Béjaïa

وزارة التعليم العالي
والبحث العلمي
جامعة عبد الرحمان ميرة
بجاية

Faculty of Technology
Department of Mechanical Engineering

Notion de systèmes asservis

Level: 3rd Year Bachelor (L3 - LMD)

Specialization: - Mechanical Construction

- Energetic

Course material prepared and taught by:

Dr. MALOUM Hakima

2024/2025

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FOREWORD

Foreword

This controls systems course material, taught in the Department of Mechanical Engineering at the Faculty of Technology, Abderrahmane Mira University of Béjaïa, is intended for 3rd-year Bachelor (L3 – LMD) students in Mechanical Engineering, specializing in Mechanical Construction and Energetic, during the 5th semester of their university studies. It is set up and designed according to the ministerial program. Nevertheless, attending the class will enable students to understand better, master, and apply the concepts presented

At the end of this course, the student will be able to:

1- Understand what a regulatory system means.
2- Learn how to model and structure first- and second-order systems.
3- To acquire knowledge about the Laplace transform and its properties.
4- Learn how to manipulate block patterns.
5- To study the response of a first-order and second-order system in the time and frequency domains.
6- Master the analysis of the stability of these systems.

To achieve these objectives, students must be equipped with prerequisites in mathematics, including calculus of partial and total derivatives, calculus of integrals, and complex numbers, as well as knowledge of physics, such as kinematics and systems dynamics.

An online course space on the Progress platform of the University of Bejaia reinforces the electronics module for better assimilation and mastery. This space is equipped with different tests for the module's prerequisites and the other chapters, learning activities, help in the form of videos for the problematic parts of the course, and a space for exchange with the teacher.

SUMMARY

Summary

Chapter IV: Study of a first-order Control system

Chapter V: Study of a second-order Control system

Chapter VI: Bode and Nyquist diagram of Control systems

CHAPTER I: CONTROL SYSTEM TERMINOLOGY

I.1. Introduction

The proper functioning of automatic systems and their control depends on the control system and its efficiency. Control systems regulate a system's behaviour (output or response) to achieve a precise or desired result. Understanding certain key concepts is essential to establishing controls or regulations. To achieve this, in this chapter, we will introduce the fundamental concepts of control systems, namely the notions of open and closed-loop systems, the different components of a regulated system, and their key characteristics.

I.2. Terminologies and definitions

- **A system** is an assembly of constituents connected in such a way as to form an entity or a whole.
- **The word "control"** is generally used in the sense of being regulated, directed, or commanded.

Combining the two definitions, we get:

A control system: A system is an assembly of physical components, connected in such a way that they can control, direct, or regulate another system or process.

I.3. The main signals

To define or identify a control system, two main terms must first be defined in the language of servo systems: Input signals and output signals.

I.3.1. Input signals

These quantities are external to the system but affect its behavior. These quantities can be classified into two types:

a) **Reference input (or setpoints):** These quantities are generated using an external energy source to obtain a specific and predefined response from the system.
b) **Disturbances** are unwanted input that affect system's performance.

I.3.2. Output signal

It represents the response developed by the system from an initial state before receiving the input signals.

I.3.3. Course study restrictions

This course only applies to systems (or parts of systems) with analog-type input and output.

Example of a control system: The furnace regulator.

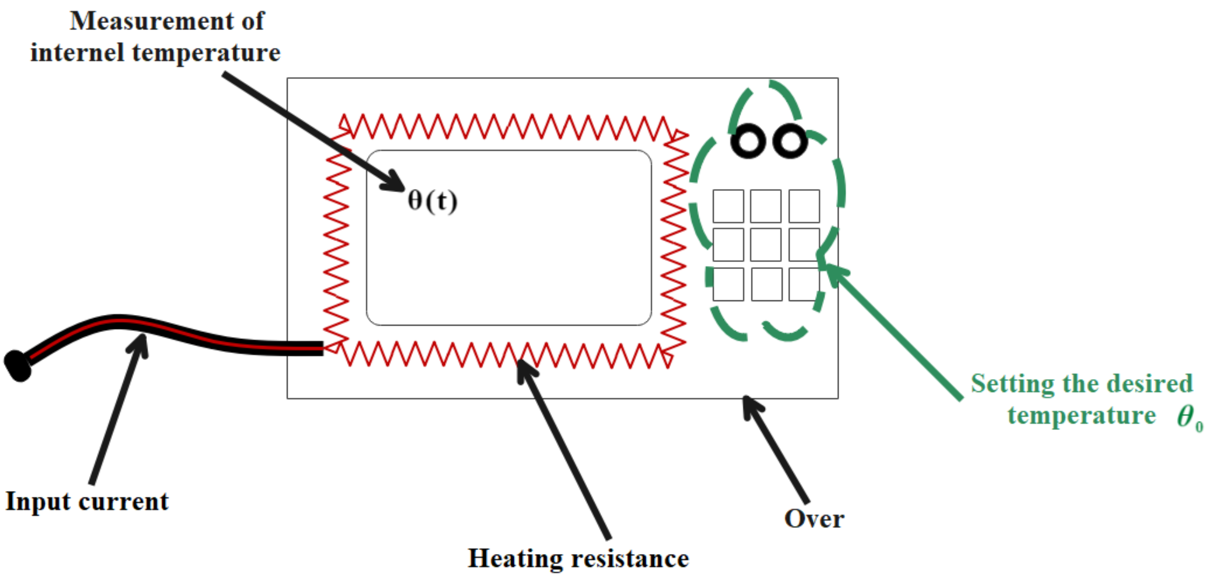


Figure I.1. Electric furnace diagram showing input and output parameters

The system can be organized as follows:

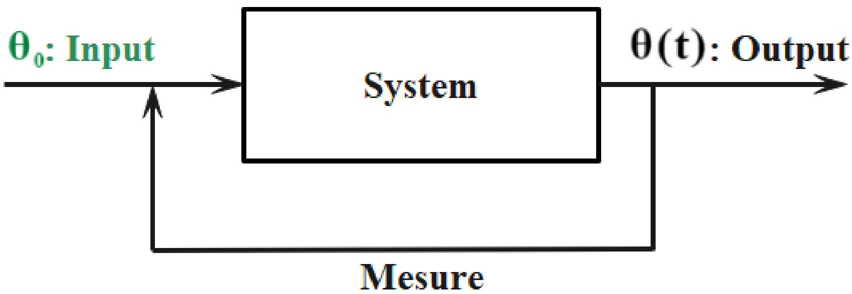


Figure I.2. Simplified diagram of an electric furnace operation

The signal identification for the system that contains the electric furnace is as follows:

- **The input signal** θ_0 **:** the desired temperature in (°C).

- **The control signal**: is the electric current that controls the Joule effect at the heating resistor.
- **The output signal:** the temperature inside the electric furnace $\theta(t)$

I.4. Open Loop (OL) / Closed Loop (CL) System

I.4.1. A system is in an open loop (direct chain)

A system is said to be in an open loop when the control is established without any information about the state of the output quantity. There is no return option for changing the size of the order. The diagram of an open-loop system is shown in Figure I.3.

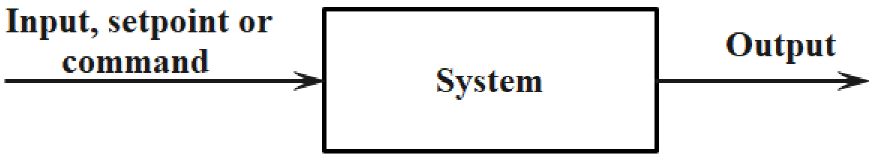


Figure I.3. Diagram of a system without control: Open-loop system

I.4.2. The system is closed-loop (return)

A system is said to be closed-loop when the control is generated based on the setpoint (the desired output value) and the measurement of the output quantity.

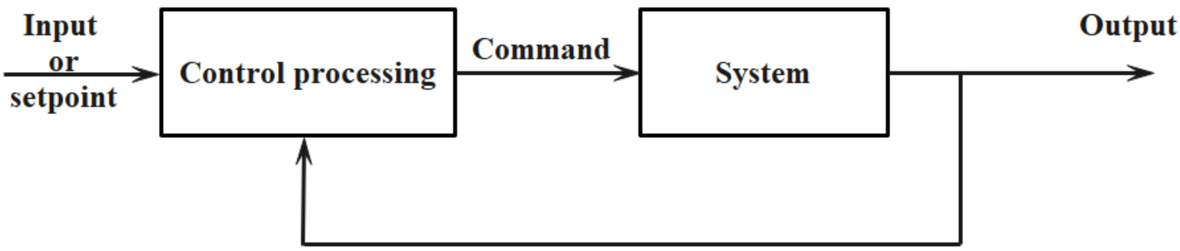


Figure I.4. Diagram of a system with control: closed-loop system

Example of the application of the concept of open loop and closed loop in daily life

- **Open-loop system:** A classic fan.
- **Closed-loop system:** an air conditioner that adjusts the room's temperature using a thermostat.

I.5. Creation of a control system

A control system, comprising its various components, can be schematized as shown in Figure I.5.

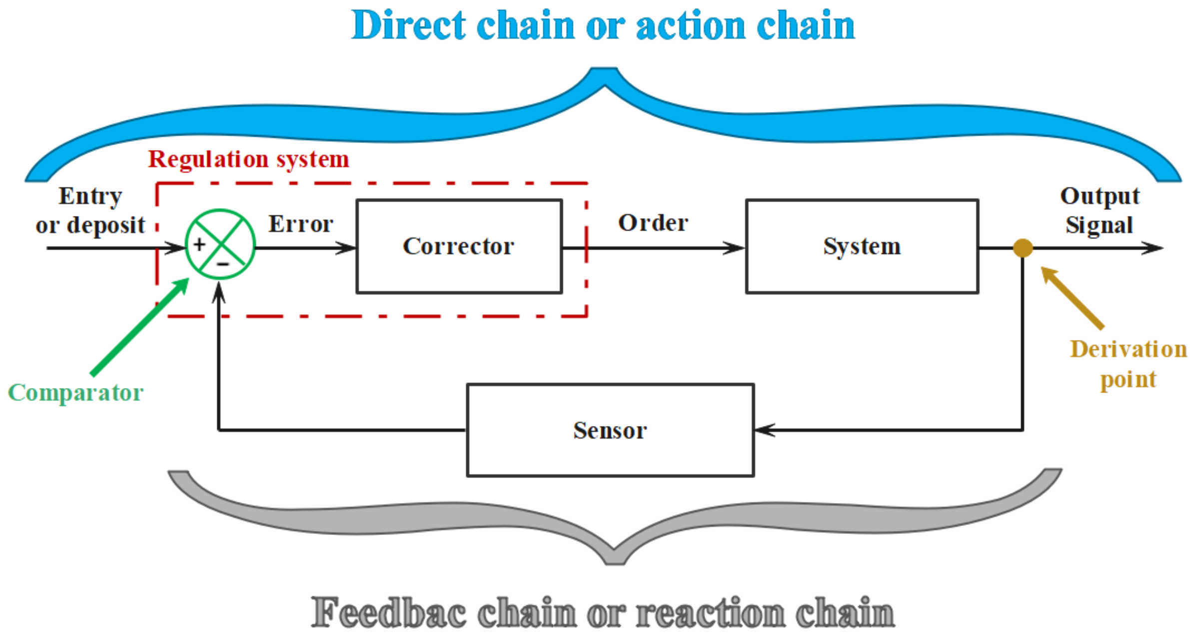


Figure I.5. Diagram of a control system

The control system can be broadly divided into two:

I.5.1. Direct chain or actions (Forward path)

The chain of action includes:

- A controller includes a comparator to compare the set point with the measurement and a controller compensator (corrector) that produces a control signal to adjust the output signal towards the set point.
- All necessary power components for the job's proper execution.
- Amplifiers.

I.5.2. Return or reaction chain (Feedback path)

In general, the Feedback patch enables the system's control to be obtained by measuring the value of the output quantity using a sensor. This measurement undergoes amplification and transformation before sending it to the direct chain comparator.

I.6. Performance of a servo system

The performance of a control system is manifested by its ability to achieve and stabilize the desired state in an optimal time. So, we can set the following three parameters:

I.6.1. Stability

A system is stable when its output tends towards the desired state. For example, the output signal is close to zero for zero input.

I.6.2. Accuracy

Accuracy represents the difference between the output $z(t)$ and setpoint $e(t)$ signals. The closer this value is to zero, the more accurate the system is. This difference is called an error $\varepsilon(t)=e(t)-z(t)$. There are two types of precision:

I.6.2.a. Transient accuracy

$\varepsilon(t)$ defined as dynamic error when the transient regime of the response is not negligible.

I.6.2.b. Steady-state accuracy

Characterizes the static error when the output reaches its final steady-state value.

I.6.3. Speed

Several methods can be used to evaluate a control system's speed of response. We cite the most used:

I.6.3.1. Evaluation of the speed of the control system with the rise time

The rise time represents the time required for the response to rise from 10% to 90% of its final value.

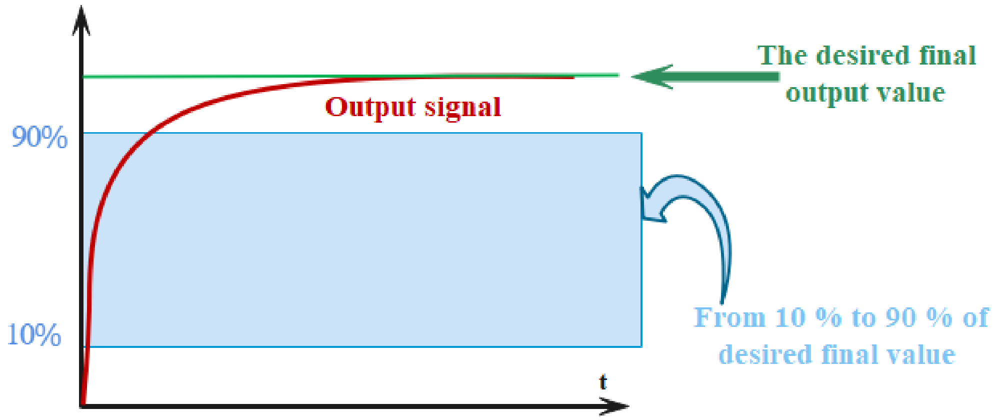


Figure I.6. Range of margin representation: 10% to 90% of desired output value

I.6.3.2. Evaluation of the speed of the control system with response time (settling time)

The settling time is the time required for the system response to remain within a specified tolerance band around the final value.

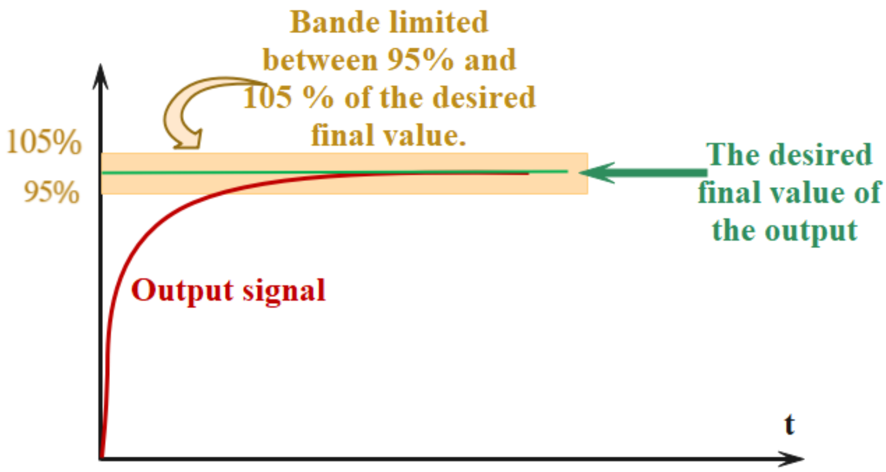


Figure I.7. Range of margin representation: 95%–105% of desired output value

I.7. Learning activity

Exercise N°1:

1- What is the difference between an open and closed-loop control system?

2- Categorize the following systems as open-loop or closed-loop, with a rationale for your answer:

a- A washing machine, when programmed for a single wash cycle.

b- The operation of an air conditioner with a thermostat for automatic temperature adjustment.

c- The operation of a coffee machine is precise when it heats a quantity of water, regardless of the amount of coffee.

Exercise N°2:

In a 12-story building, an elevator is set up to take people up and down between the different floors according to the request of the people who use it:

1. What are the different elements that make up this system (entrance, exit, sensor, etc.), and what is the role of each component in the proper functioning of the elevator?
2. Is the elevator operation controlled in an open loop or a closed loop? Justify.

Exercise N°3:

A speed regulator ensures an electric motor's speed variation by adjusting the speed according to demand by varying the supply voltage.

1. Explain the operating principle of the control system to vary the motor speed according to the desired speed.

CHAPTER II: LAPLACE TRANSFORMATION

II.1. Introduction

The Laplace transform is a mathematical tool used to analyze dynamic systems more simply. It transforms differential equations into polynomials with "s" as the variable. In this chapter, we will define the Laplace transform, present its properties, and conclude with a table of standard functions that facilitate its application.

II.2. Definitions

Let $f(t)$ be a function of the time variable t such that $t \succ 0$. The function $F(s)$ is called the Laplace transform of $f(t)$:

$$F(s) = L\left[f(t)\right]$$

$$F(s) = \int_0^{+\infty} e^{-s.t} \cdot f(t)\,dt$$

s: is a complex variable playing the role of time, so $s \succ 0$.

Example: $f(t) = 1$

$$u(t)\begin{cases} 0 \to t \prec 0 \\ 1 \to t \geq 0 \end{cases}$$

$$L[1] = \int_0^{+\infty} \left(e^{-s\cdot t} \times 1\right) dt$$

$$L[1] = \lim_{a \to +\infty} \int_0^{a} \left(e^{-s\cdot t} \times 1\right) dt$$

$$L[1] = \lim_{a \to +\infty} \left[\frac{-1}{s} \cdot e^{-s\cdot t}\right]_0^a$$

$$L[1] = \left(\frac{-1}{s} \cdot e^{-s\cdot(+\infty)} - \frac{-1}{s} \cdot e^{-s\cdot 0}\right)$$

$$L[1] = \frac{1}{s} \cdot e^{0}$$

From where:

$$L[1] = \frac{1}{s}$$

And, in general, it is noted:

$$L[cst] = \frac{cst}{s}$$

II.3. Properties

The Laplace domain has mathematical properties that simplify the analysis of dynamic systems from the time domain to the Laplace domain.

II.3.1. Uniqueness

The uniqueness property guarantees that for every function in the time domain, there is one and only one Laplace transform of the function that corresponds to it, and the converse inverse Laplace transform is also guaranteed.

$$f_1(t) \xrightarrow{L} F_1(s)$$

Time domain **Laplace domain**

$$f_2(t) \xrightarrow{L} F_2(s)$$

II.3.2. Linearity

The linearity property states that the Laplace transform of a linear combination of functions is equal to the linear combination of the Laplace transforms of these functions. This simplifies the analysis of linear systems.

Let us consider functions of time $f_1(t)$ and $f_2(t)$

Posing

$$F_1(s) = L\left[f_1(t)\right]$$

$$F_2(s) = L\left[f_2(t)\right]$$

Then

1. $L\left[f_1(t) + f_2(t)\right] = L\left[f_1(t)\right] + L\left[f_2(t)\right]$

2. $L\left[a \cdot f_1(t)\right] = a \cdot L\left[f_1(t)\right]$

Example: Determine the Laplace transform of $\cos(\omega t)$

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$L\left[\cos(\omega t)\right] = L\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)\right]$$

$$L\left[\cos(\omega t)\right] = \frac{1}{2} \cdot L\left[\left(e^{j\omega t} + e^{-j\omega t}\right)\right]$$

$$L\left[\cos(\omega t)\right] = \frac{1}{2} \cdot \left(L\left[e^{j\omega t}\right] + L\left[e^{-j\omega t}\right]\right)$$

We know

$$L\left[e^{at}\right]=\frac{1}{s-a}$$

$$L\left[e^{-at}\right]=\frac{1}{s+a}$$

So

$$L\left[\cos(\omega t)\right]=\frac{1}{2}\cdot\left(\frac{1}{s-j\omega}+\frac{1}{s+j\omega}\right)$$

$$L\left[\cos(\omega t)\right]=\frac{1}{2}\cdot\left(\frac{s+j\omega}{(s+j\omega)(s-j\omega)}+\frac{s-j\omega}{(s+j\omega)(s-j\omega)}\right)$$

$$L\left[\cos(\omega t)\right]=\frac{1}{2}\cdot\left(\frac{s+j\omega+s-j\omega}{(s+j\omega)(s-j\omega)}\right)$$

$$L\left[\cos(\omega t)\right]=\frac{1}{2}\cdot\frac{2\cdot s}{(s+j\omega)(s-j\omega)}$$

$$L\left[\cos(\omega t)\right]=\frac{s}{s^2+\omega^2}$$

II.3.3. Delay theorem

The delay theorem describes the time-shifting property of the Laplace transform.

Let $f(t)=0 \rightarrow t \prec 0$

Then

$$L\left[f(t)\right]=F(s)\rightarrow L\left[f(t-\tau)\right]=e^{-\tau\cdot s}\times L\left[f(t)\right]$$

$$L\left[f(t-\tau)\right]=e^{-\tau\cdot s}\times L\left[f(t)\right]$$

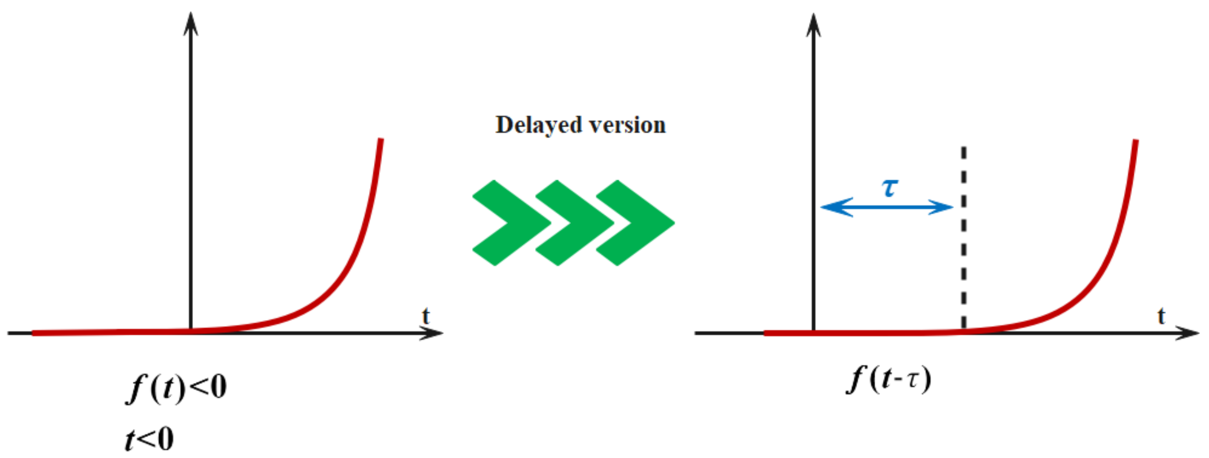


Figure II.1. Delayed function with delay τ

Example: The following figure shows a unit step with a delay τ.

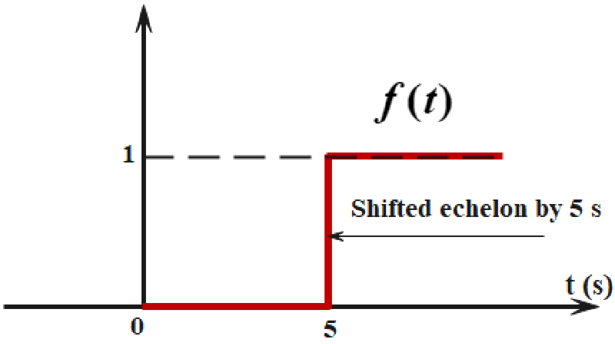


Figure II.2. Unit step with delay τ =5s

The expression of the function of a delayed step is given by $f(t)=u(t-5)$ its Laplace transform, which is calculated as follows:

$$L[f(t)]=L[u(t-5)]$$

According to the delay theorem

$$L[u(t-5)]=e^{-5t}\cdot L[u(t)]$$

$$L[u(t-5)]=e^{-5t}\cdot\frac{1}{s}$$

II.3.4. Laplace transform of a derivative

Let $f(t)$ be a function of the variable t and differentiable for $t \succ 0$.

Then:

$$L\left[\frac{df(t)}{dt}\right]=L[f'(t)]$$

$$L\left[\frac{df(t)}{dt}\right]=s\cdot F(s)-f(0)$$

$$L\left[\frac{df^2(t)}{dt}\right]=s^2\cdot F(s)-s\cdot f(0)-f'(0)$$

$f(0)=f'(0)=0 \;\rightarrow$ When the initial conditions are zero.

Demonstration:

$$L[f'(t)]=\int_0^{+\infty} f'(t)\cdot e^{-st}dt$$

$$\int_a^b U'V=[UV]_a^b-\int_a^b UV'$$

$$\int_0^{+\infty} f'(t)\cdot e^{-st}dt=\left[f(t)\cdot e^{-st}\right]_0^{+\infty}-\int_0^{+\infty} f(t)\left(-s\cdot e^{-st}\right)dt$$

$$\int_0^{+\infty} f^{'}(t)\cdot e^{-st}dt = -f(0) + s.\int_0^{+\infty} f(t)\cdot e^{-st}dt$$

$$L\left[f^{'}(t)\right] = \int_0^{+\infty} f^{'}(t)\cdot e^{-st}dt = s\cdot F(s) - f(0)$$

II.3.5. Laplace transform of an integral

Let $f(t)$ be a function of the variable of "t" then

$$L\left[f(t)\right] = F(s)$$

Therefore

$$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{g(0)}{s}$$

$g(0) = 0$ (When the initial conditions are zero).

II.3.6. initial value theorem and final value theorem

The Laplace transforms into the initial value theorem and final value theorem makes it easy to determine a function's initial value and final value in the time domain using its expression in the Laplace domain without finding its inverse transform.

- $f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to +\infty} s\cdot F(s)$
- $f(\infty) = \lim_{t\to +\infty} f(t) = \lim_{s\to 0} s\cdot F(s)$

Provided that the limits exist.

II.3.7. Using the Laplace transform

Using the Laplace transform, we can solve complex differential equations (of order n) by transforming them into simpler algebraic equations, making them easier to solve, as shown above.

Time domain

$$a_n \frac{d^n Z(t)}{dt^n} + a_{n-1}\frac{d^{n-1}Z(t)}{dt^{n-1}} + \ldots + a_1\frac{d^1 Z(t)}{dt^1} + a_0 Z(t) = b_m\frac{d^m e(t)}{dt^m} + b_{m-1}\frac{d^{m-1}e(t)}{dt^{m-1}} + \ldots + b_1\frac{d^1 e(t)}{dt^1} + b_0 e(t)$$

$\boldsymbol{L[e(t)]}$ $\qquad\qquad$ $\boldsymbol{L^{-1}[Z(s)]}$

Frequency domain

$$a_n s^n Z(s) + a_{n-1}s^{n-1}Z(s) + \ldots + a_1 s^1 Z(s) + a_0 Z(s) = b_m s^m E(s) + b_{m-1}s^{m-1}E(s) + \ldots + b_1 s^1 E(s) + b_0 E(s)$$

Namely, $n \succ m$.

II.4. Common Functions

To perform the direct or inverse Laplace transform easily, without computing the improper integral, we use common functions that have already been calculated and determined. These functions are summarized in Table II.1.

Tableau II.1. Common Functions

$f(s)$	$f(t)$ for $t \succ 0$
1	Unit impulse of duration t_0 and amplitude of $1/t_0$.
$e^{-\tau\cdot s}$	Delayed unit impulse $\delta(t-\tau)$
$\frac{1}{s}$	Unit step $u(t)$
$\frac{E_c}{s}$	Step of amplitude $E_c \cdot u(t)$
$\frac{1}{s}e^{-\tau s}$	Delayed unit step $u(t-\tau)$
$\frac{1}{s}\left(1-e^{-\tau s}\right)$	Rectangular impulse $u(t)-u(t-\tau)$
$\frac{1}{s+a}$	$e^{-at}\cdot u(t)$
$\frac{1}{1+\tau\cdot s}$	$\frac{e^{-t/\tau}}{\tau}\cdot u(t)$
$\frac{1}{s^2}$	Unit ramp: $t\cdot u(t)$
$\frac{1}{s^n}$, n: Positive integer	$\frac{t^{n-1}}{(n-1)!}\cdot u(t)$
$\frac{1}{s\cdot(s+a)}$	$\frac{1-e^{-at}}{a}\cdot u(t)$
$\frac{1}{s\cdot(1+\tau\cdot s)}$	$\left(1-e^{-t/\tau}\right)\cdot u(t)$
$\frac{1}{(s+a)^2}$	$t\cdot e^{-at}\cdot u(t)$
$\frac{1}{(1+\tau\cdot s)^2}$	$\frac{t}{\tau^2}e^{-t/\tau}\cdot u(t)$
$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}\cdot e^{-at}\cdot u(t)$
$\frac{1}{(1+\tau\cdot s)^n}$	$\frac{1}{\tau^n(n-1)!}t^{n-1}\cdot e^{-t/\tau}\cdot u(t)$
$\frac{1}{s^2\cdot(1+\tau\cdot s)}$	$\left(t-\tau+\tau\cdot e^{-t/\tau}\right)\cdot u(t)$

$\frac{1}{s\cdot(1+\tau\cdot s)^2}$	$\left(1-\left(1+\frac{1}{\tau}\right)e^{-t/\tau}\right)\cdot u(t)$
$\frac{1}{s^2\cdot(1+\tau\cdot s)^2}$	$\left(t-2\tau+(1+2\tau)e^{-t/\tau}\right)\cdot u(t)$
$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega t)\cdot u(t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)\cdot u(t)$
$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\cdot\sin(\omega t)\cdot u(t)$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cdot\cos(\omega t)\cdot u(t)$
$\frac{s+a}{s^2+\omega^2}$	$\sqrt{\frac{a^2+\omega^2}{\omega^2}}\cdot\sin(\omega t+\varphi)\cdot u(t)$ *With* $\varphi=\arctan\frac{\omega}{a}$
$\frac{1}{s\cdot(s^2+\omega^2)}$	$\frac{1-\cos(\omega t)}{\omega^2}\cdot u(t)$
$\frac{1}{(s+a)\cdot(s+b)}$	$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)\cdot u(t)$
$\frac{1}{(1+\tau_1\cdot s)\cdot(1+\tau_2\cdot s)}$	$\frac{1}{\tau_1-\tau_2}\left(e^{-t/\tau_1}-e^{-t/\tau_2}\right)\cdot u(t)$
$\frac{1}{s\cdot(1+\tau_1\cdot s)\cdot(1+\tau_2\cdot s)}$	$1-\frac{1}{\tau_1-\tau_2}\left(\tau_1e^{-t/\tau_1}-\tau_2e^{-t/\tau_2}\right)\cdot u(t)$
$\frac{1}{s^2\cdot(1+\tau_1\cdot s)\cdot(1+\tau_2\cdot s)}$	$t-(\tau_1+\tau_2)+\frac{1}{\tau_1-\tau_2}\left(\tau_1^2e^{-t/\tau_1}-\tau_2^2e^{-t/\tau_2}\right)\cdot u(t)$
$\frac{1}{s^2+2\cdot\omega_0\cdot\xi\cdot s+\omega_0^2}$ *With* $\xi\prec 1$	$\frac{1}{\omega}e^{-\xi\omega_0 t}\sin(\omega t)\cdot u(t)$ *With* $\omega=\omega_0\sqrt{1-\xi^2}$
$\frac{1}{s^2+2\cdot\omega_0\cdot\xi\cdot s+\omega_0^2}$ *With* $\xi\succ 1$	$\frac{e^{r_2\cdot t}-e^{r_1\cdot t}}{r_2-r_1}\cdot u(t)$, *With* r_2,r_1 *: Equation roots*
$\frac{1}{P\cdot\left(P^2+2\omega_0\xi P+\omega_0^2\right)}$ *With* $\xi\prec 1$	$\frac{1}{\omega_0^2}\left(1-\frac{\omega_0}{\omega}e^{-\xi\omega_0 t}\sin(\omega t+\varphi)\right)\cdot u(t)$ *With* $\varphi=\arccos(\xi)$
$\frac{1}{s\cdot\left(s^2+2\cdot\omega_0\cdot\xi\cdot s+\omega_0^2\right)}$ *With* $\xi\succ 1$	$\frac{1}{\omega_0^2}\left(1-\frac{\omega_0}{r_2-r_1}\left(\frac{e^{r_2t}}{r_2}-\frac{e^{r_1t}}{r_1}\right)\right)\cdot u(t)$
$\frac{1}{s^2\cdot\left(s^2+2\cdot\omega_0\cdot\xi\cdot s+\omega_0^2\right)}$ *With* $\xi\prec 1$	$\frac{1}{\omega_0^2}\left(1-\frac{2\xi}{\omega_0}+\frac{1}{\omega}e^{-\xi\omega_0 t}\sin(\omega t+\varphi)\right)\cdot u(t)$
$\frac{s}{(s+a)\cdot(s+b)}$	$\frac{1}{a-b}\left(a\cdot e^{at}-b\cdot e^{bt}\right)\cdot u(t)$

$\dfrac{s+c}{(s+a)\cdot(s+b)}$	$\dfrac{1}{a-b}\left((c-a)\cdot e^{-at}-(c-b)\cdot e^{-bt}\right)\cdot u(t)$

II.5. Learning activity

Exercise N°1: Applying the Laplace transforms, calculate the following Laplace transforms:

1. $f_1(t)=3\cdot e^{-2\cdot t}$
2. $f_2(t)=4\cdot t^3$
3. $f_3(t)=\sin(5t)$
4. $f_4(t)=e^{-t}\cdot\cos(3t)$
5. $f_5(t)=u(t-1)\cdot(t-1)$, *Where* $u(t-1)$ *is a delayed step function*
6. $f_6(t)=7\cdot e^{2\cdot t}$
7. $f_7(t)=t\cdot e^{-4\cdot t}$

Exercise N°2: find the temporal functions corresponding to the following Laplace transforms:

1. $F_1(s)=\dfrac{4}{s+3}$
2. $F_2(s)=\dfrac{6}{s^2}$
3. $F_3(s)=\dfrac{7}{s^2+16}$
4. $F_4(s)=\dfrac{s}{s^2+16}$
5. $F_5(s)=\dfrac{5\cdot e^{-2\cdot s}}{s+4}$
6. $F_6(s)=\dfrac{3}{(s+1)^2}$
7. $F_7(s)=\dfrac{10\cdot s}{(s^2+25)^2}$

Exercise N°3: For each system given by the following differential equation:

a. $\dfrac{d^2y(t)}{dt^2}+5\dfrac{dy(t)}{dt}+6\cdot y(t)=x(t)$

b. A mechanical system is described by $\dfrac{d^2\theta(t)}{dt^2}+2\dfrac{d\theta(t)}{dt}+3\cdot\theta(t)=T(t)$

Where $\theta(t)$ represents the angle (output), $T(t)$ the applied torque (input)

Find**: 1-** The equation in the Laplace domain (with zero initial conditions).

2- The transfer function $H(s)$ under canonical form.

CHAPTER III: TRANSFER FUNCTION

III.1. Introduction

Representing a system by its transfer function is essential in dynamic analysis. Using the Laplace transform allows us to examine the system's response (output) as a function of the reference (input). This representation in the form of a transfer function facilitates the analysis of a system's behavior in the frequency domain (which we will explore in Chapters 6 and 7). In this chapter, we will learn how to obtain a transfer function and the methods of simplification applied to a complex system to get an equivalent transfer function.

III.2. Definition

The system transfer function (or transmittance) is the ratio $H(s)$ defined by:

$$H(s)=\frac{Z(s)}{E(s)}$$

To obtain the transfer function of a system, follow the steps shown in Figure III.1

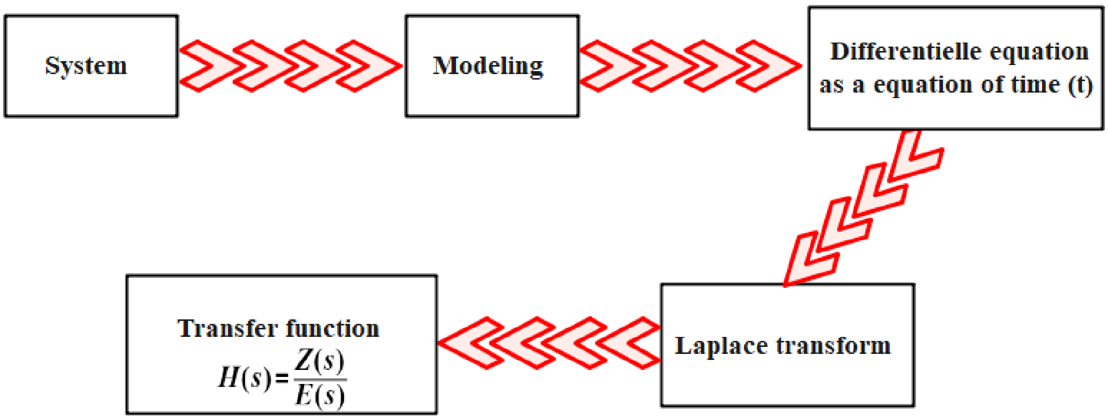


Figure III.1. The steps involved in designing a transfer function from a system

III.3. General Form

The general form of a system's transfer function is obtained as follows:

$$a_0 \cdot Z(t)+a_1\frac{dZ(t)}{dt}+...+a_n\frac{d^nZ(t)}{dt^n}=b_0 \cdot e(t)+b_1\frac{de(t)}{dt}+...+b_m\frac{d^me(t)}{dt^m}$$

With $m \prec n$

$$a_0 \cdot Z(s)+a_1 \cdot s \cdot Z(s)+...+a_n \cdot s^n \cdot Z(s)=b_0 \cdot E(s)+b_1 \cdot s \cdot E(s)+...+b_m \cdot s^m \cdot E(s)$$

$$Z(s)\left[a_0+a_1 \cdot s+...+a_n \cdot s^n\right]=E(t)\left[b_0+b_1 \cdot s+...+b_m \cdot s^m\right]$$

We note:

$$H(s)=\frac{Z(s)}{E(s)}=\frac{b_0+b_1 \cdot s+...+b_m \cdot s^m}{a_0+a_1 \cdot s+...+a_n \cdot s^n}$$

III.3.1. Pole and zero

We note:

$$H(s)=\frac{N(s)}{D(s)}=\frac{b_0+b_1\cdot s+...+b_m\cdot s^m}{a_0+a_1\cdot s+...+a_n\cdot s^n}$$

We call:

- The zeros of the transfer function $N(s)=0$.
- The poles of the transfer function $D(s)=0$.

The system is stable when its poles are in the negative real part.

III.4. Canonical form

For each transfer function, it can be written in canonical form as follows:

$$H(s)=\frac{b_m\cdot s^m+...+b_1\cdot s+b_0}{a_n\cdot s^n+...+a_1\cdot s+a_\alpha\cdot s^\alpha}$$

$$H(s)=\frac{1}{a_\alpha\cdot s^\alpha}\times\frac{b_m\cdot s^m+...+b_1\cdot s+b_0}{\frac{a_n\cdot s^n}{a_\alpha\cdot s^\alpha}+...+\frac{a_1\cdot s}{a_\alpha\cdot s^\alpha}+1}$$

We note:

$$\begin{cases}\frac{s^n}{s^\alpha}=s^{n-\alpha}\\ \frac{a_n}{a_\alpha}=a_n^{'}\end{cases}$$

We replace the expression of $H(s)$, and we obtain:

$$H(s)=\frac{1}{a_\alpha\cdot s^\alpha}\times\frac{b_m\cdot s^m+...+b_0}{a_n^{'}\cdot s^{n-\alpha}+...+1}$$

$$H(s)=\frac{1}{a_\alpha\cdot s^\alpha}\times b_0\times\frac{\frac{b_m}{b_0}\cdot s^m+...+1}{a_n^{'}\cdot s^{n-\alpha}+...+1}$$

$$H(s)=\frac{1}{s^\alpha}\times\frac{b_0}{a_\alpha}\times\frac{\frac{b_m}{b_0}\cdot s^m+...+1}{a_n^{'}\cdot s^{n-\alpha}+...+1}$$

$$H(s)=\frac{1}{s^\alpha}\times\frac{b_0}{a_\alpha}\times\frac{1+...+b_m^{'}\cdot s^m}{1+...+a_n^{'}\cdot s^{n-\alpha}}$$

$$H(s)=\frac{1}{s^{\alpha}}\times K\times\frac{1+\ldots+b_{m}^{'}\cdot s^{m}}{1+\ldots+a_{n}^{'}\cdot s^{n-\alpha}}$$

Putting $H(s)$ in canonical form consists of writing $H(s)$ in the above form. We then define:

Static gain: $K=\frac{b_0}{a_\alpha}$

The class: α

Order: n

III.4.1.Interest in the canonical form

These new coefficients will reveal quantities characteristic of the mechanism, such as:

- Time constants,
- The proper pulsations,
- Depreciation.

These quantities tell us about the system's behaviour without requiring additional in-depth calculations.

Example:

$$H(s)=\frac{s+2}{s^2+3\cdot s}$$

The canonical form of $H(s)$

$$H(s)=\frac{2}{3s}\cdot\frac{\frac{1}{2}s+1}{\frac{1}{3}s+1}$$

$$H(s)=\frac{1}{s}\cdot\frac{2}{3}\cdot\frac{\frac{1}{2}s+1}{\frac{1}{3}s+1}$$

$\frac{1}{s}$: Integrator.

$K=\frac{2}{3}$: The static gain

$\alpha=0$: The class

$n=2$: Order

III.5. Block Diagram Model

Block diagram modeling represents a system, its components, and their connection in a graphical representation. As with a transfer function $H(s)$, it has as input a signal $E(s)$ and as output another signal $Z(s)$, and we can represent this by a block diagram as follows:

$$H(s)=\frac{Z(s)}{E(s)}\rightarrow Z(s)=H(s)\cdot E(s)$$

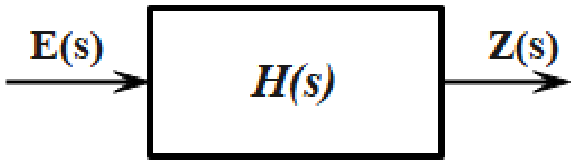


Figure III.2. Block diagram of transfer function $H(s)$

III.5.1. Connecting Blocks

The block connection is the graphical representation of the link between the various components of a system. There are several types of interaction, including:

III.5.1.a. Serial Transfer Function

Series connection is when the output of one block is connected to the input of the block that follows it, as shown in the following figure:

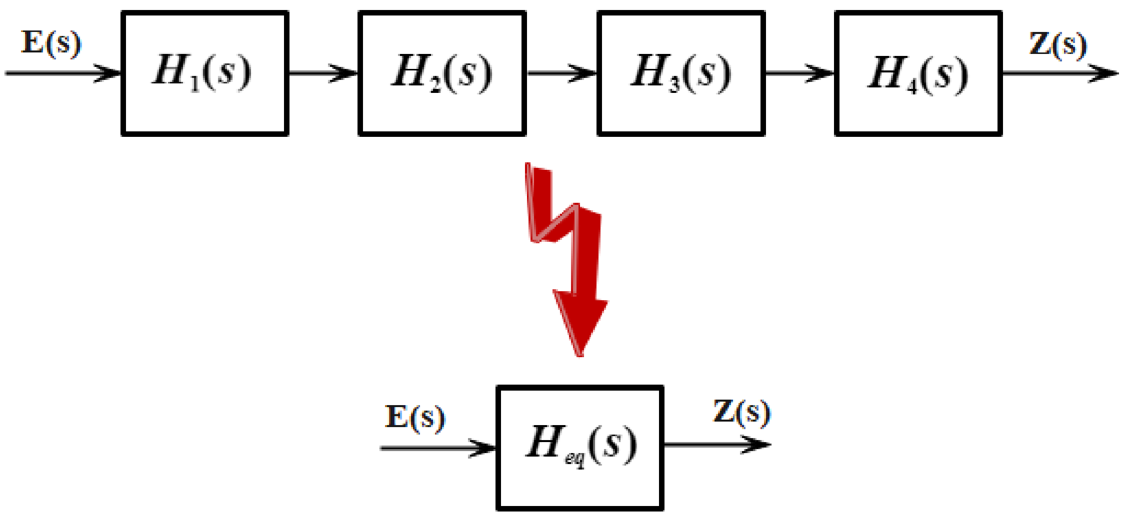


Figure III.3. Blocks connected in parallel series

The equivalent transfer function $H_{eq}(s)$ is

$$H_{eq}(s)=\frac{Z(s)}{E(s)}=H_1(s)\times H_2(s)\times H_3(s)\times H_4(s)$$

III.5.1.b. Parallel Transfer Function

Parallel connection of blocks means connecting the inputs of several blocks to the same point and their outputs to a common output point, as shown in Figure III.4.

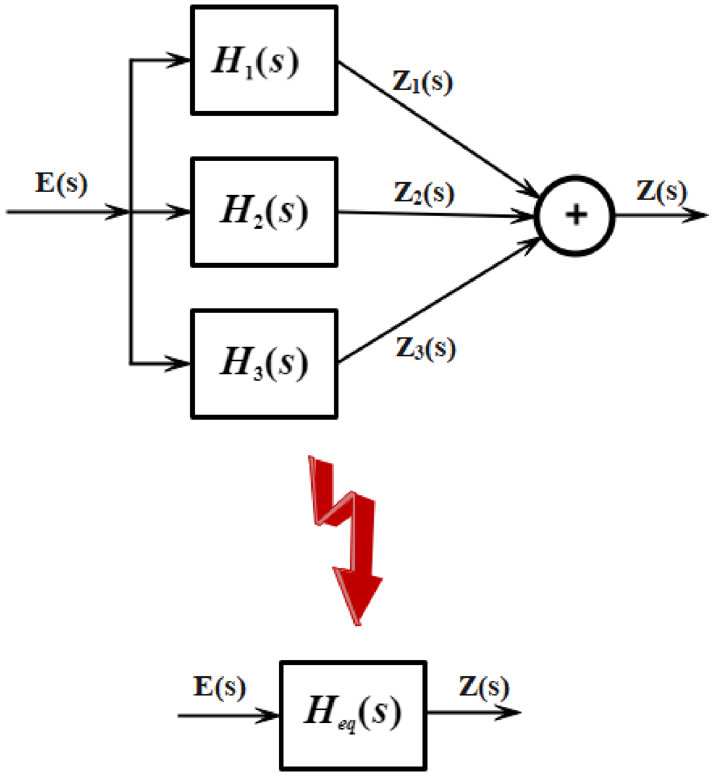


Figure III.4. Blocks connected in parallel

The equivalent function of blocks connected in parallel is determined as follows:

$$H_{eq}(s)=\frac{Z(s)}{E(s)}=H_1(s)+H_2(s)+H_3(s)$$

III.5.1.c. Addition/Subtraction

Addition and subtraction operations are performed using the block shown in Figure III.5.

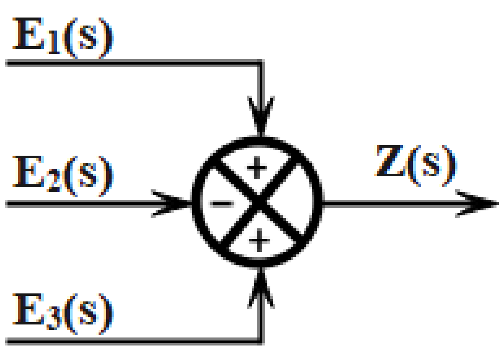


Figure III.5. Addition and subtraction block

The output of the Addition and subtraction is calculated as follows:

$$Z(s) = E_1(s) + E_3(s) - E_2(s)$$

III.5.2. Open-Loop Transfer Function (OLTF)

An open-loop system is a system with no feedback loop. Figure III.6 shows an open-loop system:

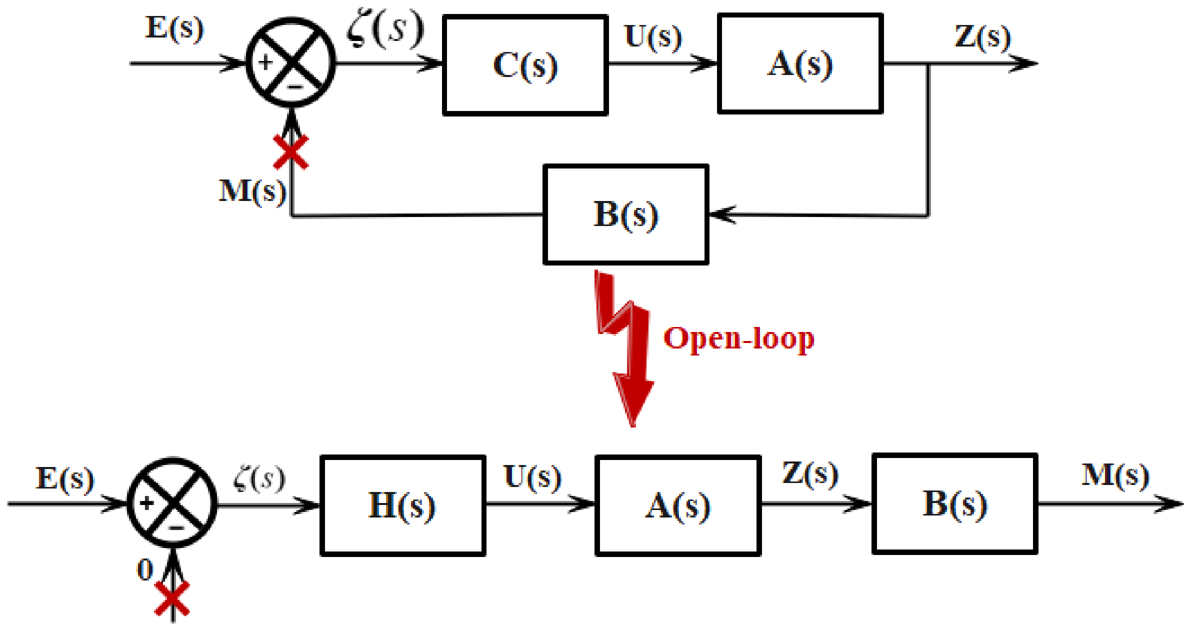


Figure III.6. Open-loop system block diagram

The equivalent function of the open-loop system is calculated as follows:

$$H_{BO}(s) = \frac{M(s)}{E(s)}$$

$$M(s) = B(s) \cdot Z(s)$$

$$M(s) = B(s) \cdot U(s) \cdot A(s) \qquad With \;\; Z(s) = U(s) \cdot A(s)$$

$$M(s) = B(s) \cdot \zeta(s) \cdot C(s) \cdot A(s) \qquad With \;\; U(s) = \zeta(s) \cdot C(s)$$

$$M(s) = B(s) \cdot E(s) \cdot C(s) \cdot A(s) \qquad With \;\; \zeta(s) = E(s) - 0 = E(s)$$

$$H_{BO}(s) = \frac{M(s)}{E(s)} = B(s) \cdot C(s) \cdot A(s)$$

III.5.3. Closed-Loop Transfer Function (CLTF)

A closed-loop system contains a feedback loop (a comparison between input and output). A distinction is made between two cases:

III.5.3.1. In negative counterreaction

Negative feedback is when the measurement is compared with the $E(s)$ input, as shown in Figure III.7.

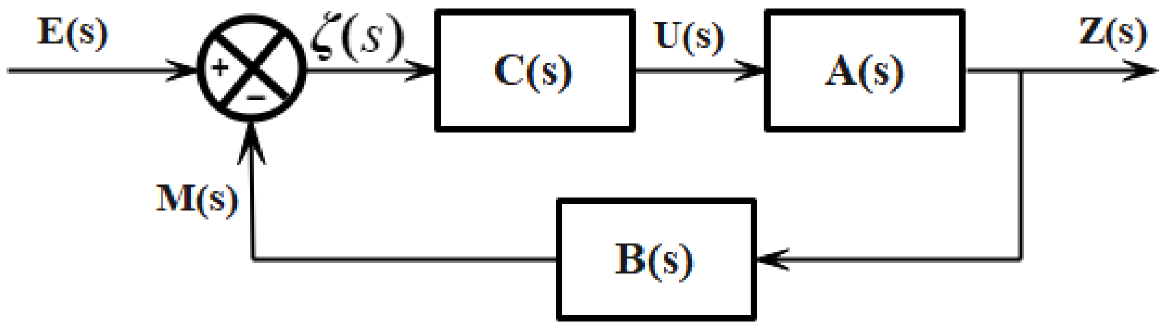


Figure III.7. Closed-loop negative feedback system

The transfer function $H(s)$, in this case, is determined as follows:

$$H_{BF}(s)=\frac{Z(s)}{E(s)}$$

$$Z(s)=A(s)\cdot U(s)$$

$$Z(s)=A(s)\cdot C(s)\cdot\zeta(s) \qquad \textit{With} \quad U(s)=C(s)\cdot\zeta(s)$$

$$Z(s)=A(s)\cdot C(s)\cdot\left[E(s)-M(s)\right] \qquad \textit{With} \quad \zeta(s)=E(s)-M(s)$$

$$Z(s)=A(s)\cdot C(s)\cdot\left[E(s)-Z(s)\cdot B(s)\right] \qquad \textit{With} \quad M(s)=Z(s)\cdot B(s)$$

$$Z(s)=A(s)\cdot C(s)\cdot E(s)-A(s)\cdot C(s)\cdot Z(s)\cdot B(s)$$

$$Z(s)+A(s)\cdot C(s)\cdot Z(s)\cdot B(s)=A(s)\cdot C(s)\cdot E(s)$$

$$Z(s)\cdot\left[1+A(s)\cdot C(s)\cdot B(s)\right]=A(s)\cdot C(s)\cdot E(s)$$

$$H_{BF}(s)=\frac{Z(s)}{E(s)}=\frac{A(s)\cdot C(s)}{1+A(s)\cdot C(s)\cdot B(s)}$$

$$H_{BF}(s)=\frac{A(s)\cdot C(s)}{1+H_{BO}(s)}$$

III.5.3.2. In positive counterreaction

Positive feedback is when the $E(s)$ input is summed with the measurement, as shown in Figure III.7.

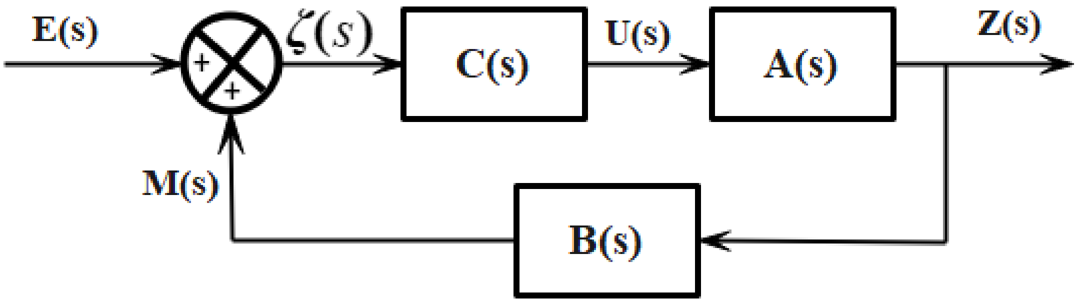


Figure III.8. Block diagram with negative feedback

The transfer function $H(s)$ is determined as follows:

$$H_{BF}(s) = \frac{Z(s)}{E(s)}$$

$$Z(s) = A(s) \cdot U(s)$$

$$Z(s) = A(s) \cdot C(s) \cdot \zeta(s) \qquad \textit{With} \quad U(s) = C(s) \cdot \zeta(s)$$

$$Z(s) = A(s) \cdot C(s) \cdot \left[E(s) + M(s)\right] \qquad \textit{With} \quad \zeta(s) = E(s) + M(s)$$

$$Z(s) = A(s) \cdot C(s) \cdot \left[E(s) + Z(s) \cdot B(s)\right] \qquad \textit{With} \quad M(s) = Z(s) \cdot B(s)$$

$$Z(s) = A(s) \cdot C(s) \cdot E(s) + A(s) \cdot C(s) \cdot Z(s) \cdot B(s)$$

$$Z(s) - A(s) \cdot C(s) \cdot Z(s) \cdot B(s) = A(s) \cdot C(s) \cdot E(s)$$

$$Z(s) \cdot \left[1 - A(s) \cdot C(s) \cdot B(s)\right] = A(s) \cdot C(s) \cdot E(s)$$

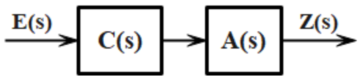
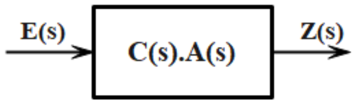
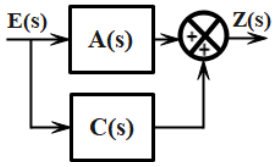
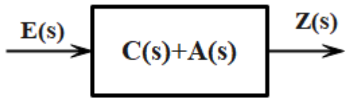
$$H_{BF}(s) = \frac{Z(s)}{E(s)} = \frac{A(s) \cdot C(s)}{1 - A(s) \cdot C(s) \cdot B(s)}$$

$$H_{BF}(s) = \frac{A(s) \cdot C(s)}{1 - H_{BO}(s)}$$

III.5.3.3. simplification of complex block diagrams

To be able to work with complex block diagrams, a simplification is necessary to obtain an equivalent transfer function. These manipulations are summarized in the following table:

Tableau III.1. Simplification of complex block diagrams:

<table>
<tr><th>Type of Block Connections</th><th>Initial diagram</th><th>Equivalent diagram</th></tr>
<tr><td>Cascade connection</td><td>
</td><td>
</td></tr>
<tr><td>Parallel connection</td><td>
</td><td>
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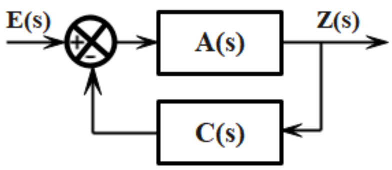
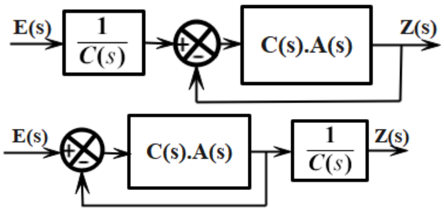
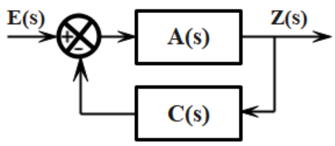
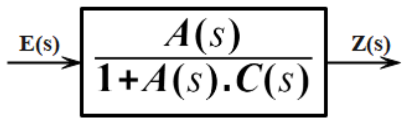
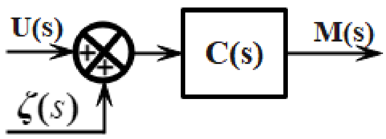
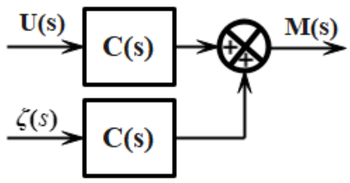
<tr><td>Unit feedback</td><td>
</td><td>
</td></tr>
<tr><td>Elimination of a feedback loop</td><td>
</td><td>
</td></tr>
<tr><td>Forward movement of a summation block or comparator</td><td>
</td><td>
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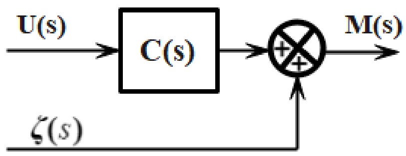
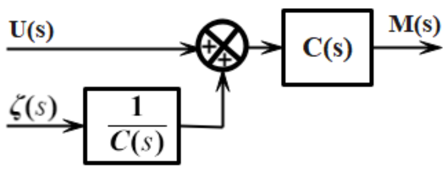
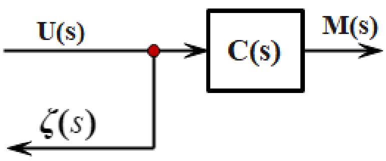
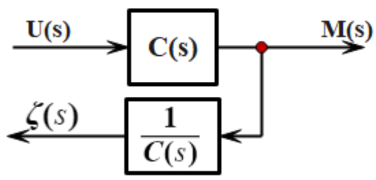
<tr><td>Backward movement of a summation block or comparator</td><td>
</td><td>
</td></tr>
<tr><td>Forward movement of a branching point</td><td>
</td><td>
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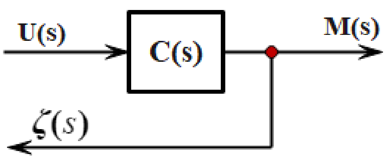
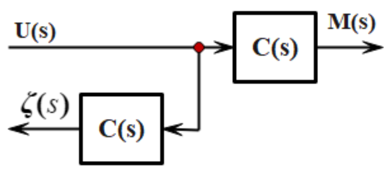
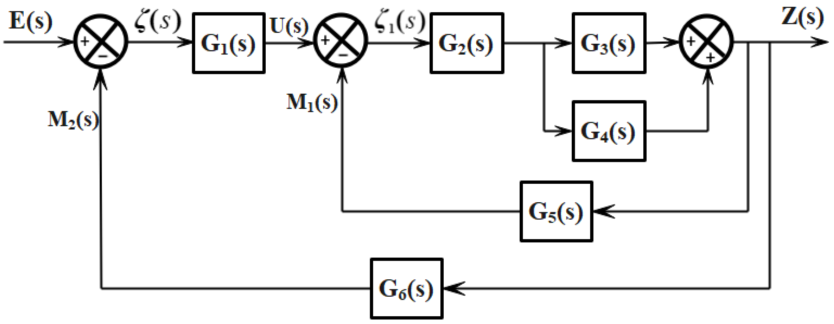
<tr><td>Forward movement of a branching point</td><td>
</td><td>
</td></tr>
</table>

III.6. Learning activity

Exercise N°1:

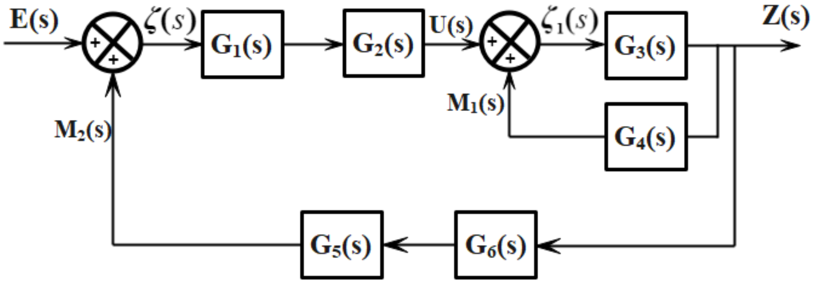
Find the equivalent transfer function for the following control system:



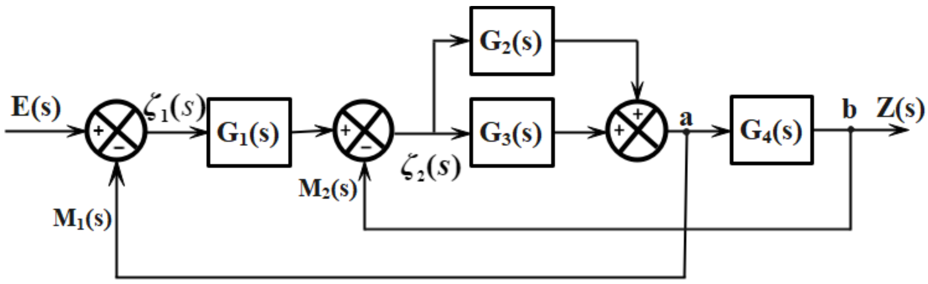
Exercise N°2:

Find the equivalent transfer function for the following control systems:

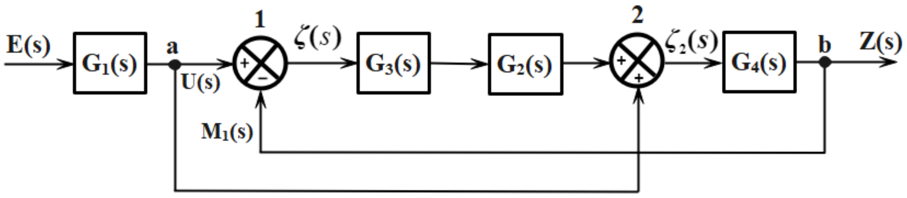
1)



2)



3)



CHAPTER IV: STUDY OF A FIRST-ORDER CONTROL SYSTEM

IV.1. Introduction

First-order systems are widespread in industrial fields, mainly mechanical engineering. They are found, for example, in vehicle suspension systems.

This chapter aims to study the temporal response of this type of system for three kinds of input: a step, an impulse, and a ramp. The study will focus on their behavior in transient and steady conditions.

IV.2. First-Order System

A first-Order System is a system governed by a differential equation of the first order of the form:

$$\tau \frac{dZ(t)}{dt} + Z(t) = K \cdot E(t)$$

$$\tau \cdot s \cdot Z(s) + Z(s) = K \cdot E(s)$$

$$Z(s)(\tau \cdot s + 1) = K \cdot E(s)$$

$$H(s) = \frac{Z(s)}{E(s)} = \frac{K}{\tau \cdot s + 1}$$

Where:

- K : Static Gain $K = H(0)$
- Time Constant $t \succ 0$

The block diagram of a first-order system is represented as follows:

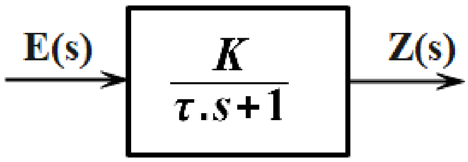


Figure IV.1: Block diagram of a first-order system

Example: The Transfer Function of an RC Circuit

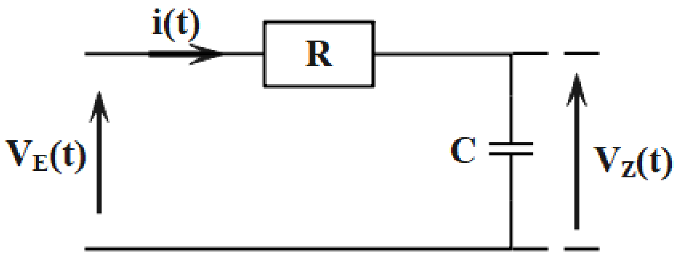


Figure IV.2. Diagram of an RC circuit

The transfer function obtained by the modeling of the RC circuit is expressed as follows:

$$H(s) = \frac{V_z(s)}{V_E(s)} = \frac{K}{1 + \tau \cdot s}$$

$$K = 1$$

$$\tau = RC$$

IV.3. Time response of a First-Order System

Two regimes govern the temporal response of a system:

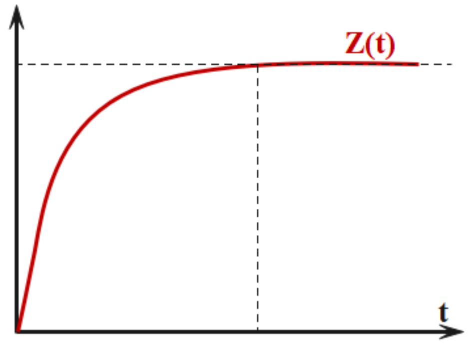


Figure IV.3. Time response of a First-Order System

The response consists of two regimes:

- **Transient**: During this stage, the system evolves from its initial to its final state, thus approaching the target value.
- **Steady-state response**: This is the final state of the response that corresponds to $t \to +\infty$.

For this course, we have studied three temporal responses corresponding to three time responses.

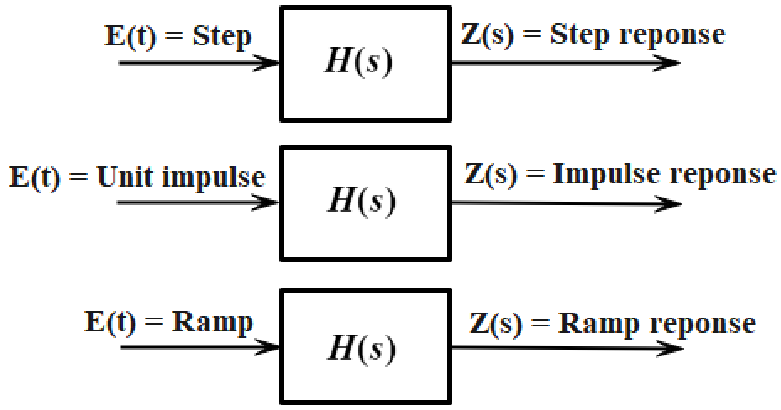


Figure IV.4. The type of response based on the input

IV.3.1. Step response

It is the temporal response of a system when the entry is a step. The study to be carried out is the same, whether for a unit step or an amplitude E_c step. The only difference lies in the constant E_c.

Table IV.1 compares the response at a unit level and an amplitude E_c step.

Table IV.1: Comparison between the response at a unit step and a step input of amplitude E_c.

Unit Step	**Echelon d'amplitude** E_c
$u(t)=0 \rightarrow t \prec 0$ $u(t)=1 \rightarrow t \geq 0$	$u(t)=0 \rightarrow t \prec 0$ $u(t)=E_c \rightarrow t \geq 0$
$E(t)=u(t)$	$E(t)=E_c \cdot u(t)$
$E(s)=L\left[e(t)\right]=L\left[u(t)\right]$	$E(s)=L\left[E(t)\right]=L\left[E_c \cdot u(t)\right]$
$E(s)=\frac{1}{s}$	$E(s)=\frac{E_c}{s}$
$H(s)=\frac{Z(s)}{E(s)}$	$H(s)=\frac{Z(s)}{E(s)}$
$Z(s)=H(s)\cdot E(s)$	$Z(s)=H(s)\cdot E(s)$
$Z(s)=\frac{K}{1+\tau \cdot s}\cdot\frac{1}{s}$	$Z(s)=\frac{K}{1+\tau \cdot s}\cdot\frac{E_c}{s}$

For this study, we will assume that the input is of amplitude E_c

So, the expression of the output (the system's response) in the case is:

$$H(s)=\frac{Z(s)}{E(s)} \rightarrow Z(s)=E(s)\cdot H(s)$$

We know that: $H(s)=\frac{K}{1+\tau \cdot s}$

$$E(t)=E_c \cdot u(t) \xrightarrow{L\left[E(t)\right]} E(s)=\frac{E_c}{s}$$

Therefore: $Z(s)=\frac{K}{1+\tau \cdot s}\cdot\frac{E_c}{s}$

To study the $Z(s)$ response, it must be decomposed into simple elements as follows:

$$Z(s)=\frac{K}{1+\tau \cdot s}\cdot\frac{E_c}{s}$$

$$Z(s)=\frac{\alpha}{s}+\frac{\beta}{1+\tau \cdot s}$$

Such as:

The calculation of the constant α is as follows:

$$\alpha = \left[s \cdot Z(s) \right]_{s=0}$$

$$\alpha = \left[s \cdot \frac{K}{1+\tau \cdot s} \cdot \frac{E_c}{s} \right]_{s=0}$$

$$\alpha = \left[\frac{K}{1+\tau \cdot s} \cdot \frac{E_c}{1} \right]_{s=0}$$

$$\alpha = \frac{K \cdot E_c}{1} \Leftrightarrow \alpha = K \cdot E_c$$

The calculation of the constant β is as follows:

$$\beta = \left[\tau \left(s + \frac{1}{\tau} \right) \cdot Z(s) \right]_{s=-\frac{1}{\tau}}$$

$$\beta = \left[\tau \left(s + \frac{1}{\tau} \right) \cdot \frac{K \cdot E_c}{\tau \cdot s \left(\frac{1}{\tau} + s \right)} \right]_{s=-\frac{1}{\tau}}$$

$$\beta = \left[\frac{\tau \left(s + \frac{1}{\tau} \right) \cdot K \cdot E_c}{\tau \cdot s \left(\frac{1}{\tau} + s \right)} \right]_{s=-\frac{1}{\tau}}$$

$$\beta = \left[\frac{K \cdot E_c}{s} \right]_{s=-\frac{1}{\tau}}$$

$$\beta = \left[\frac{K \cdot E_c}{-\frac{1}{\tau}} \right]_{s=-\frac{1}{\tau}}$$

$$\beta = -\tau \cdot K \cdot E_c$$

Replacing the constants α and β in the expression of $Z(s)$, decomposed into simple elements:

$$Z(s) = \frac{K \cdot E_c}{s} + \frac{-K \cdot E_c \cdot \tau}{1+\tau \cdot s}$$

$$Z(s) = \frac{K \cdot E_c}{s} - \frac{K \cdot E_c \cdot \tau}{1+\tau \cdot s}$$

The expression $Z(t)$ is obtained by applying the inverse Laplace transform to $Z(s)$.

$$Z(t) = L^{-1}\left[Z(s)\right] = L^{-1}\left[\frac{K \cdot E_c}{s} - \frac{K \cdot E_c \cdot \tau}{1+\tau \cdot s}\right]$$

$$Z(t) = L^{-1}\left[Z(s)\right] = L^{-1}\left[\frac{K \cdot E_c}{s}\right] - L^{-1}\left[\frac{K \cdot E_c \cdot \tau}{1+\tau \cdot s}\right]$$

$$Z(t) = K \cdot E_c L^{-1}\left[\frac{1}{s}\right] - K \cdot E_c L^{-1}\left[\frac{\tau}{1+\tau \cdot s}\right]$$

$$Z(t) = K \cdot E_c \cdot 1 - K \cdot E_c \cdot e^{-\frac{t}{\tau}}$$

$$Z(t) = K \cdot E\left(1 - e^{-\frac{t}{\tau}}\right) \cdot u(t)$$

The causality of the signal is valid only for $t \succ 0$.

IV.3.1.1. Step Response Analysis

The study of the $Z(t)$ response is carried out by applying the following steps:

IV.3.1.1.a Application of the Initial and Final Value Theorems

- **The initial value theorem:**

$$Z(0) = \lim_{t \to 0}\left(Z(t)\right) = \lim_{s \to \infty}\left(s \cdot Z(s)\right)$$

$$Z(0) = \lim_{s \to \infty}\left(s \cdot \frac{E_c}{s} \cdot \frac{K}{1+\tau \cdot s}\right)$$

$$Z(0) = \lim_{s \to \infty}\left(\frac{E_c}{1} \cdot \frac{K}{1+\tau \cdot s}\right)$$

$$Z(0) = 0 \rightarrow Zero\ initial\ value$$

- **The final value theorem :**

$$Z(\infty) = \lim_{t \to \infty}\left(Z(t)\right) = \lim_{s \to 0}\left(s \cdot Z(s)\right)$$

$$Z(\infty) = \lim_{s \to 0}\left(s \cdot \frac{E_c}{s} \cdot \frac{K}{1+\tau \cdot s}\right)$$

$$Z(\infty) = E_c \cdot K \rightarrow Final\ value\ E_c \cdot K$$

IV.3.1.1.b. Derivative of the Response and Its limits in Zero and Infinity

We calculate the derivative to find the slope a $t \rightarrow 0$ and a $t \rightarrow \infty$

$$Z^{'}(s) = s \cdot Z(s)$$

$$Z^{'}(s) = s \cdot \frac{E_c \cdot K}{s(1+\tau \cdot s)}$$

$$Z'(0)=\lim_{t\to 0} Z'(s)=\lim_{s\to\infty} s\cdot Z'(s)$$

$$Z'(0)=\lim_{s\to\infty} s\cdot\frac{E_c\cdot K}{1+\tau\cdot s}$$

$$Z'(0)=\lim_{s\to\infty}\frac{s\cdot E_c\cdot K}{\tau\cdot s}$$

$$Z'(0)=\frac{E_c\cdot K}{\tau}\to A\ slope\ diferent\ \ from\ zero$$

$$Z'(\infty)=\lim_{t\to\infty} Z'(t)=\lim_{s\to 0} s\cdot Z'(s)$$

$$Z'(\infty)=\lim_{s\to 0} s\cdot\frac{E_c\cdot K}{1+\tau\cdot s}$$

$$Z'(\infty)=\frac{0}{1}$$

$$Z'(\infty)=0\to The\ slope\ is\ zero$$

<u>Note:</u> The calculation can be performed directly, either using $Z'(t)$ or $Z(t)$ instead of its derivative ($Z'(t)$).

$$Z'(0)=\lim_{t\to 0}\frac{dZ(t)}{dt}=\lim_{s\to\infty} s\cdot\frac{dZ(s)}{dt}=\lim_{s\to\infty} s\cdot s\cdot Z(s)=\lim_{s\to\infty} s^2\cdot Z(s)$$

This study allows us to plot the step response of a first-order system for $K=1$, as shown in Figure IV.5.

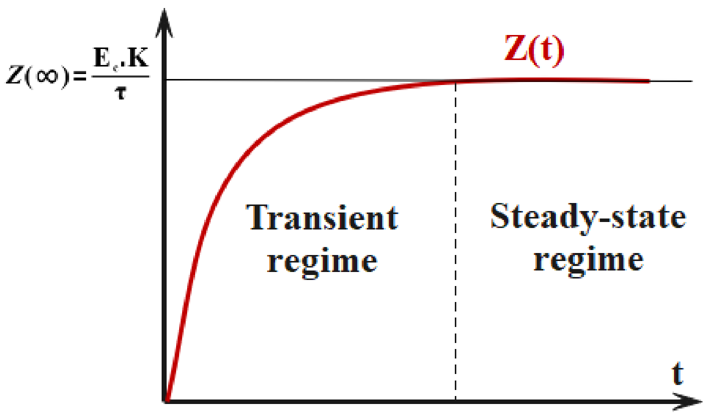


Figure IV.5: Step response of a First-Order System

In this response, we distinguish the two regimes, the transient regime and the steady-state regime, which are expected after a time $t=5\cdot\tau$.

IV.3.1.2. Transient Response Characteristics of $Z(t)$

Several characteristics define the transient regime, the most important being the time constant τ, the settling time at $\pm 5\%$ **(** t_r or $t_{\pm 5\%}$ **)**, and the rise time (t_m), which we will detail below:

IV.3.1.2.a. Time Constant τ in a First-Order System

There are two methods to determine it:

- **Graphical method:**

Draw the tangent at the origin of the response curve. τ represents the instant when this tangent intersects the final asymptote.

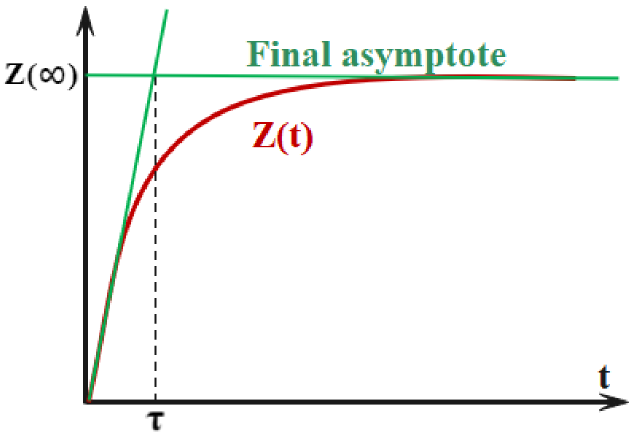


Figure IV.6. Representation of the constant τof the transient regime in the index response of a first-order system

- **Analytical method:**

We calculate $Z(\tau)$:

$$Z(t) = E_c \cdot K\left(1 - e^{-t/\tau}\right)$$

$$Z(\tau) = E_c \cdot K\left(1 - e^{-\tau/\tau}\right)$$

$$Z(\tau) = E_c \cdot K\left(1 - e^{-1}\right)$$

$$Z(\tau) = E_c \cdot K(1 - 0.37)$$

$$Z(\tau) = 0,63 \cdot E_c \cdot K$$

$$Z(\tau) = 0,63 \cdot Z(\infty)$$

The time constant τ characterizes the speed at which the system reaches its steady state. A lower time constant allows the system to respond faster.

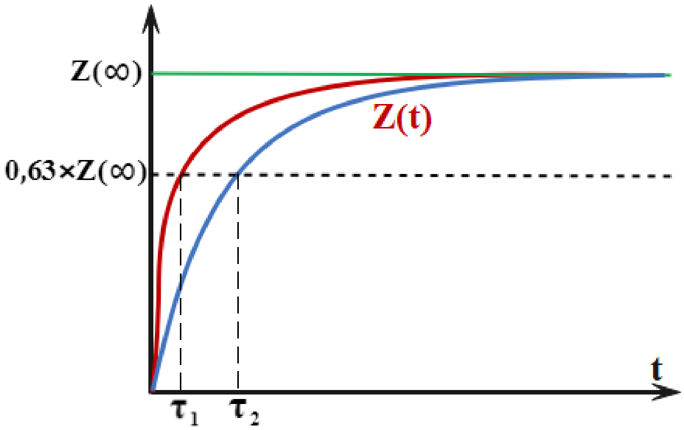


Figure IV.7: Step response speed of a first-order system under time constant variation

IV.3.1.2.b. Settling time at ±5% (t_r *or* $t_{\pm5\%}$)

The 5% settling time is the time after which the output reaches and stays within 5%. Two methods could be used to obtain it:

- **The analytical method**: It is equivalent to determining the value of the output $Z(t)$ when $t = t_r$:

$$Z(t_r) = Z(\infty) - 5\% \cdot Z(\infty)$$

$$Z(t_r) = Z(\infty) - 0.05 \cdot Z(\infty)$$

$$Z(t_r) = Z(\infty)(1 - 0.05)$$

$$Z(t_r) = Z(\infty) \cdot 0.95$$

$$E_c \cdot K\left(1 - e^{-t_r/\tau}\right) = 0.95 \cdot E_c \cdot K$$

$$1 - e^{-t_r/\tau} = 0.95$$

$$-e^{-t_r/\tau} = 0.95 - 1$$

$$-e^{-t_r/\tau} = -0.05$$

$$\ln e^{-t_r/\tau} = \ln 0.05$$

$$-\frac{t_r}{\tau} = \ln 0.05$$

$$t_r = \tau \cdot \ln\left(\frac{1}{0.05}\right)$$

$$t_r = 3 \cdot \tau$$

- **<u>The graphical method</u>:** We obtained this very graphically by drawing line a $0.95\times Z(\infty)$. The projection on the time axis of the point of intersection with the graph of $Z(t)$ gives us t_r, as shown in the following figure:

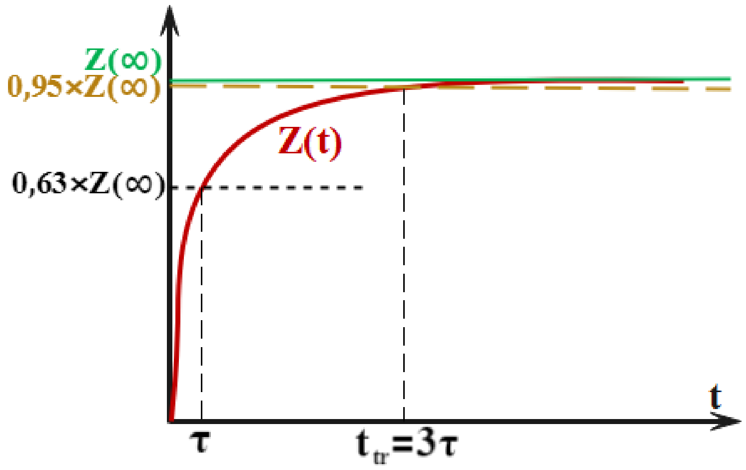


Figure IV.8. Rise Time t_r Representation in the Step Response of a First-Order System

IV.3.1.3. Rise time

The rise time is the time required for the response to rise from 10% to 90% of its final change. We denote t_1 and t_2 as the moments when the responses reach 10% and 90% of its final value, respectively, such that $t_m = t_2 - t_1$

✓ <u>Determination of t_1:</u>

$$Z(t_1) = E_c K\left(1-e^{-t_1/\tau}\right) = 0.1\cdot Z(\infty)$$

$$E_c K\left(1-e^{-t_1/\tau}\right) = 0.1\cdot E_c K$$

$$1-e^{-t_1/\tau} = 0.1$$

$$-e^{-t_1/\tau} = 0.1-1$$

$$e^{-t_1/\tau} = 0.9$$

$$\ln e^{-t_1/\tau} = \ln 0.9$$

$$-\frac{t_1}{\tau} = \ln 0.9$$

$$t_1 = \tau \ln\left(\frac{1}{0.9}\right)$$

- ✓ Determination of t_2 :

$$Z(t_2) = E_c K\left(1 - e^{-t_2/\tau}\right) = 0.9 \cdot Z(\infty)$$

$$E_c K\left(1 - e^{-t_2/\tau}\right) = 0.9 \cdot E_c K$$

$$1 - e^{-t_2/\tau} = 0.9$$

$$-e^{-t_2/\tau} = 0.9 - 1$$

$$e^{-t_2/\tau} = 0.1$$

$$t_2 = \tau \ln\left(\frac{1}{0.1}\right)$$

Where:

$$t_m = t_2 - t_1$$

$$t_m = \tau \ln\left(\frac{1}{0.1}\right) - \tau \ln\left(\frac{1}{0.9}\right)$$

$$t_m = \tau\left(\ln\left(\frac{1}{0.1}\right) - \ln\left(\frac{1}{0.9}\right)\right)$$

$$t_m = \tau\left(\ln\left(\frac{1/0.1}{1/0.9}\right)\right)$$

$$t_m = \tau\left(\ln\left(\frac{0.9}{0.1}\right)\right)$$

$$t_m = \tau(\ln 9)$$

$$t_m = 2.2 \cdot \tau$$

IV.3.2. Impulse response

It is the system's response to a Dirac impulse. The calculations are made as follows:

$$E(t) = \delta(t)$$

$$E(t) = \delta(t) \xrightarrow{Laplace\ \ transform} E(s) = L\left[E(t)\right]$$

$$E(s) = L\left[\delta(t)\right]$$

$$E(s) = 1$$

$$H(s) = \frac{Z(s)}{E(s)}$$

$$Z(s) = H(s) \cdot E(s)$$

$$Z(s)=\frac{K}{1+\tau\cdot s}\times 1$$

$$Z(t)=L^{-1}\left[\frac{K}{1+\tau\cdot s}\right]$$

$$E(t)=K\cdot L^{-1}\left[\frac{1}{1+\tau\cdot s}\right]$$

$$Z(t)=\frac{K}{\tau}e^{-t/\tau}$$

IV.3.2.a. Application of the Initial and Final Value Theorems

- **Initial Value :**

$$Z(0)=\lim_{t\to 0}Z(t)=\lim_{s\to\infty}s\cdot Z(s)$$

$$Z(0)=\lim_{s\to\infty}s\cdot\frac{K}{1+\tau\cdot s}$$

$$Z(0)=\lim_{s\to\infty}\frac{K\cdot s}{\tau\cdot s}$$

$$Z(0)=\frac{K}{\tau}$$

- **Final Value :**

$$Z(\infty)=\lim_{t\to\infty}Z(t)=\lim_{s\to 0}s\cdot Z(s)$$

$$Z(\infty)=\lim_{s\to\infty}s\cdot\frac{K}{1+\tau\cdot s}$$

$$Z(\infty)=\frac{0}{1+0}\Rightarrow Z(\infty)=0$$

IV.3.2.b. The tangent at the origin

The tangent at the origin intersects the axis of time in $t=\tau$

$$Z(\tau)=\frac{K}{\tau}e^{-\tau/\tau}$$

$$Z(\tau)=\frac{K}{\tau}e^{-1}$$

$$Z(\tau)=0.37\cdot\frac{K}{\tau}$$

$$Z(\tau)=0.37\cdot Z(0)$$

IV.3.2.c. Response time to $\pm 5\%$ $(\mathbf{t_r})$

Response time to (t_r) is determined for $\pm 5\%$ the initial value $Z(t)$, and the calculation is done as follows:

$$Z(t_r) = 5\% \cdot Z(0)$$

$$Z(t_r) = 0.05 \cdot Z(0)$$

$$\frac{K}{\tau} e^{-t_r/\tau} = 0.05 \cdot \frac{K}{\tau}$$

$$e^{-t_r/\tau} = 0.05$$

$$t_r = -\tau \ln(0.05)$$

$$t_r = 3\tau$$

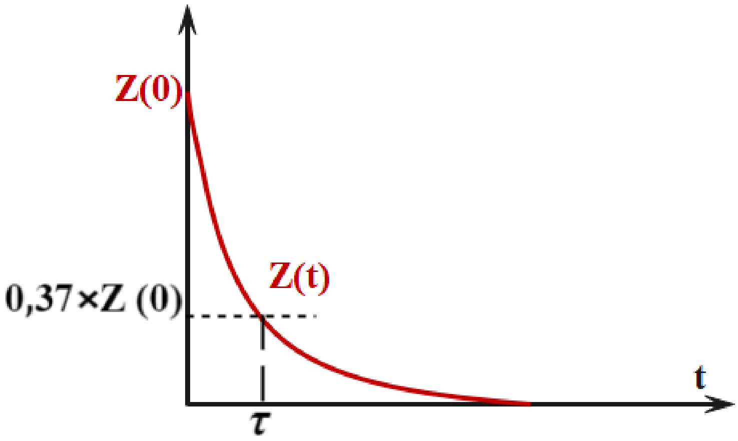


Figure IV.9. First-Order System Impulse Response

IV.3.3. Ramp Response

This is the answer to a first-order system for a unit ramp input. The response curve is as follows:

$$E(t) = t \cdot u(t)$$

$$E(s) = L\left[E(t)\right] = L\left[t \cdot u(t)\right]$$

$$E(s) = \frac{1}{s^2}$$

$$Z(s) = \frac{1}{s^2} \cdot \frac{K}{1+\tau \cdot s} \Leftrightarrow Z(s) = \frac{K}{s^2(1+\tau \cdot s)}$$

The decomposition into a simple element:

$$Z(s) = \frac{K}{s^2(1+\tau \cdot s)}$$

$$Z(s)=\frac{\alpha}{s^2}+\frac{\beta}{(1+\tau\cdot s)}+\frac{\gamma}{s}$$

The determination α is made according to the following method:

$$\alpha=\left[s^2\cdot Z(s)\right]_{s=0}=\left[s^2\cdot\frac{K}{s^2(1+\tau\cdot s)}\right]_{s=0}$$

$$\alpha=\left[\frac{K}{(1+\tau\cdot s)}\right]_{s=0}$$

$$\alpha=\frac{K}{(1+\tau\cdot 0)}=\frac{K}{1}$$

$$\alpha=K$$

β it is determined by:

$$\beta=\left[\tau\left(\frac{1}{\tau}+s\right)\cdot Z(s)\right]_{s=-\frac{1}{\tau}}$$

$$\beta=\left[\tau\left(\frac{1}{\tau}+s\right)\cdot\frac{K}{s^2(1+\tau\cdot s)}\right]_{s=-\frac{1}{\tau}}$$

$$\beta=\left[\tau\left(\frac{1}{\tau}+s\right)\cdot\frac{K}{s^2\cdot\tau\left(\frac{1}{\tau}+s\right)}\right]_{s=-\frac{1}{\tau}}$$

$$\beta=\left[\frac{K}{s^2}\right]_{s=-\frac{1}{\tau}}$$

$$\beta=\frac{K}{\left(-\frac{1}{\tau}\right)^2}$$

$$\beta=K\cdot\tau^2$$

The determination γ differs from those of the other two. It is determined by calculating the limit of $Z(s)$ at infinity, following these steps:

$$\lim_{s\to\infty}s\cdot Z(s)=\lim_{s\to\infty}\frac{s\cdot K}{s^2}$$

$$\lim_{s\to\infty}s\cdot Z(s)=0$$

$$\lim_{s\to\infty} s\cdot Z(s)=0 \Leftrightarrow \lim_{s\to\infty}\left(s\left[\frac{K}{s^2}+\frac{K\cdot\tau^2}{(1+\tau P)}+\frac{\gamma}{P}\right]\right)=0$$

$$\Leftrightarrow \lim_{s\to\infty}\left(\frac{s\cdot K}{s^2}+\frac{s\cdot K\cdot\tau^2}{(1+\tau\cdot s)}+\frac{s\cdot\gamma}{s}\right)=0$$

$$\Leftrightarrow \lim_{s\to\infty}\left(\frac{K\cdot\tau^2}{(\tau)}+\frac{\gamma}{1}\right)=0$$

$$\Leftrightarrow \left(K\cdot\tau+\frac{\gamma}{1}\right)=0$$

Where:

$$\gamma=-K\cdot\tau$$

Substituting the values of α, β and γ in the simple element decomposition of the output $Z(s)$, we obtain:

$$Z(s)=\frac{K}{s^2}+\frac{-K\cdot\tau}{s}+\frac{K\cdot\tau}{1+\tau\cdot s}$$

$Z(t)$ is obtained from the inverse Laplace transform, and its expression is:

$$Z(t)=L^{-1}\left[\frac{K}{s^2}+\frac{-K\cdot\tau}{s}+\frac{K\cdot\tau}{1+\tau\cdot s}\right]$$

$$Z(t)=K\cdot L^{-1}\left[\frac{1}{s^2}+\frac{-\tau}{s}+\frac{\tau}{1+\tau\cdot s}\right]$$

$$Z(t)=K\cdot L^{-1}\left[\frac{1}{s^2}\right]+K\cdot\tau\cdot L^{-1}\left[\frac{-1}{s}\right]+K\cdot\tau\cdot L^{-1}\left[\frac{1}{1+\tau\cdot s}\right]$$

Knowing that:

$$L^{-1}\left[\frac{1}{1+\tau\cdot s}\right]=\frac{e^{-t/\tau}}{\tau}$$

$$L^{-1}\left[\frac{1}{s}\right]=1$$

$$L^{-1}\left[\frac{1}{s^2}\right]=t$$

$$Z(t)=K\cdot t-K\cdot\tau+K\cdot\tau\frac{e^{-t/\tau}}{\tau}$$

$$Z(t) = K \cdot t - K \cdot \tau + K \cdot \tau e^{-t/\tau}$$

$$Z(t) = K\left(t - \tau + \tau e^{-t/\tau}\right)$$

IV.3.3.a. Initial and final value

$$Z(0) = \lim_{s \to +\infty} s \cdot Z(s)$$

$$Z(0) = \lim_{s \to +\infty} s \cdot \frac{K}{s^2(1 + \tau \cdot s)}$$

$$Z(0) = \lim_{s \to +\infty} \frac{K}{s \cdot (1 + \tau \cdot s)}$$

$$Z(0) = 0$$

$$Z(\infty) = \lim_{s \to 0} s \cdot Z(s)$$

$$Z(\infty) = \lim_{s \to 0} s \cdot \frac{K}{s^2(1 + \tau \cdot s)}$$

$$Z(\infty) = \lim_{s \to 0} \frac{K}{s \cdot (1 + \tau \cdot s)}$$

$$Z(\infty) = \infty \rightarrow \text{An infinite green direction}$$

IV.3.3.b. Asymptotic Tangent

The determination of the tangent to infinity is obtained by calculating the value of $Z(t)$ when $t \to \infty$ **:**

$$Z(\infty) = K\left(t - \tau + \tau e^{-\infty/\tau}\right)$$

$$Z(\infty) = K(t - \tau)$$

The tangent to a ramp having a delay $\boldsymbol{\tau}$ *for* $K = 1$.

The plot of the response to a ramp of a first-order system for case $K = 1$ is shown in Figure IV.10.

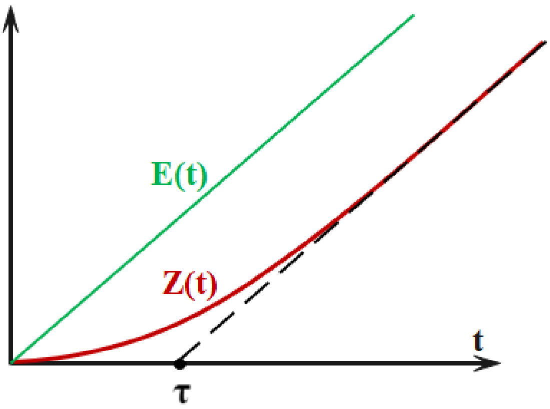


Figure IV.10. Response of a First-Order System to a Ramp Input

A first-order system's response to a ramp input is characterized by:

1. The response is the sum of two terms: a decreasing exponential function and a delayed ramp, with a delay τ.
2. The slope at the origin is zero.
3. With a delay τ, the output tends asymptotically towards the ramp $E(t)$.
4. In steady state, when t→+∞, then $Z(t) = K \cdot (t - \tau)$.
   - If $k = 1$, the delay between the output s(t) and the input is constant $\xi = \tau$. In this case, we obtain a tracking error corresponding to the difference between the input and the output.
   - If $k \neq 1$, in this case, $Z(t)$ and $E(t)$ do not have the same slope, the gap between the output and the input increases and can become infinite. The tracking error is $\xi = K\tau$

IV.4. Learning activity

Exercise N°1:

Plot the impulse response ($E(t) = \delta(t)$) of the following first-order system $H(s) = \dfrac{0,5}{1 + 0,03 \cdot s}$. Indicate the characteristic points (τ, t_r) and the tangent at the origin, and detail all the steps.

Data:

- The Laplace transforms of a unit impulse: $L[\delta(t)] = 1$
- The Laplace inverse transform : $L^{-1}\left[\dfrac{1}{1 + \tau \cdot s}\right] = \dfrac{e^{-t/\tau}}{\tau} u(t)$

Exrcice N°2:

Plot the step response by showing the characteristic points (τ, t_r, t_m) as well as the tangent while detailing all the steps for the following first-order system:

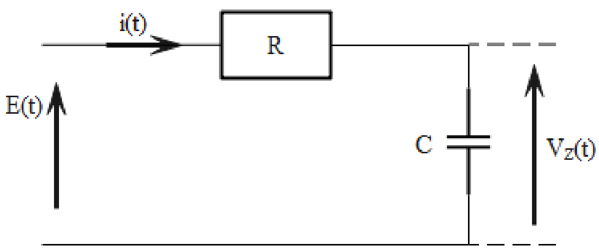
$H(s)=\dfrac{0,5}{1+0,03\cdot s}$ For an input $E(t)=5\cdot u(t)$.

Data:

- The Laplace transform of a unit pulse: $L\left[u(t)\right]=1/s$
- The Laplace inverse transform: $L^{-1}\left[\dfrac{1}{1+\tau\cdot s}\right]=\dfrac{e^{-t/\tau}}{\tau}u(t)$

Exercise N°3:

Let the following RC setup:



1- Determine the transfer function of this system $H(s)=\dfrac{V_z(s)}{E(s)}$

2- Deduce the system's nature and characteristic parameters as a function of R and C.

3- For $E(t)=10\cdot u(t)$, $R=100\Omega$, $C=10^{-6}\,F$, calculating and plot $V_z(t)$.

Data:

- The Laplace transform of: $L\left[u(t)\right]=1/s$ *and* $L\left[du(t)/dt\right]=s$
- The Laplace inverse transform : $L^{-1}\left[\dfrac{1}{s\cdot(1+\tau\cdot s)}\right]=\left(1-e^{-t/\tau}\right)\cdot u(t)$

CHAPTER V: STUDY OF A SECOND-ORDER CONTROL SYSTEM

V.1. Definition

A second-order system is governed by the following differential equation:

$$\frac{1}{\omega_0^2}\cdot\frac{d^2Z(t)}{dt^2}+\frac{2\xi}{\omega_0}\cdot\frac{dZ(t)}{dt}+Z(t)=K\cdot E(t)$$

Such as:

- ω_0 : is the undamped natural frequency of the system.
- ξ : is damping ratio.
- K : is the steady-state gain of the system.

Applying the Laplace transform to the differential equation gives:

$$\frac{1}{\omega_0^2}\cdot s^2\cdot Z(s)+\frac{2\xi}{\omega_0}\cdot s\cdot Z(s)+Z(s)=K\cdot E(s)$$

$$H(s)=\frac{Z(s)}{E(s)}=\frac{K}{\frac{1}{\omega_0^2}\cdot s^2+\frac{2\xi}{\omega_0}\cdot s+1}$$

V.2. Step response of the second-order system

When the input system is a unit step signal

$$E(t)=E_c\cdot u(t)$$

$$E(s)=L\left[E(t)\right]=L\left[E_c\cdot u(t)\right]$$

$$E(s)=\frac{E_c}{s}$$

$$H(s)=\frac{Z(s)}{E(s)}\rightarrow Z(s)=H(s)\cdot E(s)$$

$$Z(s)=\frac{K}{\frac{1}{\omega_0^2}s\cdot^2+\frac{2\xi}{\omega_0}\cdot s+1}\cdot\frac{E_c}{s}$$

This transfer function is decomposed into partial fractions. This depends on the roots of the denominator polynomial (the poles of the transfer function).

$$Z(s)=\frac{K\cdot E_c}{s\cdot\left(\frac{1}{\omega_0^2}s^2+\frac{2\xi}{\omega_0}\cdot s+1\right)}$$

$$s\cdot\left(\frac{1}{\omega_0^2}\cdot s^2+\frac{2\xi}{\omega_0}\cdot s+1\right)=0$$

$$\begin{cases} s=0 \\ \frac{1}{\omega_0^2}\cdot s^2+\frac{2\xi}{\omega_0}\cdot s+1=0 \end{cases}$$

$$\Delta=\left(\frac{2\xi}{\omega_0}\right)^2-4\cdot\left(\frac{1}{\omega_0^2}\right)\cdot 1\rightarrow\Delta=\frac{4\xi^2-4}{\omega_0^2}\rightarrow\Delta=4\frac{\xi^2-1}{\omega_0^2}$$

There are three cases:

- **If the damping is low ($\xi \prec 1$):** Then $\Delta \prec 0$ there are two conjugate complex roots.
- **If the damping is critical ($\xi = 1$):** Then $\Delta = 0$ there is a double real root.
- **If the damping is large ($\xi \succ 1$):** Then $\Delta \succ 0$ there are two distinct real roots.

V.2.a. Case 1: $\xi \succ 1$

In this case, the response is underdamped and therefore oscillatory, and the denominator of the transfer function is:

$$D(s)=\frac{1}{\omega_0^2}\cdot s^2+\frac{2\xi}{\omega_0}\cdot s+1$$

Two complex conjugate poles:

$$\begin{cases} s_1=\dfrac{\dfrac{-2\xi}{\omega_0}-\sqrt{4\dfrac{\xi^2-1}{\omega_0^2}}}{2\cdot\dfrac{1}{\omega_0^2}} \\ s_2=\dfrac{\dfrac{-2\xi}{\omega_0}+\sqrt{4\dfrac{\xi^2-1}{\omega_0^2}}}{2\cdot\dfrac{1}{\omega_0^2}} \end{cases} \xrightarrow{\textit{After simplification}} \begin{cases} s_1=-\xi\omega_0-\omega_0\sqrt{\xi^2-1} \\ s_2=-\xi\omega_0+\omega_0\sqrt{\xi^2-1} \end{cases}$$

With:

- $\xi \succ 0,\ \omega_0 \succ 0$

By defining $s_1 \prec s_2 \prec 0$, $\tau_1=-\dfrac{1}{s_1}$ and $\tau_2=-\dfrac{1}{s_2}$

So, the transfer function $H(s)$ can be written as:

$$H(s)=\frac{K\cdot E_c}{(1+\tau_1\cdot s)\cdot(1+\tau_2\cdot s)}$$

And the output:

$$Z(s)=\frac{1}{s}\frac{K\cdot E_c}{(1+\tau_1\cdot s)\cdot(1+\tau_2\cdot s)}$$

Partial fraction decomposition:

$$Z(s)=\frac{\alpha}{s}+\frac{\beta}{(1+\tau_1\cdot s)}+\frac{\delta}{(1+\tau_2\cdot s)}$$

$$\alpha=\left[s\cdot Z(s)\right]_{s=0}=\left[s\cdot\frac{1}{s}\frac{K\cdot E_c}{(1+\tau_1\cdot s)\cdot(1+\tau_2\cdot s)}\right]_{s=0}\xrightarrow{\text{After simplification}}\alpha=K\cdot E_c$$

$$\beta=\left[\tau_1\left(\frac{1}{\tau_1}+s\right)\cdot Z(s)\right]_{s=\frac{-1}{\tau_1}}=\left[\tau_1\left(\frac{1}{\tau_1}+s\right)\cdot\frac{1}{s}\frac{K\cdot E_c}{\tau_1\left(\frac{1}{\tau_1}+s\right)\cdot(1+\tau_2\cdot s)}\right]_{s=\frac{-1}{\tau_1}}$$

$$\beta=\frac{K\cdot E_c}{\frac{-1}{\tau_1}\left(1-\frac{\tau_2}{\tau_1}\right)}=\frac{-K\cdot E_c}{\left(\frac{\tau_1-\tau_2}{\tau_1^2}\right)}\xrightarrow{\text{After simplification}}\beta=\frac{-K\cdot E_c\cdot\tau_1^2}{\tau_1-\tau_2}$$

$$\delta=\left[\tau_2\left(\frac{1}{\tau_2}+s\right)\cdot Z(s)\right]_{s=\frac{-1}{\tau_2}}=\left[\tau_2\left(\frac{1}{\tau_2}+s\right)\cdot\frac{1}{s}\frac{K\cdot E_c}{(1+\tau_1\cdot s)\cdot\tau_2\left(\frac{1}{\tau_2}+s\right)}\right]_{s=\frac{-1}{\tau_2}}$$

$$\delta=\frac{K\cdot E_c}{\frac{-1}{\tau_2}\left(1-\frac{\tau_1}{\tau_2}\right)}=\frac{K\cdot E_c}{\left(\frac{\tau_1-\tau_2}{\tau_2^2}\right)}\xrightarrow{\text{After simplification}}\delta=\frac{K\cdot E_c\cdot\tau_2^2}{\tau_1-\tau_2}$$

$$Z(s)=\frac{K\cdot E_c}{s}+\frac{K\cdot E_c\cdot\tau_2^2}{\tau_1-\tau_2}\times\frac{1}{1+\tau_2\cdot s}+\frac{-K\cdot E_c\cdot\tau_1^2}{\tau_1-\tau_2}\times\frac{1}{1+\tau_1\cdot s}$$

Inverse Laplace transformation:

$$Z(t)=K\cdot E_c+\frac{K\cdot E_c\cdot\tau_2^2}{\tau_1-\tau_2}\times\frac{e^{-t/\tau_2}}{\tau_2}+\frac{-K\cdot E_c\cdot\tau_1^2}{\tau_1-\tau_2}\times\frac{e^{-t/\tau_1}}{\tau_1}$$

$$Z(t) = K \cdot E_c \left[1 - \frac{1}{\tau_1 - \tau_2} \left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2} \right) \right]$$

Hence:

$$\tau_1 = -\frac{1}{s_1} \quad and \quad \tau_2 = -\frac{1}{s_2}$$

Therefore:

$$\frac{1}{\tau_1 - \tau_2} = \frac{1}{-\frac{1}{s_1} + \frac{1}{s_2}} = \frac{s_1 \cdot s_2}{s_1 - s_2}$$

$$\frac{1}{\tau_1 - \tau_2} = \frac{\left(-\xi \cdot \omega_0 - \omega_0 \sqrt{\xi^2 - 1}\right)\left(-\xi \cdot \omega_0 + \omega_0 \sqrt{\xi^2 - 1}\right)}{-\xi \cdot \omega_0 + \omega_0 \sqrt{\xi^2 - 1} + \xi \cdot \omega_0 + \omega_0 \sqrt{\xi^2 - 1}}$$

$$\frac{1}{\tau_1 - \tau_2} = \frac{\xi^2 \cdot \omega_0^2 - \omega_0^2 \left(\xi^2 - 1\right)}{2 \cdot \omega_0 \sqrt{\xi^2 - 1}}$$

$$\frac{1}{\tau_1 - \tau_2} = \frac{\omega_0}{2\sqrt{\xi^2 - 1}}$$

Therefore:

$$Z(t) = K \cdot E_c \left[1 - \frac{\omega_0}{2\sqrt{\xi^2 - 1}} \left(\frac{e^{t \cdot s_2}}{s_2} - \frac{e^{t \cdot s_1}}{s_1} \right) \right]$$

<u>Initial and final values:</u>

$$Z_0 = \lim_{t \to 0} Z(t) = \lim_{s \to +\infty} s \cdot Z(s)$$

$$Z_0 = \lim_{s \to +\infty} s \frac{1}{s} \frac{E_c K}{(1 + \tau_1 \cdot s)(1 + \tau_2 \cdot s)}$$

$$Z_0 = 0$$

$$Z_\infty = \lim_{t\to+\infty} Z(t) = \lim_{s\to 0} s\cdot Z(t)$$

$$Z_\infty = \lim_{s\to 0} s\cdot\frac{1}{s}\frac{E_c\cdot K}{(1+\tau_1\cdot s)(1+\tau_2\cdot s)}$$

$$Z_\infty = E_c\cdot K$$

The slope:

$$Z^{'}(0) = \lim_{t\to 0} Z^{'}(t) = \lim_{s\to+\infty} s^2\cdot Z(s)$$

$$Z^{'}(0) = \lim_{s\to+\infty} s^2\cdot\frac{1}{s}\frac{E_c\cdot K}{(1+\tau_1\cdot s)(1+\tau_2\cdot s)}$$

$$Z^{'}(0) = 0 \qquad (\text{Zero slope at origin})$$

$$Z^{'}(\infty) = \lim_{t\to+\infty} Z^{'}(t) = \lim_{s\to 0} s^2\cdot Z(s)$$

$$Z^{'}(\infty) = \lim_{s\to 0} s^2\cdot\frac{1}{s}\frac{E_c\cdot K}{(1+\tau_1\cdot s)(1+\tau_2\cdot s)}$$

$$Z^{'}(\infty) = 0 \qquad (\text{Zero slope to infinity})$$

Figure V.1 shows the plot of the second-order system response to a step input for the case under consideration.

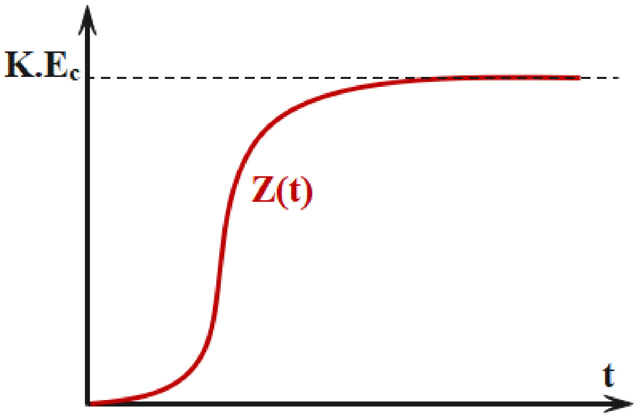


Figure V.1: Step response of a second-order system: Case $\xi \succ 1$

There is no exact formula for determining the 5% response time, but it can be approximated by the value: $t_{r5\%} = 3 \times 2\xi\omega_0$.

V.2.b. Case 2: $\xi = 1$

We have the critical damping for this case $\tau_1 = \tau_2 = \tau_0$ and a double pole.

$$Z(s) = \frac{K \cdot E_c}{s \cdot (1 + \tau_0 \cdot s)^2}$$

Where from:

$$Z(t) = K \cdot E_c \left[1 - (1 + \omega_0) e^{-t/\tau} \right]$$

Property: we do the calculations in the same way:

- Initial value: $Z(0) = 0$.
- Steady-state value: $\lim_{t \to +\infty} Z(t) = K \cdot E_c$.
- A tangent at the origin: The slope at the origin is horizontal (zero).

In this case, the response also converges to $K \cdot E_c$, and the settling time is relatively short. Figure V.2 shows the plot of the index response for the case under consideration.

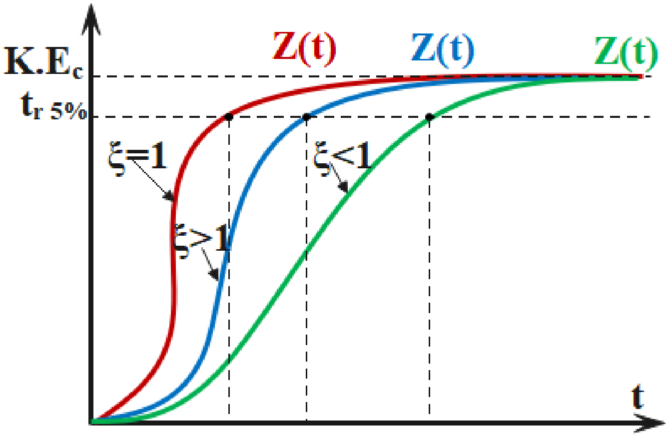


Figure V.2. Index response of a second-order system: Case $\xi = 1$

V.2.c. Case 3 : $\xi \prec 1$

For $\xi \prec 1$ the denominator, $D(s) = \frac{1}{\omega^2} s^2 + \frac{2 \cdot \xi}{\omega_0} s + 1$ has two complex conjugate roots:

$$\begin{cases} s_1 = -\xi \cdot \omega_0 + j\omega_0\sqrt{1-\xi^2} \\ s_2 = -\xi \cdot \omega_0 - j\omega_0\sqrt{1-\xi^2} \end{cases}$$

$$Z(t) = K \cdot E_c \left(1 - \frac{1}{\tau_1 + \tau_1} \left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2} \right) \right)$$

The difference between the two signals forms this response:

- A step of amplitude $K \cdot E_c$.
- Moreover, a sinusoidal signal is modulated by a decreasing and oscillating exponential envelope.

The output signal $Z(t)$ approaches a constant asymptote at infinity. The signal tends toward this asymptote with a damped oscillatory regime $K \cdot E_c$.

The step response of a first-order system in the open case is shown in Figure V.3.

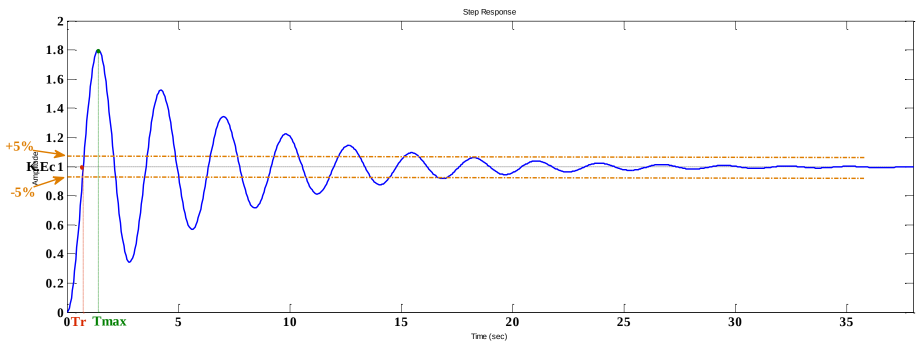


Figure V.3. Step response of a second-order system: Case $\xi \prec 1$

Characteristics of the response signal in the case:

- Steady state: $Z_\infty = K \cdot E_c$.
- A horizontal tangent at origin.
- A damped natural frequency: $\omega_P = \omega_0 \sqrt{1-\xi^2}$
- Damped oscillation period: $T_P = \dfrac{\pi}{\omega_P}$
- Peak time: $t_1 = \dfrac{T_P}{2}$
- First Overshoot in percentage: $D\% = \dfrac{Z(t_{\max}) - Z(\infty)}{Z(\infty)}$

- 5% settling time: $t_r = \dfrac{3}{\xi \cdot \omega_0}$
- The rise time: $t_m = \dfrac{T_P}{2}\left(1 - \dfrac{Q}{\pi}\right)$

V.3. Impulse response (input is a pulse)

$$E(s) = 1$$

We have:

$$Z(s) = E(s) \times H(s)$$

$$Z(s) = \frac{K}{\dfrac{s^2}{\omega_0^2} + \dfrac{2\xi}{\omega_0} s + 1} = \frac{N(s)}{D(s)}$$

$$D(s) = \frac{s^2}{\omega_0^2} + \frac{2\xi}{\omega_0} \cdot s + 1$$

$$\Delta = \left(\frac{2\xi}{\omega_0}\right)^2 - 4\frac{1}{\omega_0^2} = \frac{4\xi^2}{\omega_0^2} - 4\frac{1}{\omega_0^2}$$

$$\Delta = \frac{4}{\omega_0^2}\left(\xi^2 - 1\right)$$

V.3.a. Case 1 : $\xi \succ 1$

The system has two distinct real roots:

$$\begin{cases} s_1 = -\xi \cdot \omega_0 - \omega_0 \sqrt{\xi^2 - 1} \\ s_2 = -\xi \cdot \omega_0 + \omega_0 \sqrt{\xi^2 - 1} \end{cases}$$

Decomposition into simple elements and inverse Laplace transformation gives us:

$$Z(t) = \frac{K\omega_0}{2\sqrt{\xi^2 - 1}}\left(e^{t\cdot s_1} - e^{t\cdot s_2}\right)$$

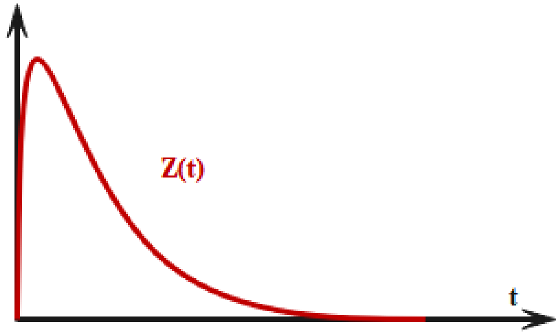


Figure V.4. Index response of a second-order system: Case $\xi \prec 1$

V.3.b. Case 2 : $\xi = 1$

For this case, $D(s)$ has a double root because $\Delta = 0$, the expression of $Z(t)$ is given by :

$$Z(t) = K\omega_0^2 e^{-\omega_0 t} \cdot t$$

The response has the same form as the one for $\xi \succ 1$.

V.3.c. Case 3 : $\xi \prec 1$

The answer has two complex conjugate roots:

$$\begin{cases} s_1 = \omega_0\left(\xi - j\sqrt{1-\xi^2}\right) \\ s_2 = \omega_0\left(\xi + j\sqrt{1-\xi^2}\right) \end{cases}$$

The temporal response is expressed as:

$$Z(t)=\frac{K\cdot\omega_0}{2\sqrt{\xi^2-1}}\left(e^{t\cdot s_1}-e^{t\cdot s_2}\right) \text{ or } Z(t)=\frac{K\omega_0}{2\sqrt{\xi^2-1}}e^{-\omega_0\xi t}\sin\left(\omega_0\sqrt{1-\xi^2}\cdot t\right)$$

This response is illustrated in Figure V.5.

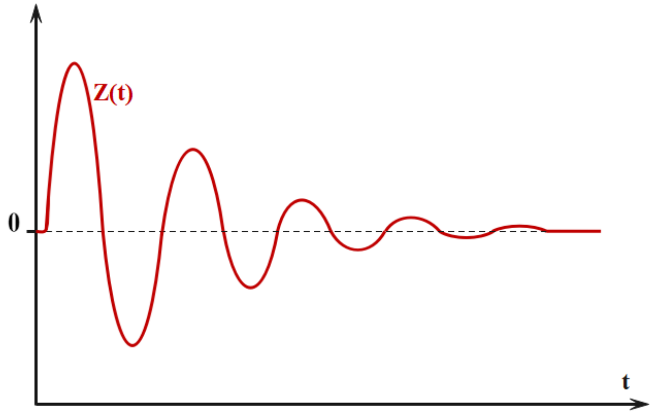


Figure V.5. Impulse response of a second-order system: case $\xi \prec 1$.

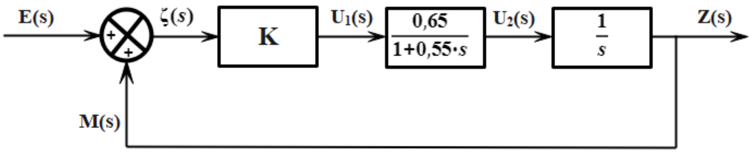
Observe that the response is both oscillating (due to $e^{-\omega_0\xi t}$) and damped (due to $\sin\left(\omega_0\sqrt{1-\xi^2}\cdot t\right)$), so note:

- Natural pulsation: $\omega_p=\omega_0\sqrt{1-\xi^2}$
- The pseudo-period: $T_p=\dfrac{2\pi}{\omega_p}$

V.4. Learning activity

Exercise N°1:

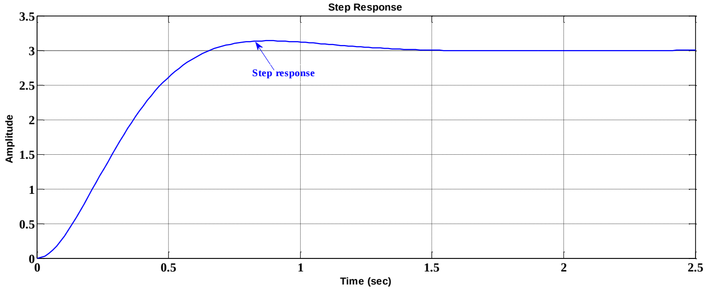
Let us consider a mechanical system modeled by the following diagram:



1) Is the system transfer function of second order?
2) Write the transfer function in its canonical form and highlight its parameters for a value of $K = 1,5$.
3) Plot the two responses (impulsive and step response) of this system.

Exercise N°2:

Let be a system with a step response as shown in the following figure:



From the figure, identify:

(a) The value of response time

(b) The percentage of overshoots

(c) The value of the output in the final state

(d) The value of the damping ratio ξ and naturel frequency ω_0 of this system

CHAPTER VI: BODE AND NYQUIST DIAGRAM OF CONTROL SYSTEMS

VI.1. Definition

Frequency analysis is essential to understanding the behavior of control systems (stability and performance) of control systems. The graphical representations of the Bode and Nyquist diagrams allow us to analyze how a system responds to different frequencies.

They also allow us to determine the phase and gain margins, are used to evaluate the robustness and stability of the system.

VI.2. Frequency response or harmonic response

The frequency or harmonic response $Z(t)$ is the response of a system $H(s)$ to a steady-state sinusoidal input of type $E(t) = E_{\max} \cdot \sin(\omega t)$. In this case, the response will be of this type $Z(t) = Z_{\max} \cdot \sin(\omega \cdot t + \varphi)$.

With:

$E_{\max}$: This is the maximum amplitude of the sinusoidal signal of the input.

$S_{\max}$: This is the maximum amplitude of the output signal.

ω: Angular frequency.

φ: The phase shift between the output and the input.

VI.3. Transfer function in the complex domain

To analyze the angular frequency ω of the input signal, the transfer function $H(s)$ is expressed in the complex domain by replacing s by $j\omega$, so the transfer function is written as:

$$H(j\omega) = \frac{Z(j\omega)}{E(j\omega)} = A(\omega)e^{j\varphi(\omega)}$$

Where:

$A(\omega)$: The magnitude of $H(j\omega)$ knowing that $A(\omega) = |H(j\omega)| = \frac{Z_{\max}}{E_{\max}}$

$\varphi(\omega)$: The argument of $H(j\omega)$ knowing that $\varphi(\omega) = \arg(H(j\omega))$

This will result in two curves, a magnitude curve ($A(\omega)$) and a phase curve ($\varphi(\omega)$), that vary as a function of ω. These two curves will allow us to study the behavior of a system for different frequencies using the Bode and Nyquist diagram.

VI.4. Bode diagram

VI.4.1. Definition of Bode diagrams

The Bode diagram plots the two characteristic curves of the open-loop transfer function in the complex domain: the magnitude plot and a phase plot. The two curves of the Bode diagram are obtained as follows:

The magnitude is expressed in decibels (dB) for the drawing of the gain curve:

$$A(\omega) = |H(j\omega)| = 20\log(|H(j\omega)|)$$

The phase for the drawing of the phase curve: $\varphi(\omega) = \arg(H(j\omega))$

For the calculation of the magnitude and the phase, we will use the simple decomposition of the magnitude and the phase as follows:

- **<u>The decomposition used to calculate the magnitude:</u>**

$$\begin{aligned} H(j\omega) = H_1(j\omega).H_2(j\omega) \xrightarrow{Magnitude} A(\omega) = H_{dB} &= |H(j\omega)| \\ &= |H_1(j\omega).H_2(j\omega)| \\ &= |H_1(j\omega)| + |H_2(j\omega)| \\ &= H_{1dB} + H_{2dB} \end{aligned}$$

$$\begin{aligned} H(j\omega) = \frac{H_1(j\omega)}{H_2(j\omega)} \xrightarrow{Magnitude} A(\omega) = H_{dB} &= |H(j\omega)| \\ &= \left|\frac{H_1(j\omega)}{H_2(j\omega)}\right| \\ &= |H_1(j\omega)| - |H_2(j\omega)| \\ &= H_{1dB} - H_{2dB} \end{aligned}$$

- **<u>The decomposition used to calculate the phase:</u>**

$$\begin{aligned} H(j\omega) = H_1(j\omega).H_2(j\omega) \xrightarrow{Phase} \varphi(\omega) &= \arg(H(j\omega)) \\ &= \arg\left(H_1(j\omega).H_2(j\omega)\right) \\ &= \arg(H_1(j\omega)) + \arg(H_2(j\omega)) \end{aligned}$$

$$H(j\omega)=\frac{H_1(j\omega)}{H_2(j\omega)} \xrightarrow{Phase} \varphi(\omega)=\arg(H(j\omega))$$
$$=\arg\left(\frac{H_1(j\omega)}{H_2(j\omega)}\right)$$
$$=\arg\left(H_1(j\omega)\right)-\arg\left(H_2(j\omega)\right)$$
$$=\left|H_1(j\omega)\right|-\left|H_2(j\omega)\right|$$

Both curves are plotted on a logarithmic scale (Figure VI.1), and an example of the plot of the Bode diagram is shown in Figure VI.2.

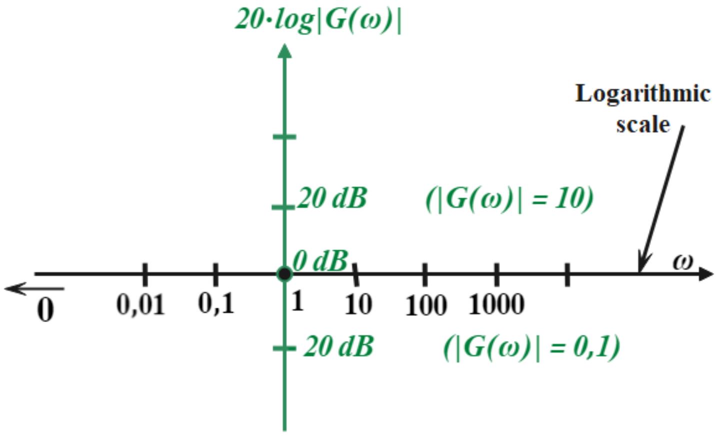


Figure VI.1. Logarithmic scale of the Bode diagram plot

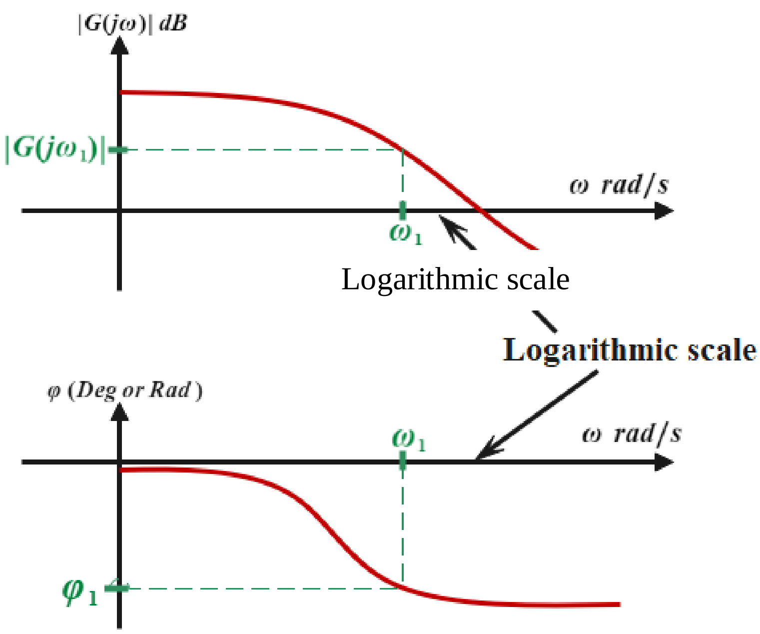


Figure VI.2. Bode diagram

To draw the curves of the Bode diagram, the Bode diagram can be constructed according to the following steps:

1- Determination of the open-loop transfer function $H(s)$.

2- Express the transfer function in the complex domain by replacing $s = j\omega$.

3-Determination of the magnitude $A(\omega) = |H(j\omega)| = 20\log(|H(j\omega)|)$ and the phase $\varphi(\omega) = \arg(H(j\omega))$

4- Study the magnitude and the phase at the limits.

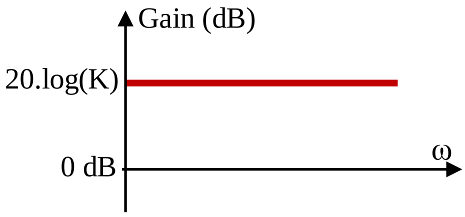
5- Determination of slopes.

6- Determination of the phase.

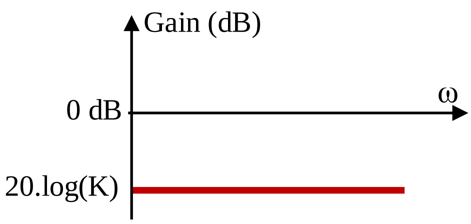
VI.4.2. Bode diagram of elementary functions

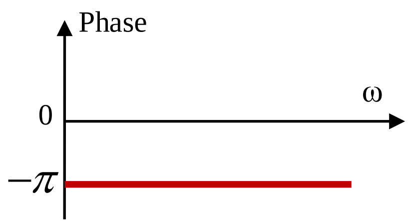
a- The transfer function is constant $H(j\omega) = K$

- **Case or K>0**

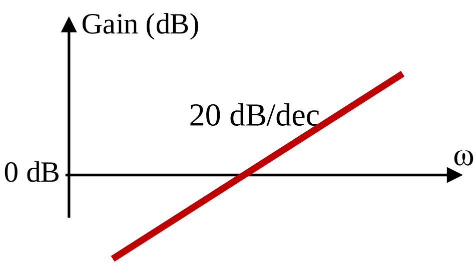


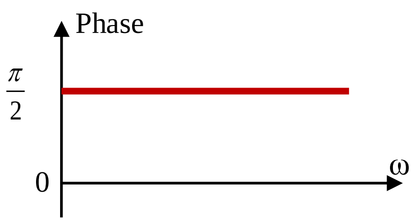
- **Case or K< 0**



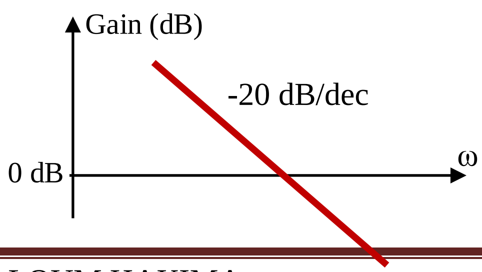


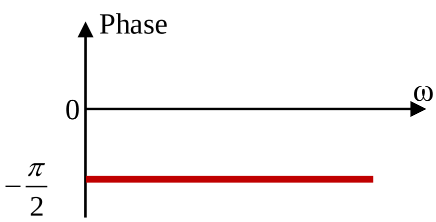
b- Differentiating Transfer Function $H(j\omega) = j\omega$



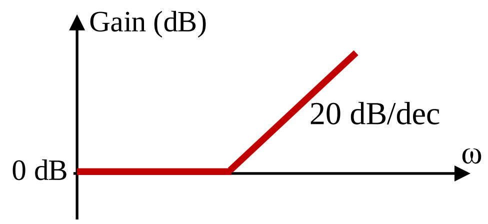


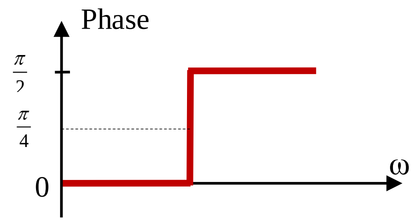
c- The integrator Transfer Function $H(j\omega) = \dfrac{1}{j\omega}$



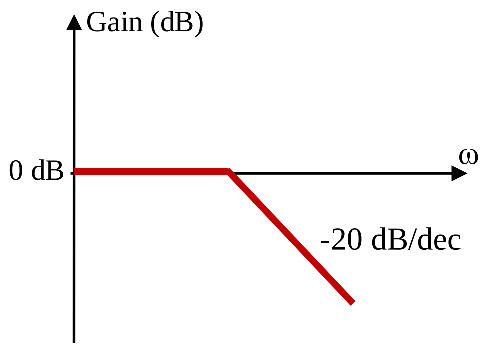


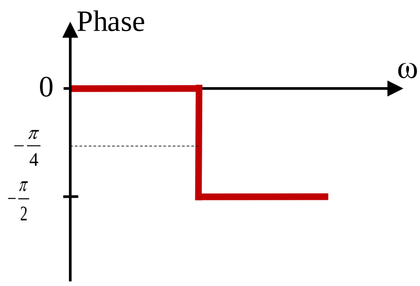
d- The Transfer Function $H(j\omega)=1+j\tau\omega$





e- The first-order transfer function $H(j\omega)=\frac{-}{1+j\tau\omega}$ *With* $K=1$





VI.4.3. Plot of the Bode diagram for a first-order system

To be able to draw the Bode diagram of the first-order system, we will follow the steps mentioned earlier in (IV-1).

VI.4.3.1. Determine the open-loop transfer function of the first-order system

We know that the transfer function of a first-order system is: $H(s)=\frac{1}{1+\tau\cdot s}$ (see Chapter IV)

VI.4.3.2. The expressions of the transfer function of the first-order system in the complex domain

Substituting $s=j\omega$, we will have the expression of the transfer function in the complex domain that is equal to:

$$H(j\omega)=\frac{1}{1+j\omega\tau}$$

VI.4.3.3. Determination of the magnitude in decibels of the transfer function of the first-order system

Determinations of the magnitude in decibels of the transfer function of the first-order system:

$$A(\omega)=|H(j\omega)|=20\log(K)-20\log\left(\sqrt{1+(\omega\tau)^2}\right)$$

$$A(\omega)=|H(j\omega)|=20\log(|H(j\omega)|)$$

$$A(\omega)=|H(j\omega)|=20\log\left(\left|\frac{K}{1+j\omega\tau}\right|\right)$$

$$A(\omega)=|H(j\omega)|=20\log\left(\frac{K}{\sqrt{1+(\omega\tau)^2}}\right)$$

$$A(\omega)=|H(j\omega)|=20\log(K)-10\times 2\log\left(\sqrt{1+(\omega\tau)^2}\right)$$

$$A(\omega)=|H(j\omega)|=20\log(K)-10\log\left(\sqrt{1+(\omega\tau)^2}\right)^2$$

$$A(\omega)=|H(j\omega)|=20\log(K)-10\log\left(1+(\omega\tau)^2\right)$$

Wherefrom:

$$A(\omega)=20\log(K)-10\log\left(1+(\omega\tau)^2\right)$$

VI.4.3.4. Determination of the phase of the transfer function of the first-order system

The argument for this first-order system is calculated as follows:

$$\begin{aligned}\varphi(\omega)&=\arg(H(j\omega))\\&=\arg\left(\frac{K}{1+j\tau\omega}\right)\\&=\arg(K)-\arg(1+j\tau\omega)\\&=-\arg(1+j\tau\omega)\quad car\ \ \arg(K)=0\\&=-\arctan\left(\frac{\tau\omega}{1}\right)\\&=-\arctan(\tau\omega)\end{aligned}$$

Hence; $\varphi(\omega)=-\arctan(\tau\omega)$

VI.4.3.5. Asymptotic behavior of the magnitude and phase

In this part, we will determine the magnitude and phase limits for three frequency values, $\omega \to 0,\ \omega \to \infty\ \ and\ \ \omega \to \omega_e$ knowing that $\omega_e = \dfrac{1}{\tau}$ this is the cut-off frequency of the system. The asymptotes are studied as follows:

a- For $\omega \to 0$:

- **The magnitude :**

$$A(\omega) \to 20\log(K) - \overbrace{10\log\left(1+\underbrace{(\omega\tau)}_{0}{}^{2}\right)}^{0} \quad \text{because } \log(1)=0$$

$$A(\omega) \to 20\log(K)$$

- **The phase :**

$$\varphi(\omega) \to -\arctan(\tau\underbrace{\omega}_{0})$$

$$\varphi(\omega) \to 0$$

b- For $\omega = \omega_e = \dfrac{1}{\tau}$

- **The magnitude :**

$$A = 20\log(K) - 10\log\left(1+\left(\frac{1}{\cancel{\tau}}\cancel{\tau}\right)^2\right)$$

$$A = 20\log(K) - 10\log\left(1+(1)^2\right)$$

$$A(\omega) \to 20\log(K) - 10\log(2)$$

$$A(\omega) \to 20\log(K) - 3$$

- **The phase :**

$$\varphi = -\arctan\left(\cancel{\tau}\frac{1}{\cancel{\tau}}\right)$$

$$\varphi = -\arctan(1)$$

$$\varphi = -45^o = -\frac{\pi}{4}$$

c- For $\omega \to \infty$

- **The magnitude :**

$$A(\omega) \to 20\log(K) - 10\log\left(1+(\omega\tau)^2\right)$$

$$A(\omega) \to 20\log(K) - 10\log(\omega\tau)^2 \quad because\ 1 \prec\prec \omega\tau$$

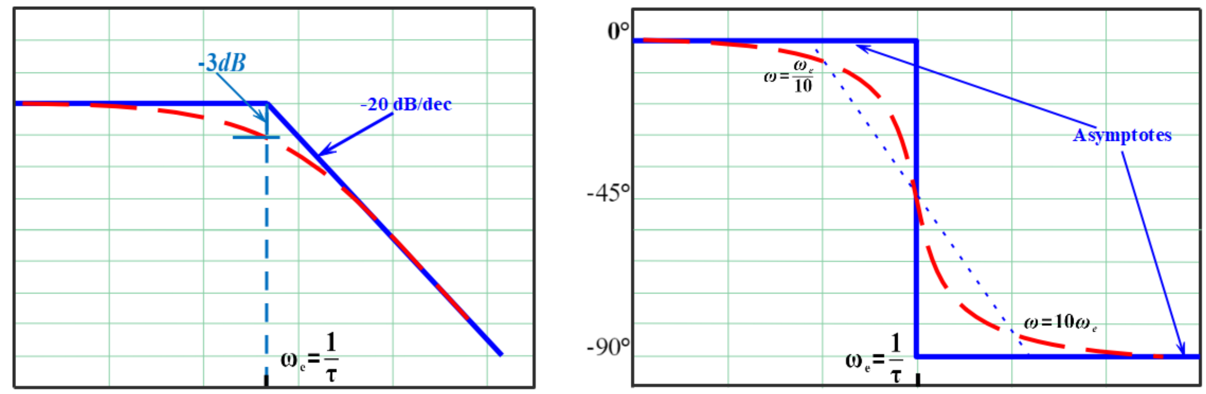
$$A(\omega) \to 20\log(K) - 10 \times 2\log(\omega\tau)$$

$A(\omega) \to 20\log(K) - 20\log(\omega\tau)$: An asymptote is a line with a slope of -20 dB/decade from $\omega = \omega_e$ the x-axis.

- **The phase :**

$$\varphi \to -\arctan(\tau\omega)$$

$$\varphi \to -\frac{\pi}{2} \qquad \left(\varphi \to -90^o\right)$$



(a) The magnitude **(b)** The phase

Figure VI.3. The plot of the Bode diagram (magnitude and phase) for the first-order System

VI.5. Bode Plot diagram for a second-order system

We will proceed similarly to the first order for the plot of the Bode diagram for the second-order system. This system has two pulsations: a cut-off pulse ω_e and a resonance pulsation ω_r. The same mathematical formula determines $\omega_e = \omega_r = \omega_0\sqrt{1 - 2\cdot\xi^2}$ when $\xi \succ \frac{1}{\sqrt{2}}$ these two pulsations. However, these two pulsations do not have the same meaning ω_r ; they designate the pulsation for which the response reaches its maximum. It is obtained when the second-order system is poorly damped $0 \prec \xi \prec \frac{1}{\sqrt{2}}$. On the other hand ω_e, it refers to the pulse for which the response begins to decrease.

VI.5.1. Determination

of the transfer function of the second-order system

We know that the transfer function of a second-order system (see Chapter V) is:

$$H(s) = \frac{K}{\frac{s^2}{\omega_0^2} + \frac{2\xi s}{\omega_0} + 1}$$

VI.5.1.1. The expressions of the transfer function of the second-order system in the complex domain

Substituting $s = j\omega$, we will have the expression of the transfer function in the complex domain, which is equal to:

$$H(j\omega) = \frac{K}{\frac{(j\omega)^2}{\omega_0^2} + \frac{2\xi j\omega}{\omega_0} + 1} = \frac{K}{-\frac{\omega^2}{\omega_0^2} + \frac{2\xi j\omega}{\omega_0} + 1} \quad With\ j^2 = -1$$

$$H(j\omega) = \frac{K}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) + j2\xi\frac{\omega}{\omega_0}}$$

VI.5.1.2. Determinations of the magnitude in decibels of the transfer function of the second-order system

The magnitude of this decibel transfer function is calculated as follows:

$$A(\omega) = |H(j\omega)|_{dB} = 20\log\left(\frac{K}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) + j\left(2\xi\frac{\omega}{\omega_0}\right)}\right)$$

$$A(\omega) = |H(j\omega)|_{dB} = 20\log(K) - 20\log\left(\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2}\right)$$

$$A(\omega) = |H(j\omega)|_{dB} = 20\log(K) - 10 \times 2\log\left(\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2}\right)$$

$$A(\omega)=|H(j\omega)|_{dB}=20\log(K)-10\times 2\log\left(\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_0}\right)^2}\right)$$

$$A(\omega)=|H(j\omega)|_{dB}=20\log(K)-10\log\left(\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_0}\right)^2}\right)^2$$

$$A(\omega)=|H(j\omega)|_{dB}=20\log(K)-10\log\left(\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_0}\right)^2\right)$$

Hence, the magnitude is given by:

$$A(\omega)=|H(j\omega)|_{dB}=20\log(K)-10\log\left(\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_0}\right)^2\right)$$

VI.5.1.2.1. Study of the magnitude of the second-order function at the boundaries

The study is carried out for limit pulsation values, as follows:

a- For $\boldsymbol{\frac{\omega}{\omega_0}\prec\prec 1\Rightarrow\omega\to 0}$:

$$A=|H(j\omega)|_{dB}=20\log(K)-10\log\left(\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_0}\right)^2\right)$$

$$A=|H(j\omega)|_{dB}=20\log(K)$$

b- For $\boldsymbol{\frac{\omega}{\omega_0}=1\Rightarrow\omega=\omega_0}$:

$$A=|H(j\omega)|_{dB}=20\log(K)-10\log\left(\left(1-\left(\frac{\cancel{\omega_0}}{\cancel{\omega_0}}\right)^2\right)^2+\left(2\xi\frac{\cancel{\omega_0}}{\cancel{\omega_0}}\right)^2\right)$$

$$A=|H(j\omega)|_{dB}=20\log(K)-10\log\left((1-1)^2+(2\xi)^2\right)$$

In this case, the value of the magnitude depends on ξ we can distinguish two cases, such as:

❖ For $\xi=\frac{\sqrt{2}}{2}$, we will have :

$$A = \left|H(j\omega)\right|_{dB} = 20\log\left(K\right) - 10\log\left(\left(\cancel{2}\times\frac{\sqrt{2}}{\cancel{2}}\right)^2\right)$$

$$A = \left|H(j\omega)\right|_{dB} = 20\log\left(K\right) - 3dB$$

❖ For $\xi \prec \frac{\sqrt{2}}{2}$ we will have :

$$A_{\max} = \left|H(j\omega_r)\right|_{dB,\max} = 20\log\left(K\right) - 10\log\left(\left(1-\left(\frac{\cancel{\omega_0}\sqrt{1-2\xi^2}}{\cancel{\omega_0}}\right)^2\right)^2 + \left(2\xi\frac{\cancel{\omega_0}\sqrt{1-2\xi^2}}{\cancel{\omega_0}}\right)^2\right)$$

$$A_{\max} = \left|H(j\omega_r)\right|_{dB,\max} = 20\log\left(K\right) - 10\log\left(\left(2\xi\sqrt{1-2\xi^2}\right)^2\right)$$

$$A_{\max} = \left|H(j\omega_r)\right|_{dB,\max} = 20\log\left(K\right) - 10\times 2\log\left(\left(2\xi\sqrt{1-2\xi^2}\right)\right)$$

$$A_{\max} = \left|H(j\omega_r)\right|_{dB,\max} = 20\log\left(K\right) - 20\log\left(2\xi\sqrt{1-2\xi^2}\right)$$

$$A_{\max} = \left|H(j\omega_r)\right|_{dB,\max} = 20\log\left(K\right) - 20\log\left(2\xi\sqrt{1-2\xi^2}\right)$$

$$A_{\max} = \left|H(j\omega_r)\right|_{dB,\max} = 20\log\left(\frac{K}{2\xi\sqrt{1-2\xi^2}}\right)$$

c- For $\boldsymbol{\frac{\omega}{\omega_0} \succ\succ 1 \Rightarrow \omega \to \infty}$:

$$A = \left|H(j\omega)\right|_{dB} = 20\log\left(K\right) - 10\log\left(\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2\right)$$

$$A = \left|H(j\omega)\right|_{dB} = 20\log\left(K\right) - 10\log\left(\left(\left(\frac{\omega}{\omega_0}\right)^2\right)^2\right) \quad Car\ 1 \prec\prec \left(\frac{\omega}{\omega_0}\right)^2 \ et\ \frac{\omega}{\omega_0} \prec\prec \left(\frac{\omega}{\omega_0}\right)^2$$

$$A = \left|H(j\omega)\right|_{dB} = -10\log\left(\left(\frac{\omega}{\omega_0}\right)^4\right)$$

$$A = \left|H(j\omega)\right|_{dB} = -10\times 4\log\left(\frac{\omega}{\omega_0}\right)$$

$$A = \left|H(j10\omega_0)\right|_{dB} = -40\log\left(\frac{10\cancel{\omega_0}}{\cancel{\omega_0}}\right) \quad For\ \ \omega = 10\omega_0$$

$A=\left|H(j10\omega_0)\right|_{dB}=-40\ dB\rightarrow$ A straight line with a slope of -40 dB.

VI.5.1.2.2. Determination of the phase of the transfer function of the second-order system

The phase for this second-order system is calculated as follows:

$$\varphi=\arg(H(j\omega))=\arg\left(\frac{K}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)+j2\xi\frac{\omega}{\omega_0}}\right)$$

$$\varphi=\arg(H(j\omega))=\arg\left(K\right)-\arg\left(\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)+j2\xi\frac{\omega}{\omega_0}\right)\quad With:\arg\left(K\right)=0$$

$$\varphi=\arg(H(j\omega))=-\arctan\left(\frac{2\xi\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$$

Hence, the phase is given by:

$$\varphi=-\arctan\left(\frac{2\xi\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$$

VI.5.1.2.3. Study of the phase limits

a- For $\frac{\omega}{\omega_0}\prec\prec1\Rightarrow\omega\rightarrow0$:

$$\varphi=-\arctan\left(\frac{2\xi\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$$

$\varphi=0^{\circ}=0rad\ \rightarrow$ A horizontal asymptote.

b- For $\frac{\omega}{\omega_0}=1\Rightarrow\omega=\omega_0$:

$$\varphi=-\arctan\left(\frac{2\xi\frac{\omega_0}{\omega_0}}{1-\left(\frac{\omega_0}{\omega_0}\right)^2}\right)$$

$$\varphi=-\arctan\left(\frac{2\xi}{1-(1)^2}\right)$$

$$\varphi=-\arctan\left(\frac{2\xi}{0}\right)$$

$$\varphi=-\arctan(\infty)$$

$$\varphi=-\frac{\pi}{2}$$

c- For $\frac{\omega}{\omega_0}\succ\succ 1\Rightarrow\omega\rightarrow\infty$:

$$\varphi=-\arctan\left(\frac{2\xi\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$$

$$\varphi=-\arctan\left(\frac{\cancel{\frac{\omega}{\omega_0}}}{\left(\frac{\omega}{\omega_0}\right)^{\cancel{2}}}\right)$$

$$\varphi=-\arctan\left(\frac{1}{\left(\frac{\omega}{\omega_0}\right)}\right)$$

$$\varphi=-\arctan\left(\frac{1}{\infty}\right)$$

$\varphi=-\arctan(0)\quad\varphi=-\pi\ \rightarrow$ A horizontal asymptote.

d- Summary table for the calculation of the magnitude at the limits

The following table summarizes the calculation of the magnitude of the boundary transfer function for the plot of the Bode diagram:

Table VI-1: Summary Table for calculating the phase at the Limits

Pulsation	Phase
$\frac{\omega}{\omega_0} \prec\prec 1 \Rightarrow \omega \to 0$	$\varphi = 0^\circ = 0\ rad$
$\frac{\omega}{\omega_0} = 1 \Rightarrow \omega = \omega_0$	$\varphi = -\frac{\pi}{2}$
$\frac{\omega}{\omega_0} \succ\succ 1 \Rightarrow \omega \to \infty$	$\varphi = -\pi$

The following figure shows the plot of the Bode diagram of the transfer function of a second-order system for several values ξ :

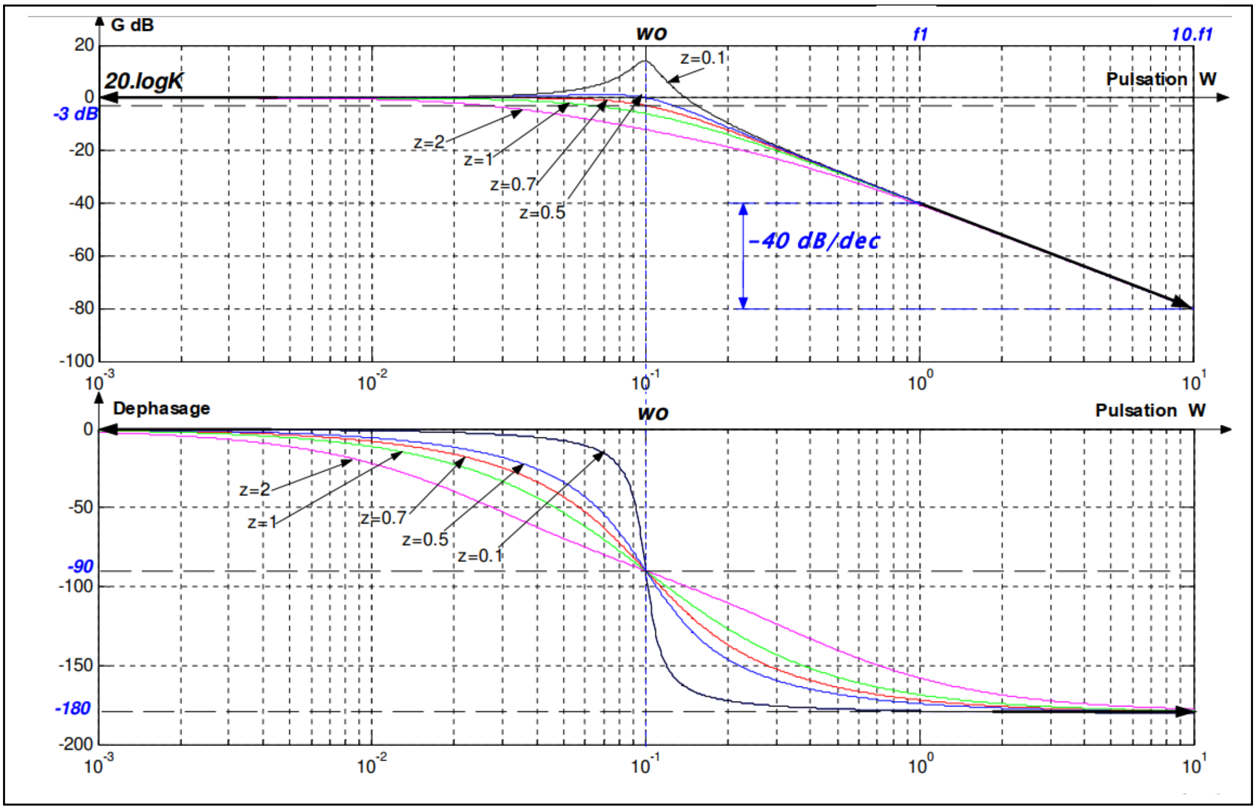


Figure VI.4: The plot of the Bode diagram of the transfer function of a second-order system

VI.6. Nyquist diagram

VI.6.1. Nyquist diagram definition

The Nyquist plot illustrates the representation of the open-loop transfer function in the complex plane for ω varying from 0 to $+\infty$ (Figure VI.5) [1-3]. The Nyquist diagram is traversed in the direction of increasing frequency. knowing that the transfer function is expressed as follows:

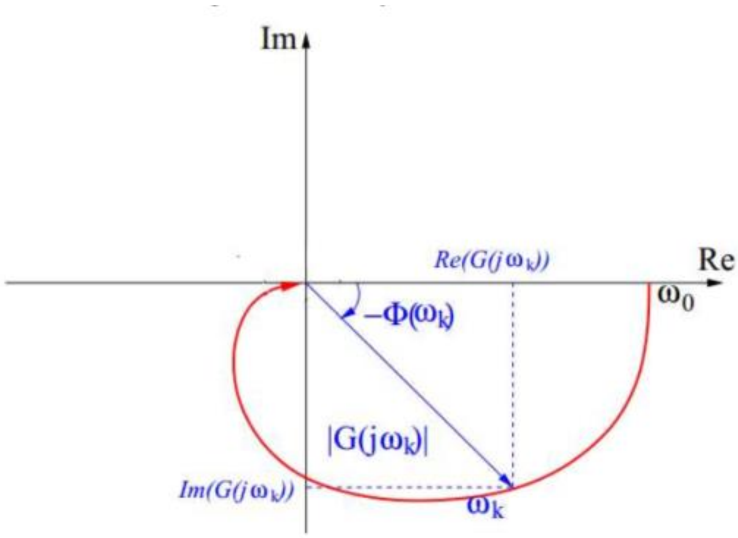


Figure VI.5. Nyquist diagram

The steps of the Nyquist diagram. To make the Nyquist diagram, we need to:

1- Decompose the open-loop transfer function into a real part and an imaginary part as:

$$H(j\omega) = \mathrm{Re}\left(H(j\omega)\right) + j\,\mathrm{Im}\left(H(j\omega)\right)$$

2- Determination of the magnitude $|H(j\omega)| = \sqrt{\left(\mathrm{Re}\left(H(j\omega)\right)\right)^2 + \left(\mathrm{Im}\left(H(j\omega)\right)\right)^2}$ and its evaluation at the limits.

3- Determination of the argument $\varphi(\omega) = \arctan\left(\dfrac{\mathrm{Im}\left(H(j\omega)\right)}{\mathrm{Re}\left(H(j\omega)\right)}\right)$ and its evaluation at the limits.

4- The plot of the Nyquist diagram.

VI.6.2. The Nyquist diagram for a first-order system

The plot of the Nyquist diagram for a first-order system, the transfer function of a first-order system as shown above, is as follows: $H(j\omega) = \dfrac{K}{1 + j\tau\omega}$

To decompose this function into real and imaginary parts, multiply and divide by the conjugate of the denominator of this function as follows:

$$H(j\omega) = \frac{K}{1 + j\tau\omega}$$

$$H(j\omega) = \frac{K}{1 + j\tau\omega} \times \frac{1 - j\tau\omega}{1 - j\tau\omega}$$

$$H(j\omega) = \frac{(K)(1 - j\tau\omega)}{(1 + j\tau\omega)(1 - j\tau\omega)}$$

$$H(j\omega) = \frac{(K)(1 - j\tau\omega)}{(1)^2 - (j\tau\omega)^2}$$

$$H(j\omega) = \frac{K - Kj\tau\omega}{1 + \tau^2\omega^2}$$

$$H(j\omega) = \frac{K}{1 + \tau^2\omega^2} - j\frac{K\tau\omega}{1 + \tau^2\omega^2}$$

By identification $H(j\omega) = \operatorname{Re}\left(H(j\omega)\right) + j\operatorname{Im}\left(H(j\omega)\right)$, we obtain the following:

$$\begin{cases} \operatorname{Re}\left(H(j\omega)\right) = \dfrac{K}{1 + \tau^2\omega^2} \\ \operatorname{Im}\left(H(j\omega)\right) = -\dfrac{K\tau\omega}{1 + \tau^2\omega^2} \end{cases}$$

VI.6.2.1. The magnitude of the first-order transfer function

The expression of the magnitude is given by:

$$|H(j\omega)| = \sqrt{\left(\frac{K}{1 + \tau^2\omega^2}\right)^2 + \left(\frac{K\tau\omega}{1 + \tau^2\omega^2}\right)^2}$$

VI.6.2.1.1. Determination of the magnitude at the limits

a- For $\omega \to 0$:

$$|H(j\omega)| = \sqrt{\left(\frac{K}{1 + \tau^2\omega^2}\right)^2 + \left(\frac{K\tau\omega}{1 + \tau^2\omega^2}\right)^2}$$

$$|H(j\omega)| = \sqrt{\left(\frac{K}{1}\right)^2}$$

$$|H(j\omega)| = K$$

b- For $\omega = \omega_e$ *with* $\omega_e = \frac{1}{\tau}$:

$$|H(j\omega)| = \sqrt{\left(\frac{K}{1+\tau^2\left(\frac{1}{\tau}\right)^2}\right)^2 + \left(\frac{K\tau\frac{1}{\tau}}{1+\tau^2\left(\frac{1}{\tau}\right)^2}\right)^2}$$

$$|H(j\omega)| = \sqrt{\left(\frac{K}{1+\cancel{\tau}^2\frac{1}{\cancel{\tau}^2}}\right)^2 + \left(\frac{K\cancel{\tau}\frac{1}{\cancel{\tau}}}{1+\cancel{\tau}^2\frac{1}{\cancel{\tau}^2}}\right)^2}$$

$$|H(j\omega)| = \sqrt{\left(\frac{K}{1+1}\right)^2 + \left(\frac{K}{1+1}\right)^2}$$

$$|H(j\omega)| = \sqrt{\left(\frac{K}{2}\right)^2 + \left(\frac{K}{2}\right)^2}$$

$$|H(j\omega)| = \sqrt{\cancel{2}\frac{K^2}{2^{\cancel{2}}}}$$

$$|H(j\omega)| = \sqrt{\frac{K^2}{2}} \rightarrow |H(j\omega)| = \frac{k}{\sqrt{2}}$$

c- For $\omega \to \infty$:

$$|H(j\omega)| = \sqrt{\left(\frac{K}{1+\tau^2\underbrace{\omega^2}_{\infty}}\right)^2 + \left(\frac{K\tau\omega}{1+\tau^2\underbrace{\omega^2}_{\infty}}\right)^2} \rightarrow |H(j\omega)| = 0$$

VI.6.2.1.2. Summary table for the calculation of the magnitude at limits

The following table summarizes the calculation of the magnitude of the boundary transfer function for the plot of the Nyquist diagram:

Table VI.2. Summary Table for calculating the magnitude at the Limits

Pulsation	The magnitude
$\omega \to 0$	$\lvert H(j\omega)\rvert = K$
$\omega = \omega_e$	$\lvert H(j\omega)\rvert = \frac{k}{\sqrt{2}}$
$\omega \to \infty$	$\lvert H(j\omega)\rvert = 0$

VI.6.2.2. The argument of the first-order transfer function

The phase for the first-order transfer function is determined as follows:

$$\varphi(\omega) = \arctan\left(\frac{\operatorname{Im}(H(j\omega))}{\operatorname{Re}(H(j\omega))}\right)$$

VI.6.2.2.1. Determination of the phase limits

The value of the phase is calculated for three limiting frequency values as follows:

a- For $\omega \to 0$:

$$\varphi(\omega) = \arctan\left(\frac{\operatorname{Im}(H(j\omega))}{\operatorname{Re}(H(j\omega))}\right)$$

$$\varphi(\omega) = \arctan\left(\frac{-\dfrac{K\tau\omega}{1+\tau^2\omega^2}}{\dfrac{K}{1+\tau^2\omega^2}}\right)$$

$$\varphi = \arctan\left(\frac{0}{\dfrac{K}{1}}\right)$$

$$\varphi = 0$$

b- For $\omega = \dfrac{1}{\tau}$:

$$\varphi(\omega) = \arctan\left(\frac{-\dfrac{K\tau\dfrac{1}{\tau}}{1+\tau^2\left(\dfrac{1}{\tau}\right)^2}}{\dfrac{K}{1+\tau^2\left(\dfrac{1}{\tau}\right)^2}}\right)$$

$$\varphi(\omega) = \arctan\left(\frac{-\dfrac{K\cancel{\tau}\dfrac{1}{\cancel{\tau}}}{1+\cancel{\tau^2}\dfrac{1}{\cancel{\tau^2}}}}{\dfrac{K}{1+\cancel{\tau^2}\dfrac{1}{\cancel{\tau^2}}}}\right)$$

$$\varphi(\omega)=\arctan\left(\frac{-\frac{K}{1}}{\frac{K}{1}}\right)$$

$$\varphi(\omega)=\arctan(-1)$$

$$\varphi(\omega)=-\frac{\pi}{4}$$

c- For $\omega\to\infty$:

$$\varphi(\omega)=\arctan\left(\frac{-\frac{K\tau\omega}{\cancel{1+\tau^2\omega^2}}}{\frac{K}{\cancel{1+\tau^2\omega^2}}}\right)$$

$$\varphi(\omega)=\arctan\left(\frac{-\cancel{K}\tau\omega}{\cancel{K}}\right)$$

$$\varphi(\omega)=\arctan\left(\frac{-\tau\omega}{1}\right)$$

$$\varphi(\omega)=\arctan(\infty)\to\varphi(\omega)=-\frac{\pi}{2}$$

VI.6.2.2.2. Summary table for the calculation of the phase limits

The following table summarizes the calculation of the phase of the boundary transfer function for the plot of the Nyquist diagram:

Table VI.3. Summary table for the calculation of the phase at the limits

The pulsation	The phase
$\omega\to 0$	$\varphi=0$
$\omega=\omega_e=\frac{1}{\tau}$	$\varphi(\omega)=-\frac{\pi}{4}$
$\omega\to\infty$	$\varphi(\omega)=-\frac{\pi}{2}$

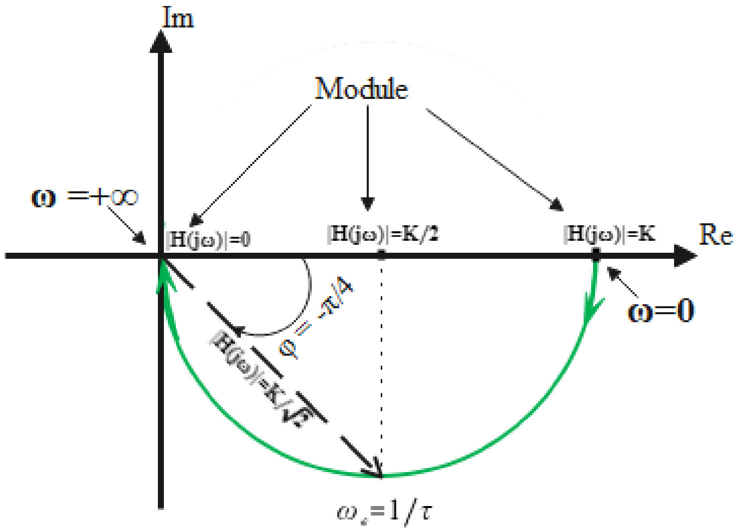


Figure VI.6. Nyquist diagram for a first-order system

VI.6.3. The Nyquist diagram for a second-order system

We will follow the same procedure for the first-order to draw the Nyquist diagram for the second-order system. Knowing that the transfer function of a second-order system in the complex domain is:

$$H(j\omega)=\frac{K}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)+j\left(2\xi\frac{\omega}{\omega_0}\right)}$$

$$H(j\omega)=\frac{K}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)+j\left(2\xi\frac{\omega}{\omega_0}\right)}\times\frac{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)-j\left(2\xi\frac{\omega}{\omega_0}\right)}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)-j\left(2\xi\frac{\omega}{\omega_0}\right)}$$

$$H(j\omega)=\frac{K\times\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)-j\left(2\xi\frac{\omega}{\omega_0}\right)\right]}{\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\right]^2-\left[j\left(2\xi\frac{\omega}{\omega_0}\right)\right]^2}$$

$$H(j\omega)=\frac{K\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)-jK\left(2\xi\frac{\omega}{\omega_0}\right)}{\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\right]^2-\left[j\left(2\xi\frac{\omega}{\omega_0}\right)\right]^2}$$

$$H(j\omega)=\frac{K\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)-jK\left(2\xi\frac{\omega}{\omega_0}\right)}{\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\right]^2+\left[\left(2\xi\frac{\omega}{\omega_0}\right)\right]^2}$$

$$H(j\omega)=\frac{K\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)}{\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\right]^2+\left[\left(2\xi\frac{\omega}{\omega_0}\right)\right]^2}-j\frac{K\left(2\xi\frac{\omega}{\omega_0}\right)}{\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)\right]^2+\left[\left(2\xi\frac{\omega}{\omega_0}\right)\right]^2}$$

We will get the Nyquist diagram for the second-order system in the same way as before. Figure VI.7 represent the plots obtained from the case studies [4].

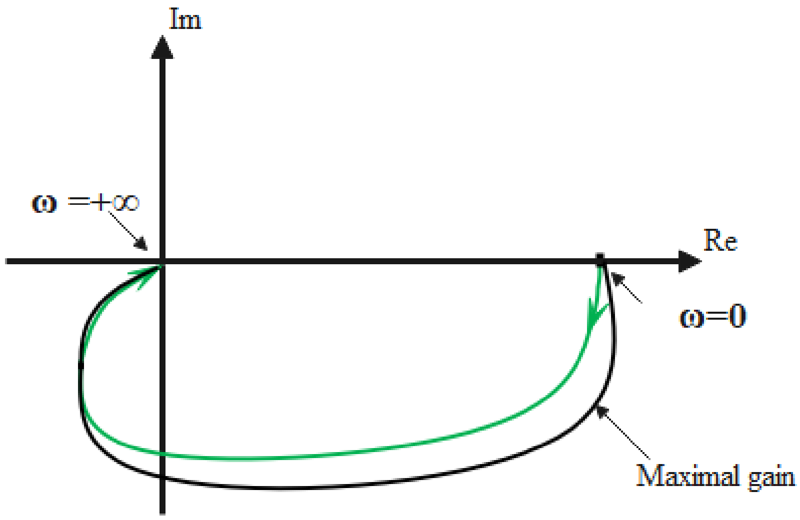
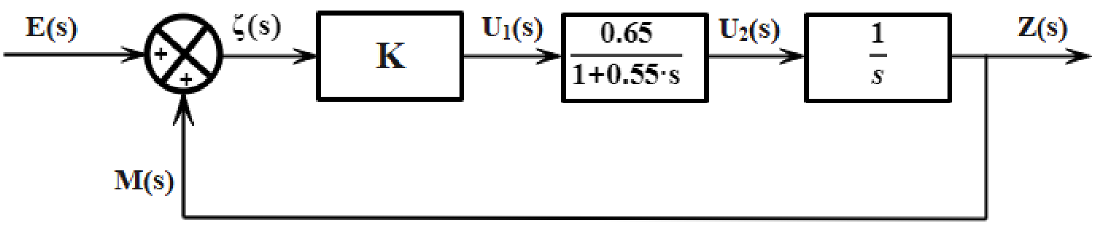


Figure VI.7. Nyquist diagram for the second-order system

VI.7. Learning activity

Exercise N°1:

Let us consider a mechanical system modeled by the following diagram:



Draw the Bode and Nyquist diagram.

Exercise N°2:

The following two transfer functions represent two electromechanical systems:

$$H_1(s)=\frac{221}{s^2+14\cdot s+23}$$

$$H_2(s)=\frac{101}{(1+0.3\cdot s)(1+0.5\cdot s)(1+2.6\cdot s)}$$

Draw the Bode plot (the gain and the phase curves) for these two transfer functions.

Exercise N°3:

Let be the following transfer function:

$$H(s)=\frac{4\cdot s}{1+4\cdot s}$$

Plot the Bode diagram (Gain and Phase) and the Nyquist diagram of this transfer function.

Exercise N°4:

$H(s)$ is the transfer function of a unitary return system:

$$H(s)=\frac{6}{-1+\tau\cdot s} \quad , \quad \tau \succ 1$$

Plot the Nyquist diagram for $G(s)$, which represents $H(s)$ direct chain open-loop transfer function.

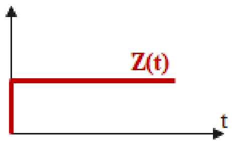
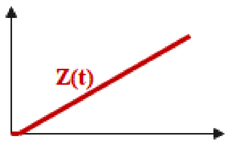
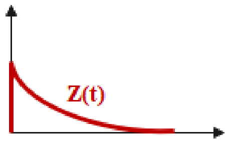
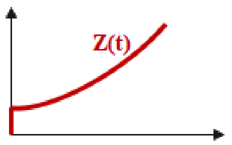
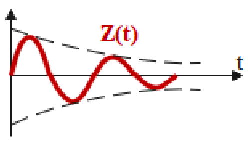
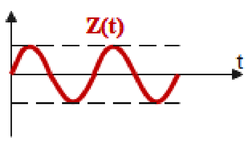
CHAPTER VII: STABILITY STUDY OF A CONTROL SYSTEM

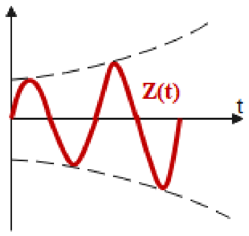
VII.1. Definition

The system is considered stable if it produces a bounded output in response to a bounded input. Stability must be analyzed before evaluating the transient response and steady-state accuracy because if the system is not stable, these performance characteristics cannot be evaluated (stability and accuracy are evaluated only for a stable system).

The stability of the system can be determined from its output response, and the following table can summarize this:

Table VII.1: System Stability and Instability by Output Signal:

Output response signal	Remark
	Stable
	Unstable
	Stable
	Unstable
	Stable
	Marginally stable

	Unstable

VII.2. Stability criteria

Several criteria are available to evaluate the stability of servo systems. In this chapter, we will examine two criteria: the analytical criterion of stability according to Routh-Hurwitz and the graphical Nyquist Bode criteria.

VII.2.1. The analytical criterion of stability according to Routh-Hurwitz

VII.2.1.1. Definition

The Routh-Hurwitz analytical criterion is a mathematical criterion that allows stability to be evaluated by constructing the Routh table with the denominator coefficients of the closed-loop transfer function.

VII.2.1.2. Statement and properties of the analytical criterion of stability according to Routh-Hurwitz

a- Utterance:

The system is stable if and only if all the terms in the first column of the table are strictly positive and there is no sign of change.

b- Properties:

1- Routh's criterion applies for $n \succ 0$.

2- The system is stable if all the coefficients of the first column are positive.

3- The system is unstable if one or more terms in the first column change signs.

4- An additional method is required if we have zeros on an entire row.

VII.2.1.3. Construction of the table for the Routh-Hurwitz stability study

The closed-loop transfer function is given by:

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s^1 + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + a_{n-3} s^{n-3} + a_{n-4} s^{n-4} + a_{n-5} s^{n-5} + a_{n-6} s^{n-6} + \ldots + a_1 s^1 + a_0 s^0}$$

The characteristic polynomial can therefore be written as:

$$D(s)=a_n s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+a_{n-3}s^{n-3}+a_{n-4}s^{n-4}+a_{n-5}s^{n-5}+a_{n-6}s^{n-6}+...+a_1 s^1+a_0 s^0$$

The table will be filled in by the coefficients of the denominator $D(s)$ as follows:

a- In The first line, we write the coefficients associated with the highest powers of $'s'$, followed by all the coefficients s^{n-2k} with $k \succ 0$.

b- On the second line, we deposit the $'s'$ with the lower-order powers, followed by all the coefficients s^{n-2k-1} with $k \succ 0$.

c- The other rows will be filled with terms calculated from these coefficients until the first column is filled.

Applying these steps, we obtain the following Routh table:

Tableau VII.2: Tableau Routh-Hurwitz.

coefficients Position	s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_2	a_0
	s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	a_3	a_1
coefficients calculation	s^{n-2}	c_{n-2}	c_{n-4}	c_{n-6}	$\cdots$	$\cdots$	$\cdots$
	s^{n-3}	d_{n-3}	d_{n-5}	d_{n-7}	$\cdots$	$\cdots$	$\cdots$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	s^0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The 3rd line is filled in by:

$$c_{n-2}=\frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} \xrightarrow{Generally} c_{n-i}=\frac{-\begin{vmatrix} a_n & a_{n-i} \\ a_{n-1} & a_{n-i-1} \end{vmatrix}}{a_{n-1}}$$

$$c_{n-2}=\frac{-\left[\left(a_n \times a_{n-3}\right)-\left(a_{n-1}\times a_{n-2}\right)\right]}{a_{n-1}} \qquad c_{n-i}=\frac{-\left[\left(a_n \times a_{n-i-1}\right)-\left(a_{n-1}\times a_{n-i}\right)\right]}{a_{n-1}}$$

The fourth row is completed as follows:

$$d_{n-3}=\frac{-\begin{vmatrix} a_{n-1} & a_{n-2} \\ c_{n-2} & a_{n-3} \end{vmatrix}}{c_{n-2}} \xrightarrow{Generally} c_{n-j}=\frac{-\begin{vmatrix} a_{n-1} & a_{n-j} \\ c_{n-2} & a_{n-j-1} \end{vmatrix}}{c_{n-2}}$$

$$c_{n-2}=\frac{-\left[\left(a_{n-1}\times a_{n-3}\right)-\left(c_{n-2}\times a_{n-2}\right)\right]}{c_{n-2}} \qquad c_{n-j}=\frac{-\left[\left(a_{n-1}\times a_{n-j-1}\right)-\left(c_{n-2}\times a_{n-j}\right)\right]}{c_{n-2}}$$

VII.2.2. The Graphical stability criteria

Geometric stability criteria use graphical or geometric representations to study the stability of a system in the frequency domain or the complex plane from the open-loop transfer function.

VII.2.2.1. Characteristic equation and critical point

Let be a servo system represented in Figure VII.1.

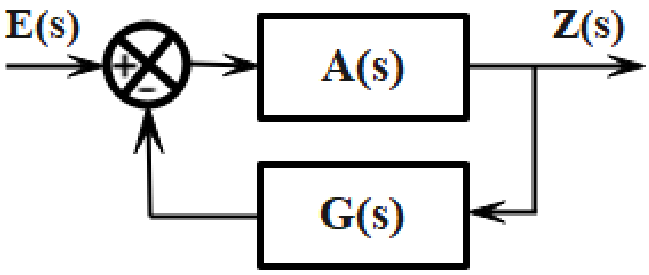


Figure VII.1. Servo system

The closed-loop transfer function of this system is given by:

$$CLTF = \frac{H(s)}{1+G(s)\times H(s)}$$

CLTF : Closed-Loop Transfer Function

We know that $CLTF = G(s)\times H(s)$ (see Chapter III) from which the CLTF takes the following form:

$$CLTF = \frac{H(s)}{1+OLTF(s)}$$

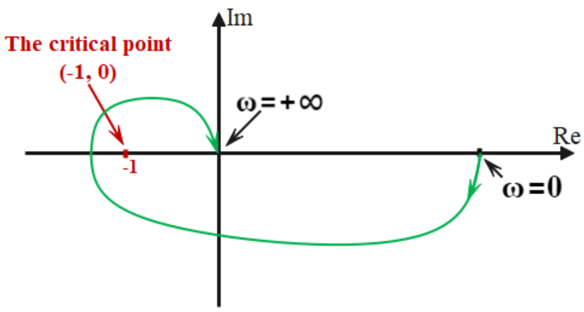
<u>The characteristic equation is therefore</u>: this system is given by $1+CLTF=0$: This system is at the limit of stability when $CLTF(s)=-1$.

<u>The critical point</u>: is the point $z=-1+j0$ with a modulus of 1 and an argument of -180°.

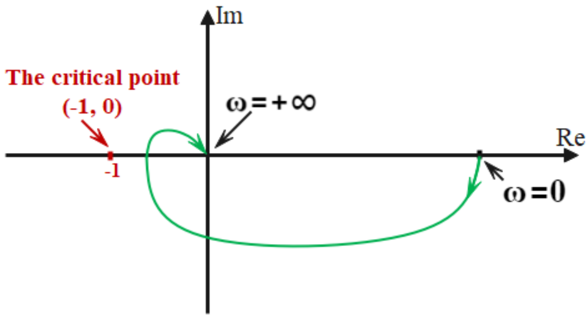
By this, the study of a closed-loop system amounts to evaluating this open-loop system concerning the critical point $(-1,0)$.

VII.2.2.2. Nyquist criterion

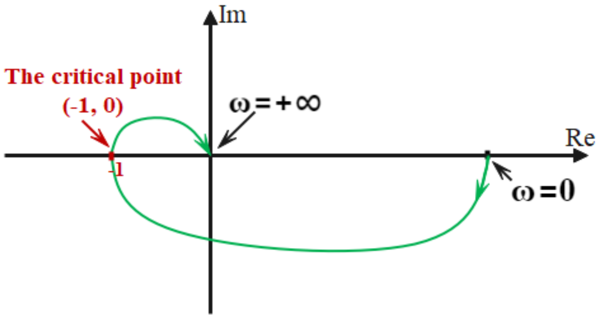
An open-loop (closed) system is stable according to the Nyquist criterion if the critical point $(-1, 0)$ is to the left of the Nyquist diagram when traversed in the direction of increasing angular frequency. Moreover, it is unstable if it is not. The system oscillates if the Nyquist locus (Nyquist plot) passes through the critical point (Figure VII.2).



(a) Unstable system



(b) Stable system



(c) Oscillatory System

Figure VII.2. Nyquist plot for system stability assessment

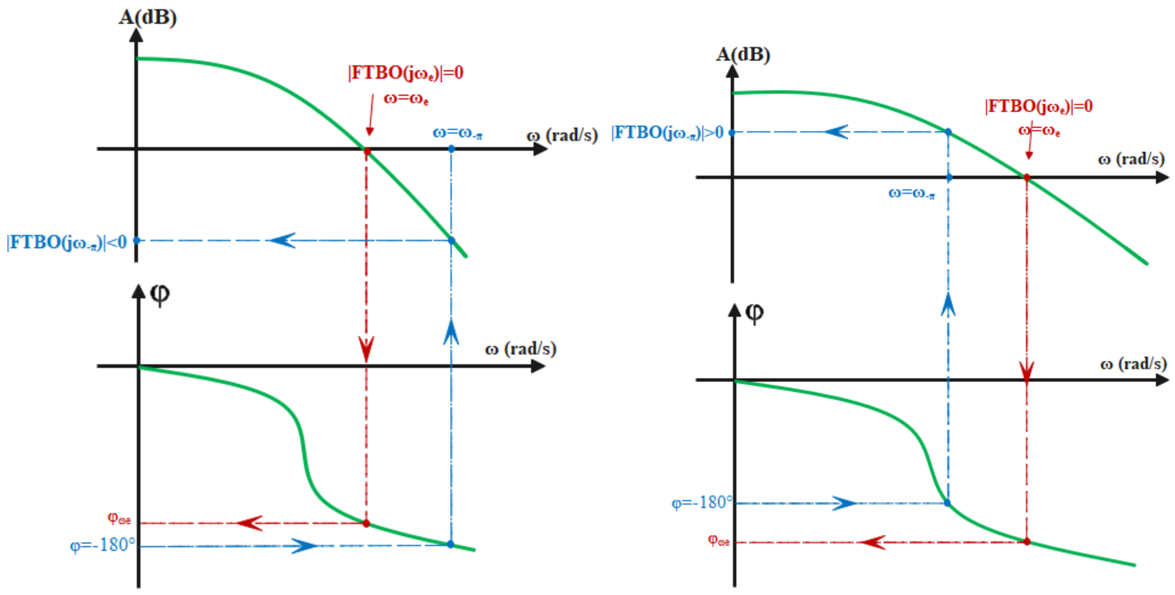
VII.2.2.3. The Bode Stability Criterion

The stability analysis the Bode criterion is more commonly used than the Nyquist criterion. This is because the Bode criterion uses the two graphs of the Bode diagram to study stability.

According to Bode's criterion, an open-loop system is stable if:

- For $\omega = \omega_e$ such that $A(dB) = |FTBO(j\omega_e)| = 0\,dB$ on the graph of the argument (magnitude), we must have on the phase graph: $\varphi_{\omega_e}(°) = \arg(FTBO(j\omega_e))$ with $\varphi_{\omega_e}(°) \succ -180°$.
- For $\omega = \omega_{-\pi}$ such that $\varphi(°) = \arg(FTBO(j\omega_{-\pi})) = -180°$, we must have on the graph of the argument $A(dB) = |FTBO(j\omega_{-\pi})|$ with $A(dB) \prec 0\,dB$.

Figure VII.3 shows the stability criterion according to the Bode diagram.



(a) Stable system **(b)** Unstable system

Figure VII.3. Graphical representation of the Bode stability criterion

VII.2.2.4. Gain margin and phase margin criterion

The gain margin $(MG)$and the phase margin $(M\varphi)$ can be determined from the Bode diagram as illustrated in Figure VII.4. System stability is then assessed as follows:

- The system is stable if: $MG \succ 0, \quad M\varphi \succ 0$
- The system is unstable if: $MG \prec 0, \quad M\varphi \prec 0$

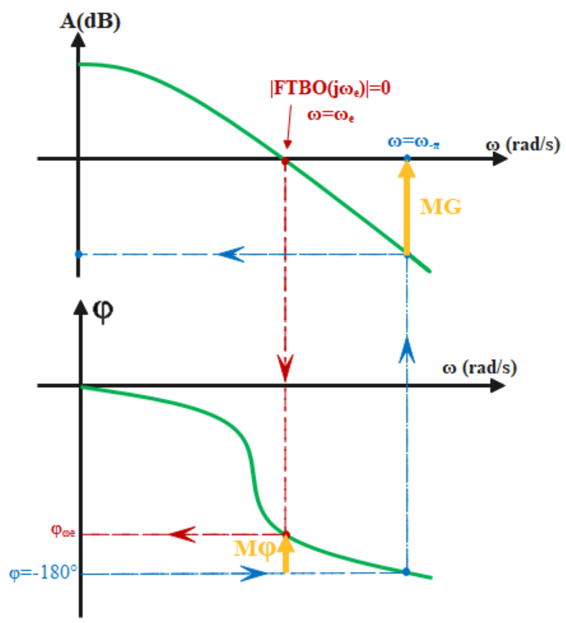


Figure VII.4. Determination of the gain margin and phase margin on the Bode diagram

VII.3. Learning activity

Exercise N°1:

To study the stability of the two systems defined by:

1- $1+H_1(s)=s^5+2\cdot s^4+3\cdot s^3+s+2=0$

2- $1+H_2(s)=5\cdot s^5+10\cdot s^4+13\cdot s^3+20\cdot s^2+s+1=0$

Exercise N°2:

Study the stability of the system represented in Figure 1 in open and closed loops, knowing that $H_1(j\omega)=\dfrac{8}{2\cdot s^3+3\cdot s^2+2\cdot s+1}$ and $H_2(s)=1+\tau\cdot s,\quad \tau=1.7$

Exercise N°3:

1- The following transfer function represents a system:

$H(j\omega)=\dfrac{K}{j\omega(1+j\cdot\tau_1\cdot\omega)(1+j\cdot\alpha\cdot\omega)}$ With $K,\ \tau_1\ and\ \alpha$ positive constants.

Using the Nyquist criterion, determine the condition under which the system is stable and the case where the system is unstable.

2- Using the Nyquist criterion, is the following system stable for $\tau \succ 0$ and $K=4$?

$$H_1=\frac{K}{1+\tau\cdot s}$$

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