People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

University of Abderrahmane Mira - Bejaia
Faculty of Technology
Department of Mechanical Engineering

Mathematics 3

This course is intended to provide second-year science and technology students.

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Preface

This course, Mathematics 3, is designed for second-year Mechanical Engineering students at the University of Bejaia. It provides comprehensive training in numerical series, multiple and improper integrals, and differential equations, while introducing essential tools such as Fourier and Laplace transforms for solving engineering problems.

Students will learn to :

- Apply integration techniques, including substitution and integration by parts;
- Evaluate double and triple integrals, using Fubini's Theorem and appropriate changes of variables;
- Study the convergence of improper integrals;
- Solve differential equations using Laplace and Fourier transforms.

The course emphasizes practical applications in engineering and includes numerous solved exercises to strengthen problem-solving skills. It aims to develop mathematical reasoning, rigor, and independence in tackling integration problems.

Chapter 1

Single Integral

1.1 Antiderivatives

1.1.1 Antiderivative of function

Definition 1.1.1. Let f be a function defined on an interval I of $\mathbb{R}$, an antiderivative of f, is a function F whose derivative is f :

$$\forall\, x \in I,\;\; F'(x) = f(x).$$

Remark 1.1.1. If F is antiderivative of f on interval I then for any real number c, $F + c$ is also an antiderivative of f on I and we write :

$$\int f(x)\, dx = F(x) + c.$$

Example 1.1.1.

- An antiderivative of $f(x) = 3x^2$ is $F(x) = x^3 + c$, and we write:

$$\int f(x)\, dx = \int 3x^2\, dx = x^3 + c, \quad c \in \mathbb{R} \quad \text{(constant of integration)}.$$

- An antiderivative of $f(x) = e^{2x}$ is $F(x) = \frac{1}{2}e^{2x} + c$, and we write:

$$\int f(x)\, dx = \int e^{2x}\, dx = \frac{1}{2}e^{2x} + c, \quad c \in \mathbb{R}.$$

Remark 1.1.2. The antiderivative of a function is not unique, it includes an arbitrary constant.

Example 1.1.2. Let the function f (x) = 4x + 2. We have :

$$F_1(x) = 2x^2 + 2x,\;\; F_2(x) = 2x^2 + 2x - 1,\;\; F_3(x) = 2x^2 + 2x + 2025,\; ...$$

are antiderivatives of the function f on $\mathbb{R}$, because :

$$F_1^{'}(x) = F_2'(x) = F_3'(x) = f(x).$$

1.1.2 properties of antiderivatives

Let F and G be antiderivatives of f and g respectively on an interval I of $\mathbb{R}$:

1. $\displaystyle\int \Big(f(x) + g(x)\Big)\, dx = \int f(x)\, dx + \int g(x)dx = F(x) + G(x) + c, \quad c \in \mathbb{R}.$

2. $\displaystyle\int \alpha f(x)\, dx = \alpha \int f(x)\, dx = \alpha F(x) + c, \quad \forall\alpha \in \mathbb{R}.$

Remark 1.1.3.

⚠ In general, the integral of a product is not equal to the product of the integrals.

$$\int \Big(f(x) \times g(x)\Big)\, dx \neq \int f(x)\, dx \times \int g(x)\, dx.$$

for example

$$\int (x^2)(2x)\, dx = \frac{1}{2}x^4 + c \neq \int x^2\, dx \times \int 2x\, dx = \frac{1}{3}x^3 \times x^2 = \frac{1}{3}x^5 + c.$$

$$\int (x^2).(2x)\, dx \neq \int x^2\, dx \times \int 2x\, dx.$$

1.1.3 Table of common antiderivatives

We could use a table of derivatives to find integrals, but the more common ones are usually found in a "Table of Integrals" such as that shown below. You could check the entries in this table using your knowledge of differentiation. Try this for yourself :

Defined on	Function f	$F(x) = \int f(x)dx$
$\mathbb{R}^*$	$\begin{cases} x^\alpha, & (\alpha \in \mathbb{R}\backslash\{-1\} \\ \frac{1}{x}, & \alpha = -1 \end{cases}$	$\begin{cases} \frac{x^{\alpha+1}}{\alpha+1} + C, \\ \ln\lvert x\rvert + C, \end{cases}$
$\mathbb{R}$	$\frac{1}{1+x^2}$	$\arctan x + C$
$\mathbb{R}\backslash\{-1, 1\}$	$\frac{1}{1-x^2}$	$\frac{1}{2}\ln\lvert\frac{1+x}{1-x}\rvert + C$
$\mathbb{R}$	$a^x\ (a > 0)$	$\frac{a^x}{\ln a} + C$
$\mathbb{R}$	$e^{\alpha x}$	$\frac{1}{\alpha}e^{\alpha x} + C$
$\mathbb{R}$	$\sin(\omega x + \alpha),\ (w \neq 0)$	$-\frac{1}{\omega}\cos(\omega x + \alpha) + C$
$\mathbb{R}$	$\cos(\omega x + \alpha),\ (w \neq 0)$	$\frac{1}{\omega}\sin(\omega x + \alpha) + C$
$\mathbb{R}\backslash\{\frac{\pi}{2} + k\pi\}$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\tan x + C$
$\mathbb{R}\backslash\{\frac{\pi}{2}\}$	$\frac{1}{\sin^2 x} = 1 + \coth^2 x$	$-\frac{1}{\tan x} + C$
$\mathbb{R}\backslash\{\frac{\pi}{2} + k\pi\}$	$\tan x$	$\ln\lvert\cos x\rvert + C$
$]-1, 1[$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$

1.1.4 Antiderivative and composition

$g(x)$	$G(x) = \int g(x)dx$
$f'(x)e^{f(x)}$	$e^{f(x)} + c$
$\frac{f'(x)}{f(x)}$	$\ln(\lvert\ f(x)\ \rvert) + c$
$f'(x)\frac{1}{\sqrt{f(x)}}$	$2\sqrt{f(x)} + c$
$f'(x)(f(x))^n$	$\frac{(f(x))^{n+1}}{n+1} + c$
$f'(x)g(f(x))$	$G(f(x)) + c$, G is antiderivative of g

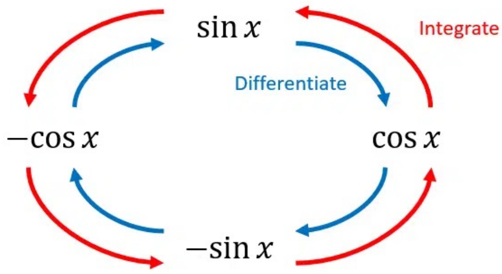


Figure 1.1: Integration differentiation cycle for Sine and Cosine.

Useful trigonometric formulas

- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

Example 1.1.3. Calculate the following antiderivatives :

1. $\displaystyle\int x^2 e^{x^3+1}\,dx = \frac{1}{3}\int 3x^2 e^{x^3+1}\,dx = \frac{1}{3}e^{x^3+1} + c,\ c \in \mathbb{R}.$
2. $\displaystyle\int \frac{\sin x}{1+\cos x}\,dx = -\int -\frac{\sin x}{1+\cos x}\,dx = -\ln|1+\cos x| + c.$
3. $\displaystyle\int \frac{1}{\sqrt{4x+1}}\,dx = \frac{1}{4}\int 4\frac{1}{\sqrt{4x+1}}\,dx = \frac{1}{4}(2\sqrt{4x+1}) + c = \frac{1}{2}\sqrt{4x+1} + c.$
4. $\displaystyle\int (9x+2)^5\,dx = \frac{1}{9}\int 9(9x+2)^5\,dx = \frac{1}{9}\frac{1}{6}(9x+2)^6 + c = \frac{(9x+2)^6}{54} + c.$
5. $\displaystyle\int x\sin(x^2+1)\,dx = \frac{1}{2}\int 2x\sin(x^2+1)\,dx = -\frac{1}{2}\cos(x^2+1) + c.$

1.2 Definite integral

1.2.1 Single integrals

Proposition 1.2.1. *Let f be a continuous function on the closed interval $[a,b]$. If F is an antiderivative of f on $[a,b]$, then*

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

This is called the definite integral of $f(x)$ between the limits a and b. Every definite integral has a unique numerical value.

Remark 1.2.1. There is a difference between the definite integral and the indefinite integral of a function :

1. $\int f(x)\,dx$ is called an indefinite integral of function f, it is an antiderivative of f.

2. $\int_a^b f(x)\,dx$ is called a definite integral of function f, it is a real number.

Remark 1.2.2. The integral $\int_a^b f(x)\,dx$ can represent an "algebraic area". It is positive if $f \geq 0$ and negative if $f \leq 0$.

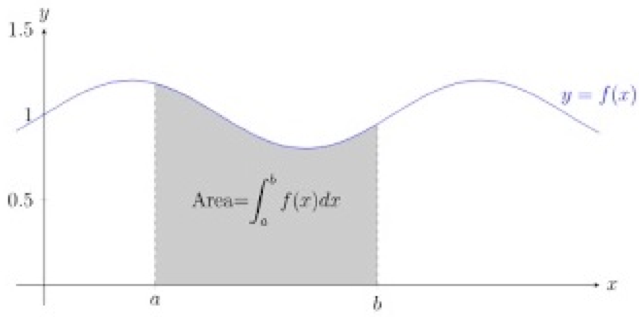


Figure 1.2: Representation of a definite integral.

Remark 1.2.3.

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a).$$

Proposition 1.2.2. *Let $f,\ g$ be two integrable functions on the interval $[a,b]$ and let $k \in \mathbb{R}$. Then :*

1. *$f+g$, $f.g$ and kf are integrable functions on $[a,b]$.*

2. $\int_a^b kf(x)\,dx = k\int_a^b f(x)dx.$

3. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)dx.$

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a<c<b.$

6. *If* $f(x)=0,\ \forall\, x\in[a,b],$ *then* $\int_a^b f(x)\,dx = 0.$

7. *if* $f(x)\le g(x),\ \forall\, x\in[a,b],$ *then* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$

8. $|\int_a^b f(x)\,dx| \le \int_a^b |f(x)|\,dx.$

Example 1.2.1. Compute the following single integrals

1. $\int_1^2 3x^2\,dx.$

$$\int_1^2 3x^2\,dx = 3\int_1^2 x^2\,dx = 3\left[\frac{x^3}{3}\right]_1^2 = \left[x^3\right]_1^2 = 7.$$

Using the formula :

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}+c.$$

2. $\int_0^{\frac{1}{2}} \frac{2x}{\sqrt{1-x^2}}\,dx.$

$$-\int_0^{\frac{1}{2}} -\frac{2x}{\sqrt{1-x^2}}\,dx = \left[-2\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \left(-2\sqrt{\frac{3}{4}}\right) - \left(-2\sqrt{1}\right) = 2-\sqrt{3}.$$

Using the formula :

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+c.$$

Second Method :

$$-\int_0^{\frac{1}{2}} -\frac{2x}{\sqrt{1-x^2}}\,dx = -\int_0^{\frac{1}{2}} -2x(1-x^2)^{-\frac{1}{2}}\,dx = -\left[\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_0^{\frac{1}{2}} = \left[-2\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = 2-\sqrt{3}.$$

Using the formula :

$$\int f'(x)f^n(x)\,dx = \frac{f^{n+1}(x)}{n+1}+c.$$

3. $\int_0^{\frac{\pi}{3}} \cos(3x)\,dx.$

$$\int_0^{\frac{\pi}{3}} \cos(3x)\,dx = \frac{1}{3}\int_0^{\frac{\pi}{3}} 3\cos(3x)\,dx = \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{3}} = \frac{1}{3}\sin(\pi) - \frac{1}{3}\sin(0) = 0$$

Using the formula :

$$\int f'(x)\,g(f(x))\,dx = G(f(x)) + c, \quad \text{where } G \text{ is an antiderivative of } g.$$

4. $\int_0^2 \frac{3x+1}{(3x^2+2x-2)^2} dx.$

$$\int_0^2 \frac{3x+1}{(3x^2+2x-2)^2}\,dx = \frac{1}{2}\int_0^2 2(3x+1)(3x^2+2x-2)^{-2}\,dx$$
$$= \frac{1}{2}\left[-\frac{1}{3x^2+2x-2}\right]_0^2 = \frac{1}{2}\left(-\frac{1}{14} - \frac{1}{2}\right) = \frac{-2}{7}$$

Using the formula :

$$\int f'(x)f^n(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c.$$

1.2.2 Methods for calculating antiderivatives and integrals

A) Integration by Parts : Integration by parts is a technique that comes from the product rule of differentiation.

If $u(x)$ and $v(x)$ are differentiable functions with continuous derivatives on an interval, then

$$(uv)' = u'v + uv', \quad \text{where } u \text{ and } v \text{ are of class } C^1.$$

Formula for indefinite integrals :

From the product rule, we get

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx = F(x) + c, \qquad c \in \mathbb{R}.$$

Formula for definite integrals :

If $u(x)$ and $v(x)$ have continuous derivatives on $[a,b]$,

$$\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx = F(b) - F(a).$$

Example 1.2.2. Compute the following integrals :

1. $\int_0^1 xe^{-x}\,dx.$

$$\begin{aligned} u(x) = x &\overset{\mathrm{d}}{\rightarrow} u'(x) = 1\,dx \\ v'(x) = e^{-x}\,dx &\underset{\int}{\rightarrow} v(x) = -e^{-x} \end{aligned}$$

we apply the formula :

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)\,dx,$$

$$\int_0^1 xe^{-x}\,dx = \Big[-xe^{-x}\Big]_0^1 + \int_0^1 e^{-x}\,dx = \Big[-xe^{-x}\Big]_0^1 + \Big[-e^{-x}\Big]_0^1$$
$$= (-e^{-1}+0)(-e^{-1}+1) = -2e^{-1}+1.$$

2. $\int x(\ln x)^2\,dx.$

$$\begin{aligned} u(x) = (\ln x)^2 &\longmapsto u'(x) = 2.\frac{1}{x}.\ln x\,dx, \\ v'(x) = x\,dx &\longmapsto v(x) = \frac{1}{2}x^2, \end{aligned}$$

We integrate by parts :

$$\int x(\ln x)^2 dx\,dx = \Big[\frac{1}{2}x^2(\ln x)^2\Big] - \int x\ln x\,dx$$

Step 2 : Second integral $\int x\ln x\,dx$

$$\begin{aligned} u(x) = \ln x &\longmapsto u'(x) = \frac{1}{x}\,dx, \\ v'(x) = x\,dx &\longmapsto v(x) = \frac{1}{2}x^2, \end{aligned}$$

We integrate by parts again :

$$\begin{aligned} \int x\ln x\,dx &= \Big[\frac{1}{2}x^2\ln x\Big] - \frac{1}{2}\int x\,dx \\ &= \Big[\frac{1}{2}x^2\ln x - \frac{1}{2}.\frac{1}{2}x^2 + c\Big] \end{aligned}$$

Finally, we substitute this result back into the original integral :

$$\begin{aligned} \int x(\ln x)^2\,dx &= \Big[\frac{1}{2}x^2(\ln x)^2\Big] - \Big[\frac{1}{2}x^2\ln x - \frac{1}{2}.\frac{1}{2}x^2\Big] + c \\ &= \frac{x^2}{2}((\ln x)^2 - \ln x - \frac{1}{2}) + c, c \in \mathbb{R} \end{aligned}$$

B) Change of variables : A change of variables (also called **substitution**) is a method for evaluating integrals by introducing a new variable that simplifies the integrand or the bounds of integration. We replace the original variable x with a new variable t defined by :

$$x = \varphi(t), \qquad dx = \varphi'(t)\,dt$$

where :

- φ is a differentiable function ;
- $\varphi'(t)$ is continuous on the interval considered;
- φ is one-to-one (invertible) on that interval.

Formula for indefinite integrals : If G is an antiderivative of g and c is the constant of integration, then :

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt = \int g(t)\,dt = G(t) + c = G(\varphi^{-1}(x)) + c.$$

Formula for definite integrals : Let f be a continuous function on the interval $[a, b]$,

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = G(\beta) - G(\alpha),$$

where :
$\alpha = \varphi^{-1}(a)$, $\beta = \varphi^{-1}(b)$and φ is invertible on $[\alpha, \beta]$ and differentiable on $]\alpha, \beta[$.
Here, α and β are the new limits of integration for the variable t.

Example 1.2.3. Compute the following integrals :

1. $\displaystyle\int_0^1 2x\cos(x^2)\,dx$
   We set $t = x^2$, whose differential is $dt = 2x\,dx$.

   We change the bounds :

   $$x = 0 \longrightarrow t = 0, \quad x = 1 \longrightarrow t = 1$$

   Then

   $$\int_0^1 \cos(t)\,dt = \Big[\sin(t)\Big]_0^1 = \sin(1) - \sin(0) = \sin(1).$$

2. $\displaystyle\int_2^3 \frac{1}{x\ln x}\,dx$

   $$t = \ln x \quad \Rightarrow \quad dt = \frac{1}{x}\,dx$$

   Also, for the bounds :

   $$x = 2 \longrightarrow t = \ln 2, \quad x = 3 \longrightarrow t = \ln 3$$

   Hence,

   $$\begin{aligned}\int_2^3 \frac{1}{x\ln x}\,dx &= \int_{\ln 2}^{\ln 3} \frac{1}{t}\,dt = \Big[\ln|t|\Big]_{\ln 2}^{\ln 3} \\ &= \ln|\ln 3| - \ln|\ln 2| = \ln\left(\frac{\ln 3}{\ln 2}\right).\end{aligned}$$

C) Integration of rational functions

C.1) Euclidean Division $\quad (\deg(P) \geq \deg(Q))$

When the degree of the numerator is greater than or equal to the degree of the denominator, perform the Euclidean division first :

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $\deg(R) < \deg(Q)$. Then reduce the integral to the form

$$\int \left(S(x) + \frac{mx+n}{x^2+px+q} \right) dx.$$

Example 1.2.4.

$$I = \int \frac{x^2+3x+1}{x+1}\, dx.$$

$$\frac{x^2+3x+1}{x+1} = x+2-\frac{1}{x+1}.$$

$$I = \int \left(x+2-\frac{1}{x+1} \right) dx = \frac{x^2}{2} + 2x - \ln|x+1| + c.$$

C.2) Double Root $\quad (\Delta = 0)$

$$If\ \ x^2+px+q = (x-\alpha)^2,\ then$$

$$\frac{ax+b}{(x-\alpha)^2} = \frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2}.$$

$$\int \frac{ax+b}{(x-\alpha)^2}\, dx = A\ln|x-\alpha| - \frac{B}{x-\alpha} + c.$$

Example 1.2.5.

$$I = \int \frac{x+1}{(x-1)^2}\, dx.$$

Decompose :

$$\frac{x+1}{(x-1)^2} = \frac{1}{x-1} + \frac{2}{(x-1)^2}.$$

$$I = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx = \ln|x-1| - \frac{2}{x-1} + c.$$

C.3) Two Distinct Roots $\quad (\Delta > 0)$

$$If\ \ x^2+px+q = (x-\alpha)(x-\beta), \qquad \alpha \neq \beta,\ \ then$$

$$\frac{ax+b}{x^2+px+q} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}.$$

$$\int \frac{ax+b}{x^2+px+q}\, dx = A\ln|x-\alpha| + B\ln|x-\beta| + c.$$

Example 1.2.6.

$$I = \int \frac{3x+1}{x^2-x-2}\,dx.$$

$$\textit{Since} \quad x^2-x-2 = (x-2)(x+1),$$

$$\textit{we obtain} \quad \frac{3x+1}{(x-2)(x+1)} = \frac{7}{3(x-2)} + \frac{2}{3(x+1)}.$$

$$I = \frac{7}{3}\ln|x-2| + \frac{2}{3}\ln|x+1| + c.$$

Chapter 2

Double Integral

The double integral of a function $f(x, y)$ defined on a region $D \subset \mathbb{R}^2$ is given by :

$$\begin{aligned} f : D \subset \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x, y) &\mapsto f(x, y) \end{aligned}$$

where f is a real-valued function of two variables, continuous on the region D.
If D is a closed and bounded subset of $\mathbb{R}^2$, delimited by simple curves, then the double integral of f over D is :

$$\iint_D f(x, y)\, dx\, dy$$

Remark 2.0.1.
The concept of a definite integral extends naturally to double and triple integrals for functions of two or three variables. These integrals provide powerful tools to compute volumes, masses, and centroids of regions in two- and three-dimensional space.

Definition 2.0.1. A region D is called **elementary (regular)** if it can be bounded by exactly two graphs of functions, either in terms of x or in terms of y, an example is shown in the following figure :

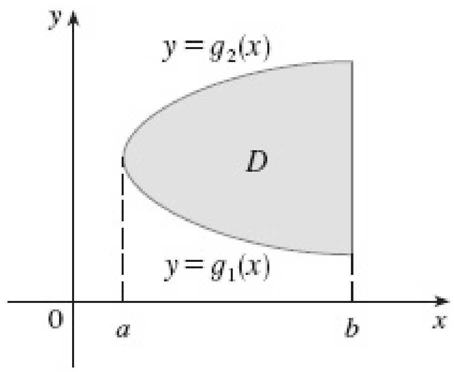


Figure 2.1: Regular region bounded by two curves $g_1(x)$ and $g_2(x)$.

2.1 Properties of the double integral

1. **Area**(D) : The area of the region D is given by :

$$Areaof(D) = \iint\limits_D dx\,dy.$$

Where $f(x,y) = 1$

2. **Positivity** : If $f(x,y) \geq 0$ for $(x,y) \in D$, then :

$$\iint\limits_D f(x,y)\,dx\,dy \geq 0.$$

3. **Linearity** : Let f and g be two functions of two variables that are integrable on D and $\lambda, \mu \in \mathbb{R}$, then :

$$\iint\limits_D \Big[\lambda f(x,y) + \mu g(x,y)\Big]\,dx\,dy = \lambda \iint\limits_D f(x,y)\,dx\,dy + \mu \iint\limits_D g(x,y)\,dx\,dy.$$

4. **Chasles' Relation** : If $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$, (D_1 and D_2 are disjoint, i.e., Area$(D_1 \cap D_2) = 0$), then :

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D_1} f(x,y)\,dx\,dy + \iint\limits_{D_2} f(x,y)\,dx\,dy.$$

5. If α is a constant, then :

$$\iint\limits_D \alpha f(x,y)\,dx\,dy = \alpha \iint\limits_D f(x,y)\,dx\,dy$$

.

6. $\displaystyle\iint\limits_D [f(x,y) + g(x,y)]\,dx\,dy = \iint\limits_D f(x,y)\,dx\,dy + \iint\limits_D g(x,y)\,dx\,dy.$

7. $\displaystyle\left|\iint\limits_D f(x,y)\,dx\,dy\right| \leq \iint\limits_D \Big|f(x,y)\Big|\,dx\,dy.$

2.2 Methods for computing double integrals

In this section we start looking at how to actually compute double integrals.

2.2.1 Double integral over a rectangle

Theorem 2.2.1. ***(Fubini's Theorem)*** *If $f(x,y)$ is continuous on $D = [a,b] \times [c,d]$ then,*

$$\iint\limits_D f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)dy\right)dx = \int_c^d \left(\int_a^b f(x,y)dx\right)dy.$$

These integrals are called ***iterated integrals****.*

Remark 2.2.1. 👉 In this case, the order of integration is not fixed; therefore, we can choose the order that simplifies the calculations.

Example 2.2.1. Compute each of the following double integrals over the indicated rectangles.

1. $I = \iint\limits_D 6xy^2\,dx\,dy,$

   where $D = \{(x, y) \in \mathbb{R}^2 \mid 2 \le x \le 4,\ 1 \le y \le 2\}$.

   Solution 1: Here, dy is the inner differential (i.e., we are integrating with respect to y first), so the inner integral must use the y-bounds. To compute this, we first evaluate the inner integral while keeping the outer integral with its bounds.

$$I = \iint\limits_D 6xy^2\,dx\,dy = \int_2^4 \left(\int_1^2 6xy^2\,dy\right) dx$$

   We integrate with respect to y, holding x constant.

$$I = \int_2^4 \left[6x \cdot \frac{y^3}{3}\right]_{y=1}^{y=2} dx = \int_2^4 14x\,dx = \left[7x^2\right]_{x=2}^{x=4} = 84.$$

   Solution 2: In this case, we integrate with respect to x first and then y.

$$I = \iint\limits_D 6xy^2\,dx\,dy = \int_1^2 \left(\int_2^4 6xy^2\,dx\right) dy$$

   We integrate with respect to x, considering y as a constant.

$$I \;=\; \int_1^2 \left[3x^2y^2\right]_{x=2}^{x=4} dy = \int_1^2 36y^2\,dy = \left[12y^3\right]_{y=1}^{y=2} = 84.$$

2. $I = \iint\limits_D \dfrac{1}{(2x+3y)^2}\,dx\,dy, \qquad \text{where } D = [0, 1] \times [1, 2].$

   Since the order of integration is not important in this situation, we start by integrating with respect to x first, and then with respect to y.

$$\begin{aligned}
I &= \iint\limits_D (2x+3y)^{-2}\,dx\,dy \\
&= \int_1^2 \left(\int_0^1 (2x+3y)^{-2}\,dx\right) dy \quad \left(\text{using } \int f'(x)f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1} + C\right) \\
&= \int_1^2 \left[\frac{-1}{2(2x+3y)}\right]_{x=0}^{x=1} dy = \int_1^2 \left(\frac{-1}{2(2+3y)} + \frac{1}{6y}\right) dy \\
&= \left[-\frac{1}{6}\ln|2+3y| + \frac{1}{6}\ln|y|\right]_{y=1}^{y=2} = -\frac{1}{6}\ln(8) + \frac{1}{6}\ln(2) + \frac{1}{6}\ln(5) \\
&= \frac{1}{6}\ln\left(\frac{5}{4}\right).
\end{aligned}$$

3. $I = \iint_D xe^{xy}\,dx\,dy$, where $D = [-1, 2] \times [0, 1]$.

💡 In this case, it is easier to integrate with respect to y first, since integrating x requires integration by parts.

$$\begin{aligned} I &= \iint_D xe^{xy}\,dx\,dy = \int_{-1}^{2}\left(\int_0^1 xe^{xy}\,dy\right)dx = \int_{-1}^{2}\left[e^{xy}\right]_{y=0}^{y=1}dx \\ &= \int_{-1}^{2}\left(e^x - 1\right)dx = \left[e^x - x\right]_{x=-1}^{x=2} \\ &= (e^2 - 2) - (e^{-1} + 1) = e^2 - e^{-1} - 3. \end{aligned}$$

Proposition 2.2.2. *Let f, f_1 and f_2 be three continuous functions :*

$$\begin{aligned} f : D \subset \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto f(x, y) = f_1(x) \times f_2(y). \end{aligned}$$

Here $D = [a, b] \times [c, d]$. Then :

$$\iint_D f(x, y)\,dx\,dy = \iint_{[a,b]\times[c,d]} (f_1(x) \times f_2(y))\,dx\,dy = \left(\int_a^b f_1(x)\,dx\right)\left(\int_c^d f_2(y)\,dy\right).$$

Example 2.2.2. Compute the following integrals :

1. $I = \iint_D (4 - x^2)\cos y\,dx\,dy$, where $D = \{(x, y) \in \mathbb{R}^2 / 0 \le x \le 2\, and\, \frac{\pi}{4} \le y \le \frac{\pi}{2}\}$.

   We easily see that f is separable (product of two independent functions) :

$$\begin{aligned} I &= \iint_D (4 - x^2)\cos y\,dx\,dy \\ &= \left(\int_0^2 (4 - x^2)\,dx\right)\left(\int_{\pi/4}^{\pi/2} \cos y\,dy\right) \\ &= \left[4x - \frac{x^3}{3}\right]_0^2 \cdot \left[\sin y\right]_{\pi/4}^{\pi/2} = \frac{16}{3}\left(1 - \frac{\sqrt{2}}{2}\right). \end{aligned}$$

2. $I = \iint_D \frac{1}{(1 + x^2)(1 + y^2)}\,dx\,dy$, where $D = [0, 1] \times [0, 1]$.

$$\begin{aligned} I &= \iint_D \frac{1}{(1 + x^2)(1 + y^2)}\,dx\,dy \\ &= \int_0^1 \frac{1}{(1 + x^2)}\,dx. \int_0^1 \frac{1}{(1 + y^2)}\,dy \\ &= \left[\arctan x\right]_{x=0}^{x=1} . \left[\arctan y\right]_{y=0}^{y=1} \\ &= (\arctan 1 - \arctan 0).(\arctan 1 - \arctan 0) \\ &= \left(\frac{\pi}{4} - 0\right).\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{16}. \end{aligned}$$

2.2.2 Double integrals over general regions

In practice, regions of integration are not always rectangles; they may be irregular or bounded by graphs (curves). To compute the double integral over such regions, we proceed as follows :

1 . Describe the region D

A region D in $\mathbb{R}^2$ can usually be described in one of two ways :

- **Region of type 1 (vertical slices) :**

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}$$

where f_1 and f_2 are continuous functions of x. Here x varies between constants a and b; for each fixed x, then y varies between two graphs $y = f_1(x)$ **(lower)** and $y = f_2(x)$ **(upper)**.

So we apply Fubini's Theorem

$$\iint_D f(x, y)\, dx\, dy = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x, y)\, dy \right) dx$$

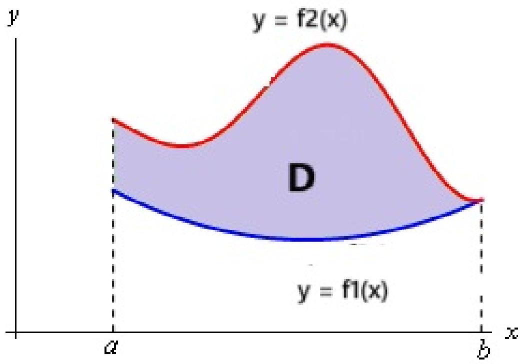


Figure 2.2: Type 1 - Area between graphs $y = f_1(x)$ and $y = f_2(x)$.

- **Region of type 2 (Horizontal slices) :**

$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ g_1(y) \leq x \leq g_2(y)\}$$

where g_1 and g_2 are continuous functions of y. Here y varies between constants c and d; for each fixed y, then x varies between two graphs $x = g_1(y)$ **(left)** and $x = g_2(y)$ **(right)**. So we apply Fubini's Theorem

$$\iint_D f(x, y)\, dx\, dy = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x, y)\, dx \right) dy$$

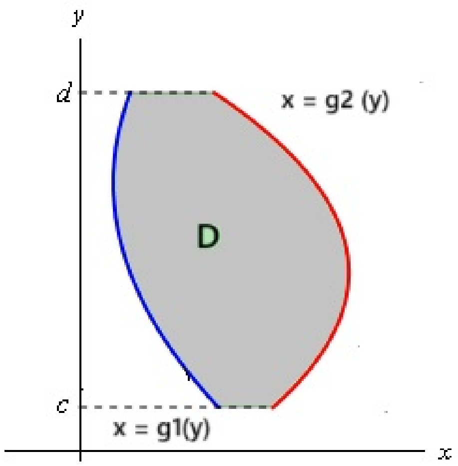


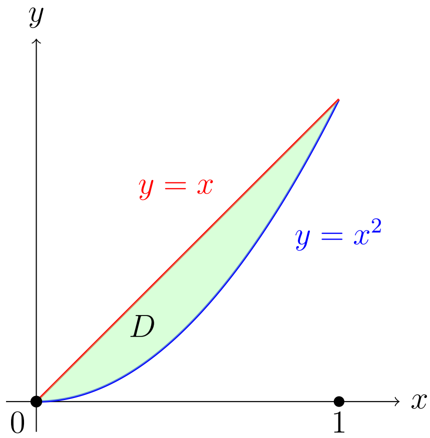
Figure 2.3: Type 2 - Area between graphs $x = g_1(y)$ and $x = g_2(y)$.

For finding the bounds of D :

Fix x between two constants then y varies between the two graphs of x
or
Fix y between two constants then x varies between the two graphs of y.

Example 2.2.3. Compute the following integrals :

1. $I = \iint_D (x+y)\,dx\,dy$, where $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le x\}$.



Since the region D is written in **type 1**, then we obtain :

$$\begin{aligned} I &= \iint_D (x+y)\,dx\,dy = \int_0^1 \left(\int_{x^2}^{x} (x+y)\,dy \right) dx = \int_0^1 \left[xy + \frac{1}{2}y^2 \right]_{y=x^2}^{y=x} dx \\ &= \int_0^1 \left(\frac{3}{2}x^2 - x^3 - \frac{1}{2}x^4 \right) dx = \left[\frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{10}x^5 \right]_{x=0}^{x=1} = \frac{3}{20}. \end{aligned}$$

2. $I = \iint_D xy\,dx\,dy$, where $D = \{(x,y) \in \mathbb{R}^2 \mid y \ge x \text{ and } y^2 \le x\}$.

Identify the graphs

To find the points of intersection, we solve : $x = \sqrt{x}$.

Squaring both sides gives : $x^2 = x$.

$$x^2 - x = 0 \quad \Longrightarrow \quad x(x-1) = 0, \quad x = 0 \quad \text{or} \quad x = 1.$$
$$For\ x = 0\ \Rightarrow\ y = 0,\ and\ for\ x = 1\ \Rightarrow\ y = 1.$$

Describe the region D

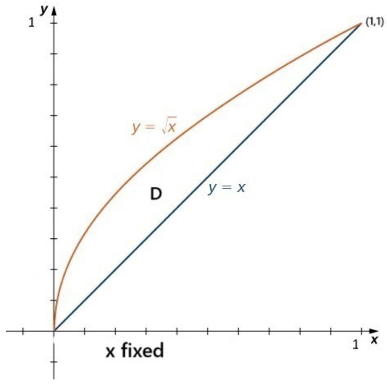


Figure 2.4: **Type 1 :** y varies between the two curves $y = x$ and $y = \sqrt{x}$.

$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ x \leq y \leq \sqrt{x}\}$.

Apply Fubini's Theorem

$$I = \iint_D xy\,dx\,dy = \int_0^1 \left(\int_x^{\sqrt{x}} xy\,dy \right) dx$$
$$= \int_0^1 x \left[\frac{y^2}{2} \right]_x^{\sqrt{x}} dx$$
$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{2} \right) dx$$
$$= \left[\frac{x^3}{6} - \frac{x^4}{8} \right]_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$$

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1,\ y^2 \leq x \leq y\}.$$

$$I = \iint_D xy\,dx\,dy = \int_0^1 \left(\int_{y^2}^{y} xy\,dx \right) dy = \int_0^1 y \left[\frac{x^2}{2} \right]_{x=y^2}^{x=y} dy$$
$$= \int_0^1 y \left(\frac{y^2}{2} - \frac{y^4}{2} \right) dy = \int_0^1 \left(\frac{y^3}{2} - \frac{y^5}{2} \right) dy = \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_{y=0}^{y=1} = \frac{1}{2} \left(\frac{1}{4} - \frac{1}{6} \right) = \frac{1}{24}.$$
$$Hence,\quad \iint_D xy\,dx\,dy = \int_0^1 \left(\int_{y^2}^{y} xy\,dx \right) dy = \int_0^1 \left(\int_x^{\sqrt{x}} xy\,dy \right) dx = \frac{1}{24}.$$

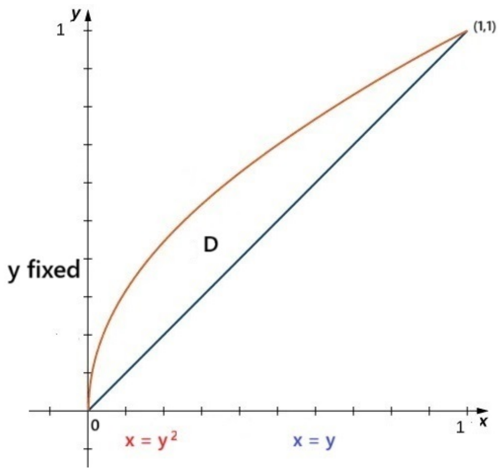
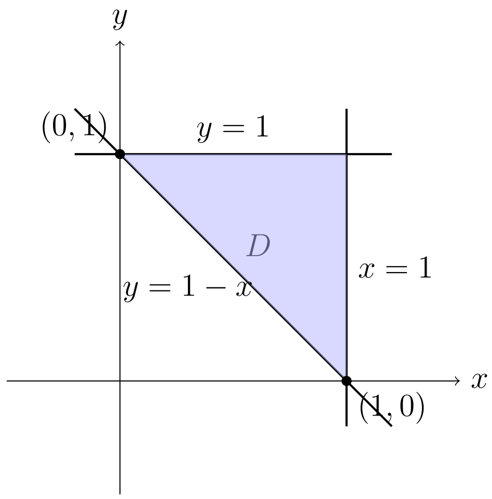


Figure 2.5: **Type 2 :** x varies between the two curves $x = y^2$ and $x = y$.

3. $I = \iint\limits_D \frac{1}{(x+y)^3}\,dx\,dy$, where $D = \{(x,y) \in \mathbb{R}^2 / x \leq 1,\ y \leq 1 \textit{ and } x+y \geq 1\}$.

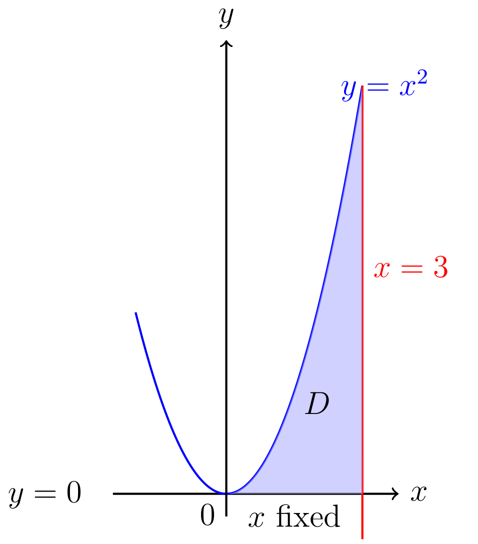


$D = \{(x,y) \in \mathbb{R}^2 / 0 \leq x \leq 1,\ 1-x \leq y \leq 1\}$.
or
$D = \{(x,y) \in \mathbb{R}^2 / 0 \leq y \leq 1,\ 1-y \leq x \leq 1\}$.

$$I = \iint\limits_D \frac{1}{(x+y)^3}\,dx\,dy = \int_0^1 \left(\int_{1-x}^1 \frac{1}{(x+y)^3}\,dy\right) dx = \int_0^1 \left(\int_{1-x}^1 (x+y)^{-3}\,dy\right) dx$$
$$= \int_0^1 \left[-\frac{1}{2(x+y)^2}\right]_{y=1-x}^{y=1} dx = \int_0^1 \left(-\frac{1}{2(x+1)^2} + \frac{1}{2}\right) dx$$
$$= \frac{1}{2}\int_0^1 \left(1 - \frac{1}{(x+1)^2}\right) dx = \frac{1}{2}\left[x + \frac{1}{x+1}\right]_{x=0}^{x=1} = \frac{1}{4}.$$

👉 Continue the example where $D = \{(x,y) \in \mathbb{R}^2 / 0 \leq y \leq 1,\ 1-y \leq x \leq 1\}$.

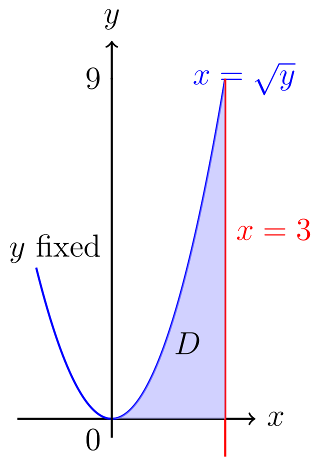
4. $I = \iint\limits_D 2x \sin y \, dx\, dy$, where $D = \{(x, y) \in \mathbb{R}^2 / x \leq 3,\ and\ y \leq x^2\}$.



$D = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 3,\ 0 \leq y \leq x^2\}$.

$$
\begin{aligned}
I &= \iint\limits_D 2x \sin y \, dx\, dy, \quad D = \{(x, y) \mid 0 \leq y \leq x^2,\, 0 \leq x \leq 3\} \\
&= \int_0^3 \int_0^{x^2} 2x \sin y \, dy\, dx = \int_0^3 2x \Big[-\cos y \Big]_{y=0}^{y=x^2} dx = \int_0^3 2x(1 - \cos x^2)\, dx \\
&= \underbrace{\int_0^3 2x\, dx}_{A} - \underbrace{\int_0^3 2x \cos x^2 \, dx}_{B} \\
A &= \int_0^3 2x\, dx = [x^2]_0^3 = 9 \\
B &= \int_0^3 2x \cos x^2 \, dx, \quad \text{let } t = x^2,\ dt = 2x\, dx \\
&= \int_0^9 \cos t\, dt = [\sin t]_0^9 = \sin 9 \\
I &= 9 - \sin 9.
\end{aligned}
$$

or



$D = \{(x, y) \in \mathbb{R}^2 / 0 \le y \le 9,\ \sqrt{y} \le x \le 3\}.$

$$I = \iint\limits_D 2x \sin y \, dx \, dy, \quad D = \{(x, y) \mid 0 \le y \le 9,\ \sqrt{y} \le x \le 3\}$$

$$= \int_0^9 \int_{\sqrt{y}}^3 2x \sin y \, dx \, dy = \int_0^9 \sin y \left[x^2\right]_{x=\sqrt{y}}^{x=3} dy = \int_0^9 (9 - y) \sin y \, dy$$

$$= \underbrace{\int_0^9 9 \sin y \, dy}_{A} - \underbrace{\int_0^9 y \sin y \, dy}_{B}.$$

$$A = 9 \int_0^9 \sin y \, dy = 9\Big[-\cos y\Big]_0^9 = 9(1 - \cos 9),$$

$B = \displaystyle\int_0^9 y \sin y \, dy,$ **Using integration by parts :** $u = y,\ dv = \sin y \, dy.$

$$B = \Big[-y \cos y\Big]_0^9 - \int_0^9 (-\cos y) \, dy = -9 \cos 9 + \int_0^9 \cos y \, dy$$

$$= -9 \cos 9 + \Big[\sin y\Big]_0^9 = -9 \cos 9 + \sin 9,$$

$$I = A - B = 9(1 - \cos 9) - \Big(-9 \cos 9 + \sin 9\Big) = 9 - \sin 9.$$

2.2.3 Area calculation of standard shapes with double integrals (via vertices)

D is Square or Rectangle

Example 2.2.4. Let D be the square with vertices : $A(0, 0)$, $B(0, 2)$, $C(2, 2)$ and $E(2, 0)$.

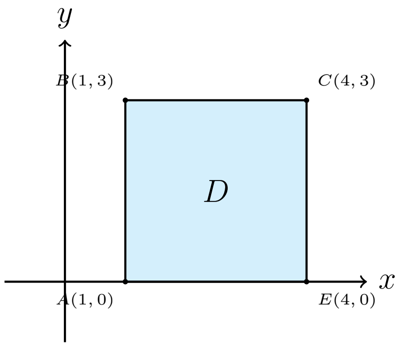
1. Draw the region D.
2. Write the mathematical expression of the region D.

3. Evaluate the double integral :

$$I = \iint_D (x^2 + y^2)\, dx\, dy.$$

Solution

1. Representation of the region D



2. The region D can be written as :

$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 4 \text{ and } 0 \leq y \leq 3\}.$$

3. Evaluate the following double integral :

$$I = \iint_D (x^2 + y^2)\, dx\, dy = \int_1^4 \left(\int_0^3 (x^2 + y^2)\, dy \right) dx$$

$$= \int_1^4 \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=3} dx = \int_1^4 \left(3x^2 + \frac{27}{3} \right) dx = \int_1^4 (3x^2 + 9)\, dx$$

$$= \left[x^3 + 9x \right]_{x=1}^{x=4} = (64 + 36) - (1 + 9) = 90.$$

D is Triangle

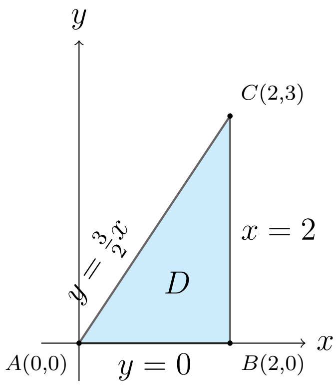
Example 2.2.5. Let D be the triangle with vertices : $A(0, 0)$, $B(2, 0)$ and $C(2, 3)$.

1. Draw the region D.
2. Write the mathematical expression of the region D.
3. Evaluate the double integral :

$$I = \iint_D \frac{x}{\sqrt{y}}\, dx\, dy.$$

Solution

1. Representation of the region D :



2. Expression of the region D :

We have $y = ax + b$ the equation of the line, we determine a and b by using the coordinates of points A and C.

Since $A(0,0) \in (AC)$, we have

$$0 = a \cdot 0 + b \implies b = 0.$$

Since $C(2,3) \in (AC)$, we get

$$3 = a \cdot 2 + 0 \implies a = \frac{3}{2}.$$

Thus, the equation of the line (AC) is :

$$y = \frac{3}{2}x.$$

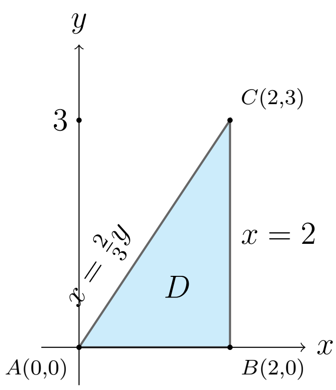
Thus the region is :

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le \frac{3}{2}x\}.$$

3. Evaluate the double integral :

$$\begin{aligned} I = \iint_D \frac{x}{\sqrt{y}}\, dx\, dy &= \int_0^2 \int_0^{\frac{3}{2}x} \frac{x}{\sqrt{y}}\, dy\, dx = \int_0^2 x\, [2\sqrt{y}]_0^{\frac{3}{2}x}\, dx \\ &= \int_0^2 x \cdot 2\sqrt{\frac{3}{2}x}\, dx = 2\sqrt{\frac{3}{2}} \int_0^2 x^{3/2}\, dx \\ &= 2\sqrt{\frac{3}{2}} \left[\frac{2}{5}x^{5/2}\right]_0^2 = \frac{4}{5}\sqrt{\frac{3}{2}} \cdot 2^{5/2} = \frac{16}{5}\sqrt{3}. \end{aligned}$$

For a fixed y



$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 3,\ \tfrac{2}{3}y \le x \le 2\}.$$

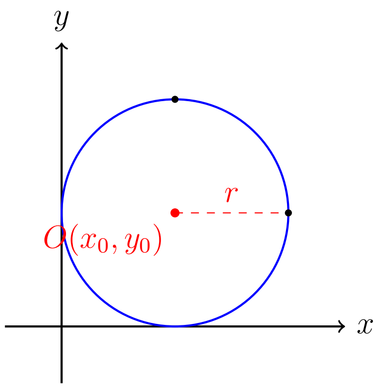
$$\begin{aligned}
I &= \iint\limits_{D} \frac{x}{\sqrt{y}}\,dx\,dy = \int_0^3 \left(\int_{\frac{2}{3}y}^{2} \frac{x}{\sqrt{y}}\,dx \right) dy \\
&= \int_0^3 \frac{1}{\sqrt{y}} \left[\frac{1}{2}x^2 \right]_{x=\frac{2}{3}y}^{x=2} dy = \int_0^3 \left(\frac{2}{\sqrt{y}} - \frac{2}{9}y^{3/2} \right) dy \\
&= 4\Big[\sqrt{y}\Big]_0^3 - \frac{4}{45}\Big[y^{5/2}\Big]_0^3 = 4\sqrt{3} - \frac{4}{5}\sqrt{3} = \frac{16}{5}\sqrt{3}.
\end{aligned}$$

D is a Circle

Summary

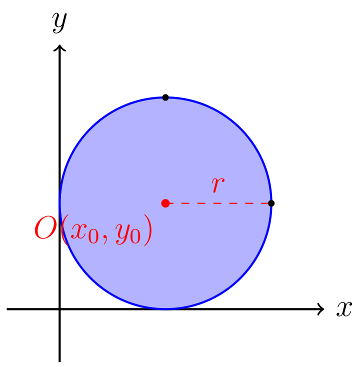
- $(x-x_0)^2 + (y-y_0)^2 = r^2$

Equation of circle of center $O(x_0, y_0)$ and radius r



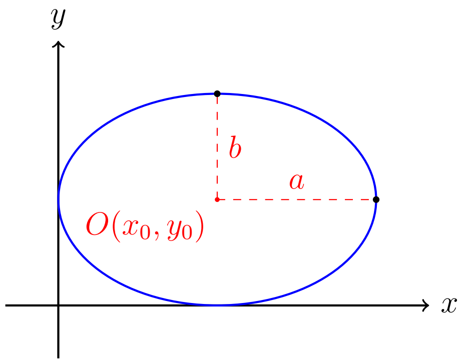
- $(x-x_0)^2 + (y-y_0)^2 \le r^2$

Equation of disk of center $O(x_0, y_0)$ and radius r



- $\dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$

Equation of ellipse with center $O(x_0, y_0)$ and axes a and b



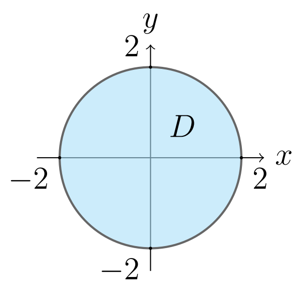
Example 2.2.6. Let D be the disk of radius $r = 2$ centered at the origin $(0,0)$.

1. Draw the region D.
2. Write the mathematical expression of the region D.
3. Evaluate the double integral :

$$I = \iint_D x\,dx\,dy$$

Solution

1. Representation of the region D :



2. Expression of the region D :

$$\begin{aligned} D &= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2,\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\}. \end{aligned}$$

3. Computation of the double integral :

$$\begin{aligned} I &= \iint_D x\,dx\,dy = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx \\ &= \int_{-2}^{2} x \Big[y\Big]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx = \int_{-2}^{2} 2x\sqrt{4-x^2}\,dx = 0. \end{aligned}$$

For a fixed y :

$D = \{(x,y) \in \mathbb{R}^2 \mid -2 \leq y \leq 2,\ -\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}\}.$

$$
\begin{aligned}
I &= \int_{y=-2}^{2} \int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x \, dx \, dy \\
&= \int_{-2}^{2} \left[\frac{x^2}{2} \right]_{x=-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}} dy = \int_{-2}^{2} 0 \, dy = 0
\end{aligned}
$$

D is a Trapezoid

Example 2.2.7. Let D be the trapezoid with vertices : $A(0,0)$, $B(1,1)$, $C(1,3)$ and $E(0,4)$.

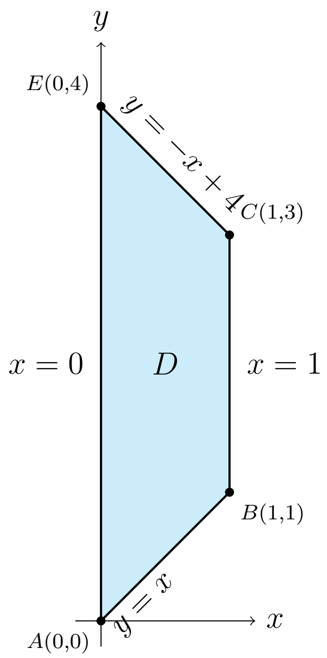
1. Draw the region D.
2. Determine the equation of the line passing through $E(0,4)$ and $C(1,3)$
3. Write the mathematical expression of the region D.
4. Evaluate the double integral :

$$
I = \iint_D x^2 y \, dx \, dy
$$

Solution

Type 1 (x is fixed)

1. Representation of the region D :



2. We have $y_{(EC)} = ax + b$.

Since $E(0,4) \in (EC)$, we have

$$
4 = a \cdot 0 + b \quad \Longrightarrow \quad b = 4.
$$

Since $C(1,3) \in (EC)$, we have

$$3 = a \cdot 1 + b \quad \Longrightarrow \quad a = -1.$$

Then, the equation of the line (EC) is

$$y = -x + 4.$$

The same for (AB): $y_{(AB)} = ax + b = x$.

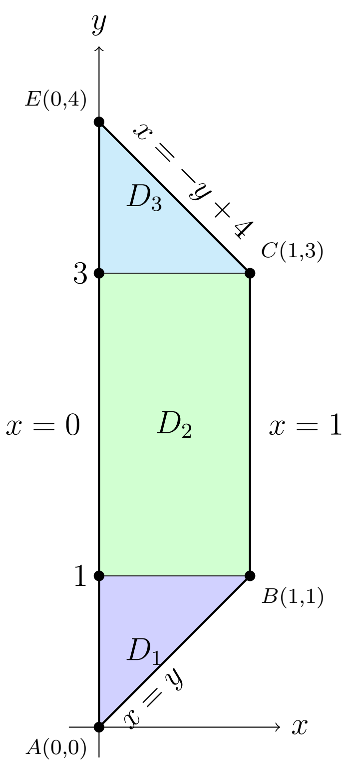
Finally, the region is

$$D = \{(x, y) \mid 0 \le x \le 1,\ x \le y \le -x + 4\}.$$

3. Evaluation of the double integral :

$$\begin{aligned}
I &= \iint_D x^2 y \, dx \, dy = \int_0^1 \int_x^{-x+4} x^2 y \, dy \, dx \\
&= \int_0^1 x^2 \left[\frac{y^2}{2}\right]_x^{-x+4} dx = \int_0^1 x^2 \cdot \frac{(-x+4)^2 - x^2}{2} \, dx \\
&= \int_0^1 x^2(-4x + 8) \, dx = \int_0^1 (-4x^3 + 8x^2) \, dx \\
&= \left[-x^4 + \frac{8x^3}{3}\right]_0^1 = \frac{5}{3}.
\end{aligned}$$

Type 2 (y is fixed) :



⚠ **Note :** The region D is *non-regular* because its boundary cannot be described with just two functions of x; instead, it requires three or more graphs.

👉 In this case, the region D is divided into regular (elementary) subregions, such as D_1, D_2, D_3, each bounded by exactly two functions.
The double integral over D is then written as the sum of the integrals over these subregions

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y)\,dx\,dy + \iint_{D_2} f(x,y)\,dx\,dy + \iint_{D_3} f(x,y)\,dx\,dy.$$

Finally, the regions are :

$$\begin{aligned} D_1 &= \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\},\\ D_2 &= \{(x,y) \mid 1 \le y \le 3,\ 0 \le x \le 1\},\\ D_3 &= \{(x,y) \mid 3 \le y \le 4,\ 0 \le x \le -y+4\}. \end{aligned}$$

$$\iint_{D_1} x^2y\,dx\,dy = \int_0^1\int_0^y x^2y\,dx\,dy = \int_0^1 y\left[\frac{x^3}{3}\right]_0^y dy = \int_0^1 \frac{y^4}{3}\,dy = \left[\frac{y^5}{15}\right]_0^1 = \frac{1}{15}.$$

$$\iint_{D_2} x^2y\,dx\,dy = \int_1^3\int_0^1 x^2y\,dx\,dy = \int_1^3 \frac{y}{3}\,dy = \left[\frac{y^2}{6}\right]_1^3 = \frac{4}{3}.$$

$$\iint_{D_3} x^2y\,dx\,dy = \int_3^4\int_0^{4-y} x^2y\,dx\,dy = \int_3^4 y\left[\frac{x^3}{3}\right]_0^{4-y} dy = \frac{1}{3}\int_3^4 y(4-y)^3\,dy.$$

Let $t = 4-y, \quad dy = -dt, \quad y = 4-t,$
$\quad y = 3 \Rightarrow t = 1, \qquad y = 4 \Rightarrow t = 0.$

$$\begin{aligned} \iint_{D_3} x^2y\,dx\,dy &= \frac{1}{3}\int_1^0 (4-t)t^3(-dt) = \frac{1}{3}\int_0^1 (4-t)t^3\,dt \\ &= \frac{1}{3}\int_0^1 (4t^3 - t^4)\,dt = \frac{1}{3}\left[t^4 - \frac{t^5}{5}\right]_0^1 = \frac{1}{3}\left(1 - \frac{1}{5}\right) = \frac{4}{15}. \end{aligned}$$

$$\begin{aligned} \iint_D x^2y\,dx\,dy &= \iint_{D_1} x^2y\,dx\,dy + \iint_{D_2} x^2y\,dx\,dy + \iint_{D_3} x^2y\,dx\,dy \\ &= \frac{1}{15} + \frac{4}{3} + \frac{4}{15} = \frac{5}{3}. \end{aligned}$$

2.2.4 Change of variables in double integrals

When evaluating double integrals, it is often useful to modify the system of coordinates. Complicated regions or integrands that are not easily handled in Cartesian coordinates (x,y) may become simpler when treated in a new system of variables. This process is called a **change of variables**.

a. Arbitrary coordinates : general case when the region D is bounded by four lines.

We consider a change of variables

$$\begin{cases} x = \varphi(u,v), \\ y = \psi(u,v), \end{cases}$$

where $(x,y) \in D$ and $(u,v) \in \triangle$. We assume that this transformation is a **bijection of class** C^1, i.e., φ and ψ have continuous partial derivatives with respect to u and v.

The **Jacobian matrix** is defined as

$$J = \begin{pmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

We change

$$f(x,y) = f(\varphi(u,v), \psi(u,v)),$$

then the **double integral transforms** as

$$\iint_D f(x,y)\,dx\,dy = \iint_\triangle f(\varphi(u,v), \psi(u,v))\,|det\,J|\,du\,dv,$$

where $det\,J$ is the **determinant of the Jacobian matrix** :

$$\det(J) = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} = \frac{\partial \varphi}{\partial u}\frac{\partial \psi}{\partial v} - \frac{\partial \varphi}{\partial v}\frac{\partial \psi}{\partial u}.$$

Example 2.2.8. Evaluate the double integral

$$I = \iint_D (x+y)^3 (x-y)^2\,dx\,dy,$$

where D is the region bounded by :

$$y = -x+1, \qquad y = x+1, \qquad y = 3-x, \qquad y = x-1.$$

Solution

We set

$$\begin{cases} u = x+y \quad \Rightarrow \quad x = \frac{u+v}{2}, \\ v = x-y \quad \Rightarrow \quad y = \frac{u-v}{2}. \end{cases}$$

The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \det J = -\frac{1}{2},$$

so that $|det\,J| = \frac{1}{2}$ and $dx\,dy = \frac{1}{2}\,du\,dv$.

Transform the boundary.

The four lines become

$$y = -x+1 \Rightarrow u = 1, \qquad y = 3-x \Rightarrow u = 3, \qquad y = x+1 \Rightarrow v = -1, \qquad y = x-1 \Rightarrow v = 1.$$

Thus $\triangle = \{(u, v) \in \mathbb{R}^2 \mid 1 \leq u \leq 3,\ -1 \leq v \leq 1\}$.

Transform the integrand.

Since $(x + y)^3(x - y)^2 = u^3v^2$, we obtain

$$I = \iint\limits_D (x + y)^3(x - y)^2\, dx\, dy = \frac{1}{2} \iint\limits_\triangle u^3v^2\, dv\, du$$

$$= \frac{1}{2} \int_1^3 u^3 \left[\frac{v^3}{3}\right]_{v=-1}^{v=1} du = \frac{1}{3} \int_1^3 u^3\, du = \frac{1}{3} \left[\frac{u^4}{4}\right]_{u=1}^{u=3} = \frac{20}{3}.$$

b. Polar coordinates :The region D is bounded by circular shapes
We now consider the change of variables from Cartesian coordinates (x, y) to polar coordinates (r, θ) defined by

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases} \qquad r \geq 0,\ 0 \leq \theta < 2\pi.$$

Conversely, r and θ can be expressed in terms of x and y (for $x \neq 0$) as :

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan(\frac{y}{x}).$$

The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

so that

$$|det\, J| = r.$$

Therefore, the double integral transforms as

$$\iint\limits_D f(x, y)\, dx\, dy = \iint\limits_\triangle f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta.$$

Example 2.2.9. Evaluate the following integrals using a change of variables (polar coordinates) :

$$I_1 = \iint\limits_D \frac{1}{\sqrt{x^2 + y^2}}\, dx\, dy,\ \ where\ \ D = \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; 1 \leq x^2 + y^2 \leq 2\Big\}.$$

Solution

Using polar coordinates : $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases}$ with $r \geq 0,\ 0 \leq \theta \leq 2\pi$.

Then $x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$.

The region in polar coordinates becomes : $1 \leq r^2 \leq 2 \quad \Rightarrow \quad 1 \leq r \leq \sqrt{2}, \quad 0 \leq \theta \leq 2\pi$.

$\triangle = \Big\{(r,\theta) \in \mathbb{R}_+ \times [0, 2\pi] \mid 1 \leq r \leq \sqrt{2},\ 0 \leq \theta \leq 2\pi\Big\}$.

the Jacobian matrix :

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

The determinant of J is :

$$\det(J) = (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

The integrand $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ becomes :

$$f(r,\theta) = \frac{1}{\sqrt{r^2}} = \frac{1}{r}.$$

$dx\,dy = |det\ J|\,dr\,d\theta = r\,dr\,d\theta.$

Thus, the integral in polar coordinates is :

$$I_1 = \iint\limits_D \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = \iint\limits_{\triangle} \frac{1}{r}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}} 1\,dr\,d\theta = \left[\int_1^{\sqrt{2}} 1\,dr\right] \cdot \left[\int_0^{2\pi} 1\,d\theta\right]$$

$$= \Big[r\Big]_{r=1}^{r=\sqrt{2}} \cdot \Big[\theta\Big]_{\theta=0}^{\theta=2\pi} = 2\pi(\sqrt{2}-1).$$

$\mathrm{I}_2 = \iint\limits_D e^{-(x^2+y^2)}\,dx\,dy$, where $D = \Big\{(x,y) \in \mathbb{R}^2 \;\Big|\; x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\Big\}$.

Solution

Using polar coordinates :

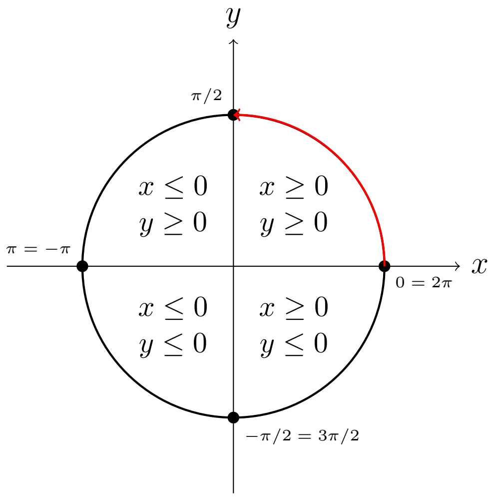
$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \qquad r \geq 0, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Then

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2.$$

The region in polar coordinates becomes :

$$0 \leq r^2 \leq 1 \quad \Rightarrow \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$



$$\triangle = \Big\{(r,\theta) \in \mathbb{R}_+ \times [0, 2\pi] \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/2\Big\}.$$

The Jacobian matrix :

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

The determinant of J is :

$$\det(J) = (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

The integrand $f(x,y) = e^{-(x^2+y^2)}$ becomes :

$$f(r,\theta) = e^{-r^2}.$$

Also, $dx\,dy = |det(J)|\,dr\,d\theta = r\,dr\,d\theta$.
Thus, the integral in polar coordinates is :

$$I_2 = \iint_D e^{-(x^2+y^2)}\,dx\,dy = \iint_\triangle e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 re^{-r^2}\,dr\,d\theta = \left[\int_0^1 re^{-r^2}\,dr\right] \cdot \left[\int_0^{\pi/2} 1\,d\theta\right]$$

$$= \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=1} \cdot \Big[\theta\Big]_{\theta=0}^{\theta=\pi/2} = \frac{1}{2}(1-e^{-1}) \cdot \frac{\pi}{2} = \frac{\pi}{4}(1-e^{-1}).$$

Chapter 3

Triple Integral

We discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space $\mathbb{R}^3$.
Let V be a closed and bounded region in $\mathbb{R}^3$. Consider the function

$$\begin{aligned} f : V \subset \mathbb{R}^3 &\longrightarrow \mathbb{R}, \\ (x, y, z) &\longmapsto f(x, y, z), \end{aligned}$$

which is continuous on the region V.
The triple integral of f over V is given by :

$$\iiint\limits_V f(x, y, z)\, dx\, dy\, dz.$$

3.1 Properties of the triple integral

1.**Volume**(V) : If $f(x, y, z) = 1$, then :

$$Volume(V) = \iiint\limits_V dx\, dy\, dz.$$

2. **Positivity** : If $f(x, y, z) \geq 0$ for all $(x, y, z) \in V$, then :

$$\iiint\limits_V f(x, y, z)\, dx\, dy\, dz \geq 0.$$

3. **Linearity** : Let f and g be two functions of three variables that are integrable over V, and let $\lambda, \mu \in \mathbb{R}$, then :

$$\iiint\limits_V [\lambda f(x, y, z) + \mu g(x, y, z)]\, dx\, dy\, dz = \lambda \iiint\limits_V f(x, y, z)\, dx\, dy\, dz \quad + \mu \iiint\limits_V g(x, y, z)\, dx\, dy\, dz.$$

4. **Chasles' relation** : If $V = V_1 \cup V2$ and $V_1 \cap V_2 = \emptyset$, (V_1 and V_2 are two disjoint regions such that, Volume$(V_1 \cap V_2) = 0$), then :

$$\iiint\limits_V f(x, y, z)\, dx\, dy\, dz = \iiint\limits_{V_1} f(x, y, z)\, dx\, dy\, dz + \iiint\limits_{V_2} f(x, y, z)\, dx\, dy\, dz.$$

3.2 Methods of computing triple integrals

We present here the practical evaluation of triple integrals on multiple regions with different geometries

3.2.1 Triple integral over a parallelepiped

Theorem 3.2.1. ***(Fubini's Theorem)*** *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be continuous on the rectangular box $V = [a, b] \times [c, d] \times [e, s]$, with $a, b, c, d, e, s \in \mathbb{R}$, then :*

$$\begin{aligned}
\iiint\limits_V f(x, y, z)\, dx\, dy\, dz &= \int_a^b \Big(\int_c^d \Big(\int_e^s f(x, y, z)\, dz \Big) dy \Big) dx \\
&= \int_a^b \Big(\int_e^s \Big(\int_c^d f(x, y, z)\, dy \Big) dz \Big) dx \\
&= \int_c^d \Big(\int_a^b \Big(\int_e^s f(x, y, z)\, dz \Big) dx \Big) dy \\
&= \int_c^d \Big(\int_e^s \Big(\int_a^b f(x, y, z)\, dx \Big) dz \Big) dy \\
&= \int_e^s \Big(\int_a^b \Big(\int_c^d f(x, y, z)\, dy \Big) dx \Big) dz \\
&= \int_e^s \Big(\int_c^d \Big(\int_a^b f(x, y, z)\, dx \Big) dy \Big) dz.
\end{aligned}$$

👉 Here, the order of integration is not fixed. However, one may choose an order that simplifies the computations

Example 3.2.1. Evaluate the triple integral :

$$I = \iiint\limits_V (3xyz^2)\, dx\, dy\, dz$$

where

$$V = \Big\{ (x, y, z) \in \mathbb{R}^3 \;\Big|\; 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3 \Big\}.$$

We could use any of the six possible orders of integration.

$$I = \iiint\limits_V 3xyz^2\, dx\, dy\, dz = \int_0^1 \Big(\int_{-1}^2 \Big(\int_0^3 3xyz^2\, dz \Big) dy \Big) dx = \int_0^1 \left(\int_{-1}^2 3xy \left[\frac{z^3}{3} \right]_{z=0}^{z=3} dy \right) dx$$

$$= \int_0^1 \Big(\int_{-1}^2 27xy\, dy \Big) dx = \int_0^1 \left[27x \cdot \frac{y^2}{2} \right]_{y=-1}^{y=2} dx = \int_0^1 \frac{81}{2} x\, dx = \left[\frac{81}{4} x^2 \right]_{x=0}^{x=1} = \frac{81}{4}.$$

Or

$$I = \iiint\limits_V 3xyz^2\, dx\, dy\, dz = \int_0^1 \Big(\int_0^3 \Big(\int_{-1}^2 3xyz^2\, dy \Big) dz \Big) dx = \int_0^1 \left(\int_0^3 3xz^2 \left[\frac{y^2}{2} \right]_{y=-1}^{y=2} dz \right) dx$$

$$= \int_0^1 \Big(\int_0^3 3xz^2 \cdot \frac{3}{2} dz \Big) dx = \int_0^1 \Big(\int_0^3 \frac{9}{2} xz^2\, dz \Big) dx = \int_0^1 \frac{9}{2} x \left[\frac{z^3}{3} \right]_{z=0}^{z=3} dx$$

$$= \int_0^1 \frac{9}{2} x \cdot 9\, dx = \int_0^1 \frac{81}{2} x\, dx = \left[\frac{81}{4} x^2 \right]_{x=0}^{x=1} = \frac{81}{4}.$$

Or

$$I = \iiint_V 3xyz^2\,dx\,dy\,dz = \int_{-1}^{2}\int_0^3\int_0^1 3xyz^2\,dx\,dz\,dy$$
$$= \int_{-1}^{2}\int_0^3 3yz^2\left[\frac{x^2}{2}\right]_0^1 dz\,dy$$
$$= \int_{-1}^{2}\int_0^3 \frac{3}{2}yz^2\,dz\,dy$$
$$= \int_{-1}^{2}\frac{3}{2}y\left[\frac{z^3}{3}\right]_0^3 dy$$
$$= \int_{-1}^{2}\frac{3}{2}y\cdot 9\,dy = \int_{-1}^{2}\frac{27}{2}y\,dy$$
$$= \left[\frac{27}{4}y^2\right]_{-1}^{2} = \frac{27}{4}(4-1) = \frac{81}{4}.$$

Proposition 3.2.2. *Let f, f_1, f_2 and f_3 be four continuous functions ;*

$$\begin{array}{rcl} f : V \subset \mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & f(x,y,z) = f_1(x)\times f_2(y)\times f_3(z). \end{array}$$

Here, $V = [a,b]\times[c,d]\times[e,s]$, then :

$$\begin{aligned} \iiint_V f(x,y,z)\,dx\,dy\,dz &= \iiint_{[a,b]\times[c,d]\times[e,s]} f_1(x)\times f_2(y)\times f_3(z)\,dx\,dy\,dz \\ &= \left(\int_a^b f_1(x)\,dx\right)\times\left(\int_c^d f_2(y)\,dy\right)\times\left(\int_e^s f_3(z)\,dz\right). \end{aligned}$$

Example 3.2.2. Evaluate the following triple integral :

1. $I = \iiint_V xe^y z\,dx\,dy\,dz$, where

$$V = \Big\{(x,y,z)\in\mathbb{R}^3/0\le x\le 1,\quad 1\le y\le 2 \text{ et } 0\le z\le 1\Big\}.$$

   we can separate $f(x,y,z)$ into three independent functions :

$$\begin{aligned} I = \iiint_V xe^y z\,dx\,dy\,dz &= \left(\int_0^1 xdx\right)\left(\int_1^2 e^y dy\right)\left(\int_0^1 zdz\right) \\ &= \left[\frac{1}{2}x^2\right]_{x=0}^{x=1}\times\left[e^y\right]_{y=1}^{y=2}\times\left[\frac{1}{2}z^2\right]_{z=0}^{z=1} \\ &= \frac{1}{2}(e^2-e)\frac{1}{2} = \frac{1}{4}(e^2-e). \end{aligned}$$

3.2.2 Triple integral over a general bounded region $V \subset \mathbb{R}^3$

We now define the triple integral over a general bounded solid region $V \subset \mathbb{R}^3$ using a procedure similar to that for double integrals.

The region V can be described in three possible forms :

Type I : z **bounded between two surfaces**

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

- D is the projection of V on the xy-plane.
- For each $(x,y) \in D$, z varies between $z = u_1(x,y)$ and $z = u_2(x,y)$.

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \iint\limits_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dx\,dy$$

Type II : x **bounded between two surfaces**

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid (y,z) \in D,\ u_1(y,z) \leq x \leq u_2(y,z)\}$$

- D is the projection of V on the yz-plane.
- For each $(y,z) \in D$, x varies between $x = u_1(y,z)$ and $x = u_2(y,z)$.

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \iint\limits_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dy\,dz$$

Type III : y **bounded between two surfaces**

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid (x,z) \in D,\ u_1(x,z) \leq y \leq u_2(x,z)\}$$

- D is the projection of V on the xz-plane.
- For each $(x,z) \in D$, y varies between $y = u_1(x,z)$ and $y = u_2(x,z)$.

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \iint\limits_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right) dx\,dz$$

Projections regions

i) If D in the xy-plane is **Type 1** :

$$D = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\},$$

then the triple integral over a type 1 region becomes

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

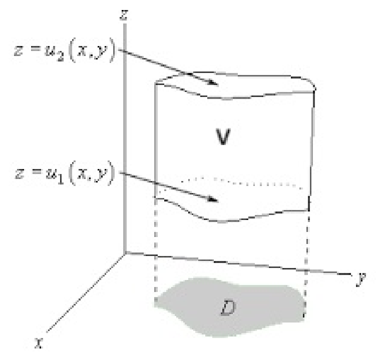
ii) If D in the xy-plane is **Type 2** :

$$D = \{(x,y) \mid c \leq y \leq d,\ g_1(y) \leq x \leq g_2(y)\},$$

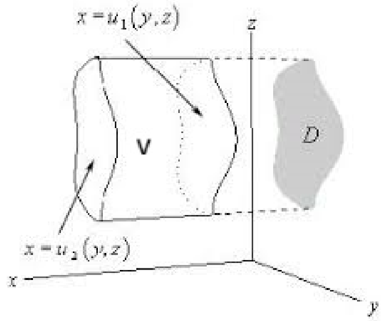
then the triple integral over a type 2 region becomes

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

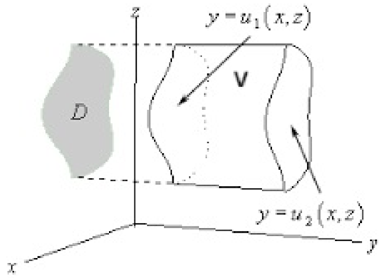
👉 Triple integrals can be computed by changing the order of integration depending on the projection and the bounding surfaces to simplify calculations.



The solid region V of type I



The solid region $V$of type II



The solid region V of type III

Example 3.2.3. Evaluate triple integrals :

1. I= $\iiint\limits_V (x^2 + yz)\, dx\, dy\, dz,$

where
$$V = \Big\{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le 2,\ 0 \le y \le z,\ 0 \le x \le y\Big\}$$

$$\begin{aligned} I = \iiint\limits_V (x^2 + yz)\, dx\, dy\, dz &= \int_0^2 \int_0^z \int_0^y (x^2 + yz)\, dx\, dy\, dz \\ &= \int_0^2 \int_0^z \left[\frac{x^3}{3} + xyz\right]_{x=0}^{x=y} dy\, dz = \int_0^2 \int_0^z \left(\frac{y^3}{3} + y^2 z\right) dy\, dz \\ &= \int_0^2 \left[\frac{y^4}{12} + \frac{y^3 z}{3}\right]_{y=0}^{y=z} dz = \int_0^2 \left(\frac{z^4}{12} + \frac{z^4}{3}\right) dz \\ &= \int_0^2 \frac{5z^4}{12}\, dz = \left[\frac{5z^5}{60}\right]_0^2 = \frac{8}{3}. \end{aligned}$$

2. Volume of the solid region V,

where
$$V = \Big\{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le 2,\ 0 \le y \le 1,\ 0 \le x \le y + z\Big\}$$

$$\begin{aligned} \mathrm{Vol}(V) = \iiint\limits_V 1\, dx\, dy\, dz &= \int_0^2 \int_0^1 \int_0^{y+z} 1\, dx\, dy\, dz \\ &= \int_0^2 \int_0^1 (y + z)\, dy\, dz = \int_0^2 \left[\frac{y^2}{2} + yz\right]_0^1 dz = \int_0^2 \left(\frac{1}{2} + z\right) dz \\ &= \left[\frac{z}{2} + \frac{z^2}{2}\right]_0^2 = 3. \end{aligned}$$

3. I=$\iiint\limits_V z dx\, dy\, dz,$

where
$$V = \Big\{(x, y, z) \in \mathbb{R}^3 \;\Big|\; x \ge 0,\ y \ge 0,\ z \ge 0,\ x + y + z \le 1\Big\}.$$

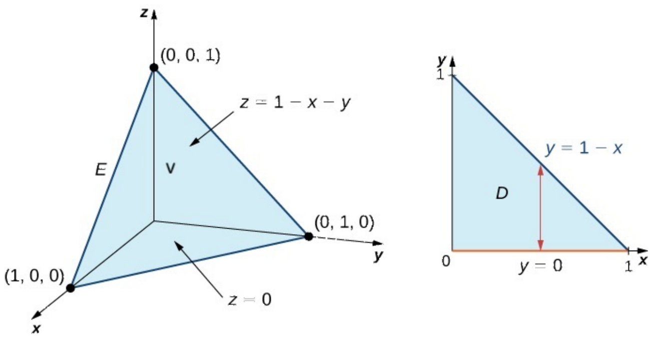


Figure 3.1: The tetrahedron V and its projection D onto the xy-plane of type 1.

Type I : bounded by z

The solid region V is a tetrahedron bounded by the coordinate planes

$$x = 0, \quad y = 0, \quad z = 0,$$

and the plane

$$x + y + z = 1.$$

From the inequality $x + y + z \leq 1$, we can solve for z:

$$0 \leq z \leq 1 - x - y.$$

Thus, the projection D of V onto the xy-plane is given by

$$\begin{aligned} D &= \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; x \geq 0,\, y \geq 0,\, x + y \leq 1\Big\} \\ &= \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; 0 \leq x \leq 1,\, 0 \leq y \leq 1 - x\Big\}. \end{aligned}$$

The region D is the right triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Therefore,

$$I = \iint_D \left(\int_0^{1-x-y} (5x - 3y)\, dz\right) dx\, dy.$$

$$V = \Big\{(x, y, z) \in \mathbb{R}^3 \,\Big|\, 0 \leq x \leq 1,\, 0 \leq y \leq 1 - x,\, 0 \leq z \leq 1 - x - y\Big\}.$$

$$\begin{aligned} I &= \iiint_V z\, dx\, dy\, dz, \quad V = \{(x, y, z) \mid 0 \leq x \leq 1,\, 0 \leq y \leq 1 - x,\, 0 \leq z \leq 1 - x - y\} \\ &= \iint_D \int_0^{1-x-y} z\, dz\, dx\, dy = \iint_D \frac{(1 - x - y)^2}{2}\, dx\, dy, \\ D &= \{(x, y) \mid 0 \leq x \leq 1,\, 0 \leq y \leq 1 - x\}, \\ I &= \int_0^1 \int_0^{1-x} \frac{(1 - x - y)^2}{2}\, dy\, dx = \int_0^1 \frac{1}{2} \int_0^{1-x} (1 - x - y)^2\, dy\, dx \\ &= \int_0^1 \frac{1}{2} \cdot \frac{(1 - x)^3}{3}\, dx = \int_0^1 \frac{(1 - x)^3}{6}\, dx \\ &= \frac{1}{6} \int_0^1 (1 - x)^3\, dx = \frac{1}{6} \cdot \frac{(1 - x)^4}{4}\Big|_0^1 = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}. \end{aligned}$$

Second method (analytical)

Type I : bounded by x

$$\begin{aligned} V &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \leq 1-x-y\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ 0 \leq z \leq 1-x-y\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ y \leq 1-x,\ 0 \leq z \leq 1-x-y\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ 0 \leq 1-x,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ x \leq 1,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y\}. \end{aligned}$$

Type II : bounded by x

$$\begin{aligned} V &= \{(x,y,z) \in \mathbb{R}^3 \mid y \geq 0,\ z \geq 0,\ x \leq 1-y-z\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq y \leq 1,\ 0 \leq z \leq 1-y,\ 0 \leq x \leq 1-y-z\}. \end{aligned}$$

Type III : bounded by y

$$\begin{aligned} V &= \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ z \geq 0,\ y \leq 1-x-z\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1,\ 0 \leq z \leq 1-x,\ 0 \leq y \leq 1-x-z\}. \end{aligned}$$

3.2.3 Change of variables

Triple integrals are used to compute volumes, mass, and other quantities over three-dimensional regions. In many situations, direct computation is difficult, so we use change of variables to simplify the integral.
Let

$$(x,y,z) = h(u,v,w),$$

where h is a bijection of class $\mathcal{C}^1$ on $\Delta = h^{-1}(V)$.
We define $F(u,v,w) = f\Big(h(u,v,w)\Big)$. Then, the triple integral transforms as

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \iiint\limits_\Delta F(u,v,w)\,|\det(J)|\,du\,dv\,dw.$$

where $|\det(J)|$ is the absolute value of the determinant of the Jacobian matrix

$$\det(J) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

1. **Change of variables by cylindrical Coordinates**

We consider the transformation to cylindrical coordinates

$$\begin{cases} x = x(r,\theta) = r\cos(\theta), \\ y = y(r,\theta) = r\sin(\theta), \\ z = z \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2+y^2}, \\ \theta = \arctan(\dfrac{y}{x}), \quad Apoint \\ z = z. \end{cases}$$

M=(x,y,z)inR^3 can be described in cylindrical coordinates by : $r = \sqrt{x^2+y^2}$ with $r \geq 0$, the angle $\theta = \arctan\left(\dfrac{y}{x}\right)$ for $r \neq 0$ (in general $0 \leq \theta < 2\pi$), and the height $z \in \mathbb{R}$.
Thus, $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.

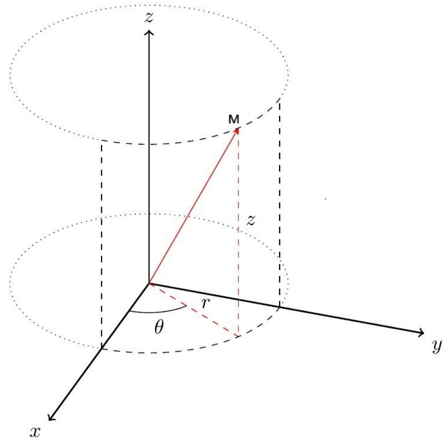


Figure 3.2: Cylindrical coordinates.

Using cylindrical coordinates, the associated Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \quad |\det(J)| = r.$$

Therefore,

$$\begin{aligned} \iiint_V f(x,y,z)\,dx\,dy\,dz &= \iiint_\Delta f(r\cos\theta,\ r\sin\theta,\ z)\,|\det(J)|\,dr\,d\theta\,dz \\ &= \iiint_\Delta f(r\cos\theta,\ r\sin\theta,\ z)\,r\,dr\,d\theta\,dz. \end{aligned}$$

where $\Delta = h^{-1}(V)$.

Example 3.2.4. Evaluate (using cylindrical coordinates) :

1. $\displaystyle\iiint_V (x^2+y^2)\,dx\,dy\,dz$, where

$$V = \Big\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 4 \text{ and } 0 \leq z \leq 2\Big\}.$$

In cylindrical coordinates, we set

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z, \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2+y^2}, \\ \theta = \arctan\left(\dfrac{y}{x}\right), \\ z = z. \end{cases}$$

We have

$$\begin{cases} 0 \le x^2 + y^2 \le 4, \\ 0 \le z \le 2, \end{cases} \qquad \Rightarrow \qquad \begin{cases} 0 \le r \le 2, \\ 0 \le \theta \le 2\pi, \\ 0 \le z \le 2. \end{cases}$$

So

$$\Delta = \{(r, \theta, z) \in \mathbb{R}^3 \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 2\}.$$

Moreover,

$$dx\, dy\, dz = r\, dr\, d\theta\, dz, \qquad f(x, y, z) = x^2 + y^2 = r^2.$$

The change to cylindrical coordinates gives

$$\begin{aligned} I &= \iiint_V (x^2 + y^2)\, dx\, dy\, dz \\ &= \iiint_\Delta r^2 \cdot r\, dr\, d\theta\, dz \\ &= \iiint_\Delta r^3\, dr\, d\theta\, dz \\ &= \left(\int_0^2 r^3\, dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 dz\right) \\ &= \left[\frac{r^4}{4}\right]_0^2 \cdot 2\pi \cdot 2 = 4 \cdot 2\pi \cdot 2 = 16\pi. \end{aligned}$$

2. **Change of variables by spherical coordinates**

Consider the following spherical coordinate transformation :

$$\begin{cases} x = x(r, \theta, \varphi) = r \cos\theta \cos\varphi, \\ y = y(r, \theta, \varphi) = r \sin\theta \cos\varphi, \\ z = z(r, \theta, \varphi) = r \sin\varphi. \end{cases}$$

A point $M = (x, y, z)$ is characterized by :

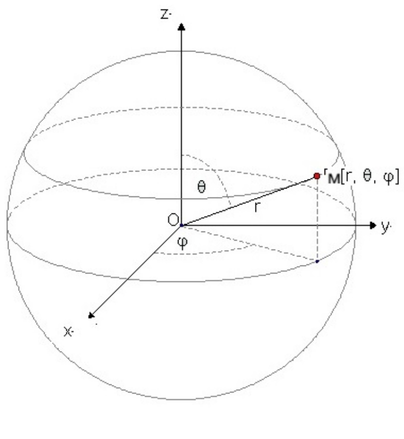


Figure 3.3: Spherical coordinates.

$$\begin{cases} \text{Distance from the origin : } r = \sqrt{x^2+y^2+z^2}, \quad r \geq 0, \\ \text{Longitude : } 0 \leq \theta < 2\pi, \\ \text{Latitude : } -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}. \end{cases}$$

The Jacobian matrix of this transformation is

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\varphi & 0 & r\cos\varphi \end{pmatrix}.$$

Its determinant is :

$$|\det(J)| = r^2\cos\varphi.$$

Thus, for a region $V \subset \mathbb{R}^3$, we have

$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \iiint\limits_\Delta f(r\cos\theta\cos\varphi,\ r\sin\theta\cos\varphi,\ r\sin\varphi)\,r^2\cos\varphi\,dr\,d\theta\,d\varphi,$$

where $\Delta = h^{-1}(V)$.

Example 3.2.5. Compute the following triple integral using spherical coordinates

1. $\iiint\limits_V dx\,dy\,dz$, where

$$V = \Big\{(x,y,z) \in \mathbb{R}^3 / 1 \leq x^2+y^2+z^2 \leq 4\Big\}.$$

In spherical coordinates

$$\begin{cases} x = x(r,\theta,\varphi) = r\cos\theta\cos\varphi, \\ y = y(r,\theta,\varphi) = r\sin\theta\cos\varphi, \\ z = z(r,\theta,\varphi) = r\sin\varphi. \end{cases}$$

$$dx\,dy\,dz = r^2\cos\varphi\,dr\,d\theta\,d\varphi.$$

$$\Delta = \Big\{(r,\theta,\varphi) \in \mathbb{R}^3 / 1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi \ \text{ and } \ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\Big\}.$$

Then

$$\begin{aligned} \iiint\limits_\Delta r^2\cos(\varphi)\,dr\,d\theta\,d\varphi &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_1^2 r^2\cos(\varphi)\,dr\,d\theta\,d\varphi \\ &= \int_1^2 r^2 dr \times \int_0^{2\pi} d\theta \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\varphi)d\varphi \\ &= \left[\frac{1}{3}r^3\right]_{r=1}^{r=2} \times \Big[\theta\Big]_{\theta=0}^{\theta=2\pi} \times \Big[\sin(\varphi)\Big]_{\varphi=-\frac{\pi}{2}}^{\varphi=\frac{\pi}{2}} \\ &= \frac{7}{3} \times 2\pi \times 2 = \frac{28\pi}{3}. \end{aligned}$$

Chapter 4

Improper Integral

Up to this point, we have computed definite integrals of the form

$$\int_a^b f(x)\,dx,$$

where f is a continuous function defined on a closed and bounded interval of $\mathbb{R}$.

In this chapter, we will extend the concept of definite integrals where one or both of the boundaries is at infinity, and integrals with infinite discontinuity :

Definition 4.0.1. The integral $\displaystyle\int_a^b f(x)\,dx$ is called an *improper integral* if :

1. One or both of the limits of integration are infinite, i.e. $a = -\infty$ or $b = +\infty$ or both.
2. a case where the integrand is discontinuous, that is, a definite integral in which the function is not defined at some point of the interval.

4.1 Integrals over intervals of the form $[a, b[,]a, b]$ or $]a, b[$

Definition 4.1.1. Let f be a continuous function on a semi-open interval of $\mathbb{R}$ (for example, $[a, b[$), then we define

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

This integral is called an **improper integral(or generalized)**, and we can also write

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = \lim_{x \to b^-} F(x) - F(a),$$

where F is an antiderivative of f on $[a, b[$.

4.1.1 The nature of the improper integral

Let f be a continuous function on $[a, b[$.

1) The integral $\displaystyle\int_a^b f(x)\,dx$ **converges** if $\displaystyle\lim_{t\to b^-}\int_a^t f(x)\,dx$ exists and is finite.

2) The integral $\displaystyle\int_a^b f(x)\,dx$ **diverges** if $\displaystyle\lim_{t\to b^-}\int_a^t f(x)\,dx$ does not exist or is infinite.

Example 4.1.1. Compute the improper integral

$$\int_0^1 \frac{1}{x}\,dx.$$

The function $f(x) = \dfrac{1}{x}$ is continuous on the interval $]0, 1]$, but it is unbounded at $x = 0$. Thus, we write the integral as a limit :

$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx\\ &= \lim_{t\to 0^+}\Big[\,\ln(x)\Big]_t^1\\ &= \lim_{t\to 0^+}(\ln 1 - \ln t)\\ &= \lim_{t\to 0^+}(-\ln t) = +\infty.\end{aligned}$$

Since the limit is infinite, the improper integral diverges.

4.2 Integrals over an unbounded interval

Definition 4.2.1. Let f be a continuous function on $[a, +\infty[$, $a \in \mathbb{R}$, then :

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = \lim_{t\to+\infty} F(t) - F(a).$$

This integral is called an **improper integral**

Remark 4.2.1. If the limits of integration are both infinite, that is

$$\int_{-\infty}^{+\infty} f(x)\,dx,$$

the improper integral is defined as the sum of two integrals :

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{t\to+\infty}\int_a^t f(x)\,dx,$$

where a is any real number.

4.3 Techniques for computing improper integrals

4.3.1 Using antiderivatives

Let f be a continuous function on $]a, b]$ and which admit an antiderivative, then :

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} [F(b) - F(t)] = F(b) - \lim_{t\to a^+} F(t),$$

Example 4.3.1. Compute the following improper integral :
$I = \displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx.$
The function $x \mapsto \dfrac{1}{\sqrt{3-x}}$ is continuous on the interval $[2, 3[$.
Since it is not defined at $x = 3$, the integral is **improper at the upper limit**, hence

$$I \;=\; \int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3}\int_2^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3}\Big[-2\sqrt{3-x}\Big]_2^t = \lim_{t\to 3}(-2\sqrt{3-t}+2) = 2.$$

The limit exists and is finite, therefore I converges.

4.3.2 Using integration by parts

Let u and v whose derivatives are continuous on $]a, b]$:

$$\int_a^b u(x)v'(x)\,dx = \lim_{t\to a^+}\Big(\Big[u(x)v(x)\Big]_t^b - \int_t^b v(x)u'(x)\,dx\Big).$$

Example 4.3.2. Compute the following improper integral :

1. $I_1 = \displaystyle\int_0^1 \ln x\,dx.$
   the function $x \mapsto \ln(x)$is continuous on $]0, 1]$. we have a problem at 0.

$$\begin{aligned} u(x) = \ln(x) \quad &\to \quad u'(x) = \frac{1}{x}\,dx \\ v'(x) = 1\,dx \quad &\to \quad v(x) = x, \end{aligned}$$

   we integrate by parts :

$$\begin{aligned} \int_0^1 \ln(x)\,dx \;&=\; \lim_{t\to 0^+}\int_t^1 \ln(x)\,dx = \lim_{t\to 0^+}\Big[x\ln x\Big]_t^1 - \int_t^1 1\,dx \\ &=\; \lim_{t\to 0^+}\Big[x\ln x - x\Big]_t^1 = \lim_{t\to 0^+}(-1 - t\ln t - t) = -1. \end{aligned}$$

   As $t \to 0^+$, we have $\ln t \to -\infty$, so the expression is of the indeterminate form $0 \times (-\infty)$.

$$t\ln t = \frac{\ln t}{1/t}.$$

Applying L'Hospital's rule :

$$\lim_{t\to 0^+}\frac{\ln t}{1/t}=\lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}}=\lim_{t\to 0^+}(-t)=0.$$

Then I_1 converges.

4.3.3 Using substitution

Let f be a continuous function on $]a,b]$ and φ be a differentiable function.
If we set $x=\varphi(t)$ we obtain :

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt,$$

where $\alpha=\varphi^{-1}(a)$ and $\beta=\varphi^{-1}(b)$.

Example 4.3.3. Calculate the following integral improper :

1. $I=\displaystyle\int_{-1}^{3}\frac{1}{\sqrt{x+1}}\,dx.$

   $I=\displaystyle\int_{-1}^{3}\frac{1}{\sqrt{x+1}}\,dx=\lim_{t\to -1^+}\int_t^3\frac{1}{\sqrt{x+1}}\,dx$

   The function $x\mapsto\frac{1}{\sqrt{x+1}}$ is continuous on $]-1,3]$, there is a problem at $x=-1$.

   $We\ set\ y=\sqrt{x+1}\Rightarrow y^2=x+1\Rightarrow 2y\,dy=dx,$

   if $x=3\rightarrow$y $=\sqrt{3+1}=2$ and $x\rightarrow t\Rightarrow$y$=\sqrt{t+1}$,

$$\begin{aligned}\lim_{t\to -1^+}\int_t^3\frac{1}{\sqrt{x+1}}\,dx &= \lim_{t\to -1^+}\int_{\sqrt{t+1}}^{2}2\,dy\\ &= \lim_{t\to -1^+}\Big[2y\Big]_{\sqrt{t+1}}^{2}\\ &= \lim_{t\to -1^+}\left(4-2\sqrt{t+1}\right)\\ &= 4-0=4.\end{aligned}$$

   The limit is finite, so the integral I converges.

4.4 Integration of positive functions

If f is negative on an interval I, thens $-f$ is positive on I The convergence of

$$\int_a^b f(x)\,dx$$

is the same as that of

$$\int_a^b -f(x)\,dx$$

.

4.4.1 Riemann integral

$$I_\alpha = \int_1^{+\infty} f_\alpha(x)dx,$$

where

$$\begin{array}{rcl} f_\alpha : [1, +\infty[& \to & \mathbb{R}_*^+ \\ (x) & \mapsto & f_\alpha(x) = \dfrac{1}{x^\alpha}, \end{array}$$

a continuous function, and $\alpha \in \mathbb{R}$.

First case : If $\alpha = 1$

$$\begin{aligned} I_1 &= \int_1^{+\infty} f_1(x)\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x}\,dx \\ &= \lim_{t\to+\infty} \Big[\ln x\Big]_1^t = \lim_{t\to+\infty} (\ln t - \ln 1) = +\infty. \end{aligned}$$

Hence, the integral $I_1 = \displaystyle\int_1^{+\infty} \frac{1}{x}\,dx$ diverges.

Second case : If $\alpha \neq 1$

$$\begin{aligned} I_\alpha &= \int_1^{+\infty} f_\alpha(x)\,dx = \int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \int_1^{+\infty} x^{-\alpha}\,dx \\ &= \lim_{t\to+\infty} \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^t = \lim_{t\to+\infty} \left(\frac{t^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}\right) \\ &= \begin{cases} -\dfrac{1}{1-\alpha} = \dfrac{1}{\alpha-1}, & \text{if } \alpha > 1, \\ +\infty, & \text{if } \alpha < 1. \end{cases} \end{aligned}$$

Thus, the integral

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \text{converges}, & \text{if } \alpha > 1, \\ \text{diverges}, & \text{if } \alpha < 1. \end{cases}$$

Summary : Let α be a real number.

1. $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx$ converges if $\alpha < 1$, and diverges if $\alpha \geq 1$.
2. $\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ converges if $\alpha > 1$, and diverges if $\alpha \leq 1$.

In the general case, there are two types of improper Riemann integrals, with $a > 0$:

First type : $\displaystyle\int_a^{+\infty} \frac{1}{x^\alpha}\,dx$ converges if $\alpha > 1$.

Second type : $\displaystyle\int_0^a \frac{1}{x^\alpha}\,dx$ converges if $\alpha < 1$.

4.4.2 Comparison test

Theorem 4.4.1. *Suppose that f and g are continuous functions on the interval $[a, +\infty[$ such that :*

$$0 \leq f(x) \leq g(x), \quad \textit{for all } x \in [a, +\infty[.$$

Then :

1. *If $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ also converges.*
2. *If $\int_a^{+\infty} f(x)\,dx$ diverges, then $\int_a^{+\infty} g(x)\,dx$ also diverges.*

Example 4.4.1. Determine the nature of the following improper integrals :

1. $I_1 = \int_1^{+\infty} e^{-x^2}\,dx.$

   First, the function $x \mapsto e^{-x^2}$ is continuous and positive on $[1, +\infty[$; the only issue is at $+\infty$.
   For $x \geq 1$ we have $x^2 \geq x$, hence $-x^2 \leq -x$ and therefore

   $$e^{-x^2} \leq e^{-x}, \qquad \forall x \geq 1.$$

   Consequently,

   $$\int_1^{+\infty} e^{-x^2}\,dx \leq \int_1^{+\infty} e^{-x}\,dx.$$

   But

   $$\int_1^{+\infty} e^{-x}\,dx = \lim_{t\to+\infty} \int_1^t e^{-x}\,dx = \lim_{t\to+\infty} \Big[-e^{-x}\Big]_1^t = \lim_{t\to+\infty} (-e^{-t}) + e^{-1} = e^{-1},$$

   so $\int_1^{+\infty} e^{-x}\,dx$ converges.
   According to the Comparison Test,

   $$I_1 = \int_1^{+\infty} e^{-x^2}\,dx$$

   also converges.

2. $I_2 = \int_0^1 \frac{e^x}{x^3}\,dx.$

   The function $f(x) = \frac{e^x}{x^3}$ is positive and continuous on $]0, 1]$, but not defined at $x = 0$.
   So we check the behavior near $x = 0$.
   For $x \in (0, 1)$, we know that $1 \leq e^x \leq e$. Thus,

   $$\frac{1}{x^3} \leq \frac{e^x}{x^3} \leq \frac{e}{x^3}.$$

   We compare $f(x)$ with $\frac{1}{x^3}$. We compute

   $$\int_0^1 \frac{1}{x^3}\,dx = \lim_{t\to0^+} \int_t^1 x^{-3}\,dx = \lim_{t\to0^+} \Big[-\frac{1}{2x^2}\Big]_t^1 = \lim_{t\to0^+} \Big(-\frac{1}{2} + \frac{1}{2t^2}\Big) = +\infty,$$

   Since $\frac{1}{x^3} \leq \frac{e^x}{x^3}$ and $\int_0^1 \frac{1}{x^3}\,dx$ diverges, then by the **Comparison test**, the integral $I_2 = \int_0^1 \frac{e^x}{x^3}\,dx$ also diverges.

4.4.3 Equivalence criterion

Theorem 4.4.2. *Let f and g be two continuous and strictly positive functions on the interval $]a, b]$. We say that*

$$f \sim_a g \iff \lim_{\substack{x \to a \\ x > a}} \frac{f(x)}{g(x)} = 1.$$

Then, the two integrals

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b g(x)\,dx$$

are of the same nature, that is :

$$\text{If } \int_a^b g(x)\,dx \text{ converges, then } \int_a^b f(x)\,dx \text{ converges.}$$

$$\text{If } \int_a^b g(x)\,dx \text{ diverges, then } \int_a^b f(x)\,dx \text{ diverges.}$$

Example 4.4.2. Determine the nature of the following improper integrals :

1. $I_1 = \displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x^3 + 2x}}\,dx.$

   The function is positive and continuous on $[1, +\infty[$

$$x^3 + 2x = x^3 \left(1 + \frac{2}{x^2}\right).$$

   Thus,

$$\frac{1}{\sqrt{x^3 + 2x}} = \frac{1}{x^{3/2}\sqrt{1 + \frac{2}{x^2}}}.$$

   Since

$$\lim_{x \to +\infty} \frac{\frac{1}{\sqrt{x^3+2x}}}{\frac{1}{x^{3/2}}} = \lim_{x \to +\infty} \frac{1}{\sqrt{1 + \frac{2}{x^2}}} = 1,$$

   we have

$$\frac{1}{\sqrt{x^3 + 2x}} \sim_{+\infty} \frac{1}{x^{3/2}}.$$

   By the equivalence criterion, the two integrals

$$\int_1^{+\infty} \frac{1}{\sqrt{x^3 + 2x}}\,dx \quad \text{and} \quad \int_1^{+\infty} \frac{1}{x^{3/2}}\,dx$$

   are of the same nature.
   $\displaystyle\int_1^{+\infty} \frac{1}{x^{3/2}}\,dx,$
   Riemann integral $\alpha = \dfrac{3}{2} > 1$, then $\displaystyle\int_1^{+\infty} \frac{1}{x^{3/2}}\,dx$ converges.
   We conclude that $I_1 = \displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x^3 + 2x}}\,dx$ is convergent.

2. $I_2 = \displaystyle\int_0^2 \frac{\sin(x)}{x\sqrt{x}}\, dx$.

The function $x \mapsto \dfrac{\sin(x)}{x\sqrt{x}}$ is continuous and positive on $(0, 2]$, but we have problem at 0.

$$f(x) = \frac{\sin(x)}{x\sqrt{x}} \sim_0 g(x) = \frac{x}{x\sqrt{x}} = \frac{1}{\sqrt{x}},$$

and

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1.$$

Since the integral $\displaystyle\int_0^2 \frac{1}{\sqrt{x}}\, dx$ is convergent (it is an improper Riemann integral of the second type with $\alpha = \dfrac{1}{2} < 1$), we deduce, by the equivalence criterion, that $\displaystyle\int_0^2 \frac{\sin(x)}{x\sqrt{x}}\, dx$ converges.

4.4.4 Improper integrals of functions of arbitrary Sign

Definition 4.4.3. Let f be a continuous function on $(a, b]$.

$$\begin{aligned} \text{If } \int_a^b |f(x)|\, dx \text{ converges } \quad &\Rightarrow \quad \int_a^b f(x)\, dx \text{ is absolutely convergent} \\ &\Rightarrow \quad \int_a^b f(x)\, dx \text{ is convergent.} \end{aligned}$$

Example 4.4.3. Determine the nature of the following improper integral :

1. $I_2 = \displaystyle\int_2^{+\infty} \frac{\cos(x^2)}{x^2}\, dx$.

The function $x \mapsto \frac{\cos(x^2)}{x^2}$ is continuous on $[2, +\infty)$.

Since $|\cos(x^2)| \leq 1$ for all x, we have

$$\left| \frac{\cos(x^2)}{x^2} \right| \leq \frac{1}{x^2}, \qquad x \geq 2.$$

The integral

$$\int_2^{+\infty} \frac{1}{x^2}\, dx = \lim_{t \to +\infty} \int_2^t \frac{1}{x^2}\, dx = \lim_{t \to +\infty} \left[-\frac{1}{x} \right]_2^t = \lim_{t \to +\infty} \left(-\frac{1}{t} + \frac{1}{2} \right) = \frac{1}{2} \textit{ converges}$$

(Riemann integral with $\alpha = 2 > 1$, then $\displaystyle\int_2^{+\infty} \frac{1}{x^2}\, dx$ is convergent)

By the comparison test, it follows that

$\displaystyle\int_2^{+\infty} \frac{\cos(x^2)}{x^2}\, dx$ also converges absolutely, thus converges.

Chapter 5

Differential Equations

5.1 Review of ordinary differential equations

Differential equations are a fundamental tool in mathematics, physics, economics, and engineering sciences. In general, a differential equation relates an unknown function $y(x)$ to its derivatives with respect to the independent variable x.

General form of a differential equation

The most common form of a differential equation is

$$\frac{dy}{dx} = F(x, y),$$

where $F(x, y)$ is a given function of the independent variable x and the unknown function $y(x)$.

This expresses a relationship between the derivative y' and the quantities x and $y(x)$. The goal is to determine the function $y(x)$ that satisfies this relation.

Remark 5.1.1. In general, we use the notation y' to denote the derivative of y with respect to x. However, we will sometimes encounter the notation $\dfrac{dy}{dx}$ instead of y' (since $y' = \dfrac{dy}{dx}$).

5.2 Ordinary differential equation (ODE)

Definition 5.2.1. An **ordinary differential equation (ODE)** is a relation between a real independent variable x, an unknown function $y(x)$, and its derivatives $y', y'', \ldots, y^{(n)}$, where $n \in \mathbb{N}$.

The order of an ODE is defined as the order of the highest derivative appearing in the equation. Thus, an n-th order differential equation can be written as :

$$F\Big(x, y, y', y'', \ldots, y^{(n)}\Big) = 0.$$

Example 5.2.1

a) $y' + 3y = 6x$ is a first-order differential equation.
b) $y'' + 3y' + 2y = e^{2x}$ is a second-order differential equation.
c) $y^{(9)} - xy'' = x^2$ is a ninth-order differential equation.

5.2.1 First-order ordinary differential equations

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. A *first-order ordinary differential equation* is an equation of the form

$$y' = f(x, y).$$

A differentiable function $y_0 : I \subset \mathbb{R} \to \mathbb{R}$ is called a solution of the first-order differential equation

$$y' = f(x, y)$$

on the interval I if it satisfies

$$y_0'(x) = f\Big(x,\, y_0(x)\Big), \qquad \forall\, x \in I.$$

Example 5.2.1. The function $y : \mathbb{R} \to \mathbb{R}$, defined by

$$y(x) = e^x,$$

is a solution of the following first-order differential equation on $\mathbb{R}$:

$$y' = y$$

because

$$y'(x) = e^x = y(x), \quad \forall x \in \mathbb{R}.$$

5.2.2 Differential equations with separable variables

A first-order differential equation is called *separable* if it can be written as

$$f(y)\, y' = g(x),$$

where f and g are continuous functions.

To solve such an equation, we integrate both sides :

$$\int f(y)\, dy = \int g(x)\, dx + C,$$

where C is an arbitrary constant.

Let F be an antiderivative of f and G an antiderivative of g. Then we can write

$$F(y) = G(x) + C.$$

Finally, solving for y gives the general solution :

$$y = F^{-1}\Big(G(x) + C\Big),$$

where F^{-1} denotes the inverse function of F.

Example 5.2.2. Solve the differential equation :

$$x^2\,dx - \cos y\,dy$$

1. Separate the variables :
$$x^2\,dx = \cos y\,dy$$
2. Integrate both sides :
$$\int x^2\,dx = \int \cos y\,dy$$
3. Compute the integrals :
$$\frac{x^3}{3} + C_1 = \sin y + C_2$$
4. Combine constants and solve for y :
$$\sin y = \frac{x^3}{3} + C, \quad C = C_1 - C_2 \in \mathbb{R}$$
5. Final solution :
$$y = \arcsin\left(\frac{x^3}{3} + C\right)$$

5.3 Linear ordinary differential equations

Definition 5.3.1. A *linear differential equation* of order n is an equation of the form

$$a_0(x)y + a_1(x)y' + \cdots + a_n(x)y^{(n)} = f(x), \tag{5.1}$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $f(x)$ are given functions of x.

The associated *homogeneous equation* is

$$a_0(x)y + a_1(x)y' + \cdots + a_n(x)y^{(n)} = 0. \tag{5.2}$$

Proposition 5.3.2. *Let y_1 and y_2 be two solutions of the homogeneous linear differential equation* (5.2)*. Then $y_1 + y_2$ and $c\,y_1$*
(where c is a constant) are also solutions of (5.2)*.*

Example 5.3.1. Let $y_1(x) = x$ and $y_2(x) = 5$ be two solutions of the differential equation

$$y''' + y'' = 0.$$

Then the functions $y_3(x) = x + 5$ and $y_4(x) = 10x$ are also solutions.

5.3.1 First-order linear differential equation

Definition 5.3.3. A first-order linear differential equation is any differential equation that can be written in the form

$$a(x)y' = b(x)y + c(x),$$

where $a(x)$, $b(x)$, and $c(x)$ are continuous functions defined on a suitable interval.

The equation is said to be *normalized* if $a(x) = 1$ for all x, that is, if it can be written in the form

$$y' = b(x)y + c(x).$$

Remark 5.3.1. When the equation is not normalized, it can be transformed into a normalized form by dividing through by $a(x)$ on any interval where $a(x) \neq 0$.

a) First-order linear equations without a second member (homogeneous)

We first consider equations of the form

$$y' = b(x)y,$$

which is a *separable equation.* The general solution can be obtained as follows :

$$\frac{y'}{y} = b(x) \quad \Longrightarrow \quad \frac{1}{y}\frac{dy}{dx} = b(x)$$

Separating variables :

$$\frac{dy}{y} = b(x)\,dx$$

Integrating both sides :

$$\int \frac{dy}{y} = \int b(x)\,dx$$

$$\ln|y| = B(x) + c, \quad c \in \mathbb{R}$$

Exponentiating both sides :

$$|y| = e^{B(x)+c} = e^c\,e^{B(x)}$$

Setting $k = \pm e^c \in \mathbb{R}$, we obtain the general solution :

$$y_h(x) = k\,e^{B(x)},$$

where $B(x)$ is an antiderivative of $b(x)$ and $k \in \mathbb{R}$.

Example 5.3.2. We solve the differential equation :

$$(1+x^2)y' - 2xy = 0.$$

$$(1+x^2)y' = 2xy \quad \Longrightarrow \quad \frac{y'}{y} = \frac{2x}{1+x^2} \quad \Longrightarrow \quad \frac{dy}{y} = \frac{2x}{1+x^2}\,dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1+x^2}\,dx \quad \Longrightarrow \quad \ln|y| = \ln|1+x^2| + C, \quad C \in \mathbb{R}$$

$$|y| = e^{\ln|1+x^2|+C} = e^C(1+x^2)$$

Set $k = \pm e^C \in \mathbb{R}$, we get the general solution :

$$y_h(x) = k(1+x^2), \quad k \in \mathbb{R}.$$

b) First-order linear equations with a second member (non-homogeneous)

Proposition 5.3.4. *Consider the non-homogeneous first-order linear differential equation*

$$y' = b(x)\,y + c(x).$$

Its general solution is the sum of :

1. *the general solution y_h of the corresponding homogeneous equation*
$$y' = b(x)\,y,$$
2. *and a particular solution y_p of the non-homogeneous equation.*
Thus, the general solution can be written as
$$y(x) = y_p(x) + y_h(x) = y_p(x) + k\,e^{B(x)}, \quad k \in \mathbb{R},$$
where
$$B(x) = \int b(x)\,dx.$$

b.1) Determination of a Particular Solution (using the Method of variation of constants)

This technique is known as the *method of variation of constants*. We begin by solving the homogeneous equation (that is, the equation without the non-homogeneous term). Then we assume that the integration constant c becomes a function depending on the variable x. By substituting this expression into the given differential equation, we can determine this function. In this way, we obtain the desired general solution of the non-homogeneous equation.

Example 5.3.3. Solve the following differential equation

1. $(x^2+1)\,y' - x\,y - 3x = 0.$

   Write it in standard linear form :
$$y' - \frac{x}{x^2+1}\,y = \frac{3x}{x^2+1}.$$
$$y' - \frac{x}{x^2+1}\,y = 0,$$
$$B(x) = \int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1),$$
   so the homogeneous solution is
$$y_h = k\sqrt{x^2+1}, \qquad k \in \mathbb{R}.$$
   Using the method of variation of constants, set $y_p = k(x)\sqrt{x^2+1}$.
$$k'(x)\sqrt{x^2+1} = \frac{3x}{x^2+1},$$
$$\textit{so}\;\; k'(x) = \frac{3x}{(x^2+1)^{3/2}}.$$
$$k(x) = \int \frac{3x}{(x^2+1)^{3/2}}\,dx = -3(x^2+1)^{-1/2} + C.$$
$$y_p = (-3(x^2+1)^{-1/2} + C)\sqrt{x^2+1} = -3 + C\sqrt{x^2+1},$$
$$\textit{Hence}\;\; y(x) = y_h(x) + y_p(x) = -3 + \alpha\sqrt{x^2+1}, \qquad \alpha = C + k \in \mathbb{R}.$$

5.3.2 Bernoulli equations

Definition 5.3.5. A **Bernoulli equation** is any differential equation that can be written in the form :

$$a(x)y' + b(x)y + c(x)y^{\alpha} = 0,$$

where $a(x)$, $b(x)$, and $c(x)$ are continuous functions on a given interval, α is a fixed real number, and $\alpha \neq 0$, $\alpha \neq 1$.

Method of solving Bernoulli equations

Consider the Bernoulli equation

$$a(x)y' + b(x)y + c(x)y^{\alpha} = 0, \qquad \alpha \neq 0, 1.$$

Introduce the substitution

$$t = y^{1-\alpha}.$$

$$Then \;\; t' = (1-\alpha)y^{-\alpha}y'.$$

Dividing the original equation by y^{α} and using the substitution gives :

$$\begin{aligned} a(x)\frac{y'}{y^{\alpha}} + b(x)\frac{y}{y^{\alpha}} + c(x) = 0 &\Longrightarrow a(x)\frac{y'}{y^{\alpha}} + b(x)y^{1-\alpha} + c(x) = 0, \\ &\Longrightarrow a(x)\frac{t'}{1-\alpha} + b(x)t + c(x) = 0. \end{aligned}$$

This is now a **first-order linear equation in** t, which can be solved by standard methods.

Remark 5.3.2. To solve a Bernoulli differential equation, one first divides the equation by y^{α}, and then introduces the substitution

$$t = y^{1-\alpha}.$$

This transforms the original nonlinear equation into a linear first-order differential equation in t, which can then be solved by standard methods.

Example 5.3.4. solve the differential equation on $\mathbb{R}$:

$$y' - \frac{y}{2x} = 5x^2y^5.$$

Identify α and divide by y^5

Here, $\alpha = 5$. Dividing the equation by y^5 gives :

$$\frac{y'}{y^5} - \frac{1}{2xy^4} = 5x^2.$$

Substitution

$$t = y^{-4} \quad \Longrightarrow \quad t' = -4\,y^{-5}y'.$$

$$\frac{y'}{y^5} = -\frac{t'}{4}.$$

Substituting into the equation :

$$-\frac{t'}{4} - \frac{1}{2x}t = 5x^2 \quad \Longrightarrow \quad t' + \frac{2}{x}t = -20x^2.$$

This is now a **first-order linear equation** in t.

a). Homogeneous equation

Consider

$$t' + \frac{2}{x}t = 0.$$

$$\frac{t'}{t} = -\frac{2}{x} \quad \Longrightarrow \quad \int \frac{1}{t}dt = \int -\frac{2}{x}dx \quad \Longrightarrow \quad \ln|t| = -2\ln|x| + C \quad \Longrightarrow \quad t = k\frac{1}{x^2}, \quad k \in \mathbb{R}.$$

b). Particular solution by variation of constant

Assume $t_h = k(x)\dfrac{1}{x^2}$.

$$t_h' = k'(x)\frac{1}{x^2} - k(x)\frac{2}{x^3}.$$

Substitute into the nonhomogeneous equation :

$$t_h' + \frac{2}{x}t_h = k'(x)\frac{1}{x^2} - k(x)\frac{2}{x^3} + \frac{2}{x}k(x)\frac{1}{x^2} = k'(x)\frac{1}{x^2}.$$

Set equal to $-20x^2$:

$$k'(x)\frac{1}{x^2} = -20x^2 \quad \Longrightarrow \quad k'(x) = -20x^4.$$

$$k(x) = \int -20x^4 dx = -4x^5 + C.$$

Hence, the particular solution :

$$t = t_h = k(x)\frac{1}{x^2} = \frac{-4x^5 + C}{x^2} = -4x^3 + \frac{C}{x^2}.$$

Back-substitution

Recall $t = y^{-4}$, so

$$y^4 = \frac{1}{t} \quad \Longrightarrow \quad y = \left(\frac{1}{-4x^3 + \frac{\alpha}{x^2}}\right)^{\frac{1}{4}}, \quad \alpha \in \mathbb{R}.$$

Thus, the general solution of the original equation is :

$$y(x) = \left(\frac{1}{-4x^3 + \frac{\alpha}{x^2}}\right)^{1/4}, \quad \alpha \in \mathbb{R}.$$

5.3.3 Riccati Equations

Definition 5.3.6. A **Riccati equation** is any differential equation that can be written in the form

$$y' = a(x)y^2 + b(x)y + c(x),$$

where $a(x)$, $b(x)$ and $c(x)$ are continuous functions on a given interval.

Remark 5.3.3. This type of equation can only be solved if a particular solution y_1 is already known.

Method of solving Riccati equation

We set

$$y = y_1 + \frac{1}{t},$$

where y_1 is a particular solution satisfying

$$y_1' = a(x)y_1^2 + b(x)y_1 + c(x),$$

and where t does not vanish on the interval of resolution.

$$Since \ \ y' = y_1' - \frac{t'}{t^2},$$

we substitute into the Riccati equation :

$$y' = a(x)y^2 + b(x)y + c(x),$$

to obtain

$$y_1' - \frac{t'}{t^2} = a(x)\left(y_1 + \frac{1}{t}\right)^2 + b(x)\left(y_1 + \frac{1}{t}\right) + c(x).$$

Using the fact that $y_1' = a(x)y_1^2 + b(x)y_1 + c(x)$, this simplifies to

$$-\frac{t'}{t^2} = \frac{2a(x)y_1}{t} + \frac{b(x)}{t},$$

which gives

$$t' + \Big(2a(x)y_1 + b(x)\Big)t = -a(x).$$

Thus, we obtain a first-order linear differential equation in t.

Example 5.3.5. solve on the interval $]0, +\infty[$ the differential equation

$$2x^2y' = (x-1)(y^2 - x^2) + 2xy.$$

We rewrite it as

$$y' = \frac{x-1}{2x^2}\,y^2 + \frac{1}{x}y - \frac{x-1}{2}.$$

We observe that $y_1 = x$ is a particular solution. We use the substitution

$$y = y_1 + \frac{1}{t} = x + \frac{1}{t},$$

where t never vanishes on $]0, +\infty[$.

$$y' = 1 - \frac{t'}{t^2}.$$

Substituting into the differential equation :

$$1 - \frac{t'}{t^2} = \frac{x-1}{2x^2}\left(x + \frac{1}{t}\right)^2 + \frac{1}{x}\left(x + \frac{1}{t}\right) - \frac{x-1}{2}.$$

$$1 - \frac{t'}{t^2} = \frac{x-1}{2x^2}\left(x^2 + \frac{2x}{t} + \frac{1}{t^2}\right) + 1 + \frac{1}{xt} - \frac{x-1}{2}.$$

After simplification :

$$-\frac{t'}{t^2} = \frac{1}{t} + \frac{x-1}{2x^2}\frac{1}{t^2}.$$

Multiplying by t^2:

$$-t' = t + \frac{x-1}{2x^2}.$$

Thus we obtain a first-order linear differential equation :

$$t' + t = \frac{1}{2x^2} - \frac{1}{2x}.$$

a) Homogeneous equation

$$t' + t = 0.$$

$$t' = -t, \quad \frac{t'}{t} = -1, \quad \frac{dt}{t} = -dx,$$
$$\ln|t| = -x + C, \quad t = Ke^{-x}, \ K \in \mathbb{R}.$$

b) General solution of the original equation

$$t = ce^{-x} - \frac{1}{2x}.$$

$$Thus \ \ y = x + \frac{1}{t} = x + \frac{1}{ce^{-x} - \frac{1}{2x}}, \qquad c \in \mathbb{R}.$$

This is the general solution on $]0, +\infty[$.

5.3.4 Second-order linear differential equations

Homogeneous equations

Definition 5.3.7. A second-order linear differential equation with constant coefficients and without a second member is any differential equation that can be written in the form

$$ay'' + by' + cy = 0,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$ (otherwise we are in the first-order linear case).

Proposition 5.3.8. *Depending on the sign of $\Delta = b^2 - 4ac$, we have the following cases*

1. $\Delta > 0$*: The characteristic equation (CE) has two distinct real roots $r_1 \neq r_2$. The general solution of the homogeneous equation (HE) is*

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad c_1, c_2 \in \mathbb{R}.$$

2. $\Delta = 0$*: The characteristic equation has a double root $r \in \mathbb{R}$. The general solution is*

$$y(x) = (c_1 x + c_2) e^{rx}, \quad c_1, c_2 \in \mathbb{R}.$$

3. $\Delta < 0$*: The characteristic equation has two complex conjugate roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ $(\alpha, \beta \in \mathbb{R}, \beta \neq 0)$. The general solution is*

$$y(x) = \Big(c_1 \cos \beta x + c_2 \sin \beta x\Big) e^{\alpha x}, \quad c_1, c_2 \in \mathbb{R}.$$

Example 5.3.6. Solve the following differential equations :

1. $y'' - 4y' + 3y = 0$. The characteristic equation is

$$r^2 - 4r + 3 = 0,$$

with two distinct real roots $r_1 = 1$ and $r_2 = 3$. The general solution is

$$y(x) = c_1 e^{x} + c_2 e^{3x}, \quad c_1, c_2 \in \mathbb{R}.$$

2. $y'' + 2y' + y = 0$. The characteristic equation is

$$r^2 + 2r + 1 = 0,$$

with a double root $r = -1$. The general solution is

$$y(x) = (c_1 x + c_2) e^{-x}, \quad c_1, c_2 \in \mathbb{R}.$$

3. $y'' + 2y' + 4y = 0$. The characteristic equation is

$$r^2 + 2r + 4 = 0,$$

with discriminant $\Delta = -12$. The roots are complex: $r_1 = -1 + \sqrt{3}i$, $r_2 = -1 - \sqrt{3}i$, and the general solution is

$$y(x) = \Big(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\Big) e^{-x}, \quad c_1, c_2 \in \mathbb{R}.$$

Nonhomogeneous equation

Any equation of the form

$$ay'' + by' + cy = f(x)$$

where a, b, and c are real constants and $f(x) \neq 0$, is called a nonhomogeneous linear second-order equation with constant coefficients.

Method of solving a nonhomogeneous equation

Consider the equation
$$ay'' + by' + cy = f(x)$$

Particular solution to (E)

1. If $f(x) = P(x)$, where $\beta \in \mathbb{R}$ and P is a polynomial, we look for a solution of the form
$$y = x^k Q(x)$$
where Q is a polynomial with $\deg Q = \deg P$, and
$$k = \begin{cases} 0, & \text{if 0 is not a root of the characteristic equation (CE),} \\ 1, & \text{if 0 is a simple root of (CE),} \\ 2, & \text{if 0 is a double root of (CE).} \end{cases}$$

2. If $f(x) = e^{\beta x} P(x)$, where $\beta \in \mathbb{R}$ and P is a polynomial, we look for a solution of the form
$$y = x^k e^{\beta x} Q(x)$$
where Q is a polynomial with $\deg Q = \deg P$, and
$$k = \begin{cases} 0, & \text{if } \beta \text{ is not a root of the characteristic equation (CE),} \\ 1, & \text{if } \beta \text{ is a simple root of (CE),} \\ 2, & \text{if } \beta \text{ is a double root of (CE).} \end{cases}$$

3. If $f(x) = \cos(\beta x)$ or $f(x) = \sin(\beta x)$, we look for a solution of the form
$$y = A\cos(\beta x) + B\sin(\beta x),$$
where A and B are constants to be determined.

   Remark 5.3.4. - The derivatives of $\cos(\beta x)$ and $\sin(\beta x)$ are still linear combinations of $\cos(\beta x)$ and $\sin(\beta x)$, so substituting y into the equation gives a system of equations for A and B.

   - If $\pm i\beta$ is a root of the characteristic equation (CE), we multiply the particular solution by x (or x^2 if it is a double root) to avoid duplication with the homogeneous solution.

Example 5.3.7. Solve the differential equation

1. $y'' - 4y' + 4y = (x^2 + 1)e^x$.

   Solve the homogeneous equation

$$\textit{homogeneous equation is } \; y'' - 4y' + 4y = 0.$$
$$\textit{The characteristic equation is } \; r^2 - 4r + 4 = 0.$$

   We have a double root $r = 2$, so the general solution of the homogeneous equation is
$$y_h = (c_1 x + c_2)e^{2x}, \quad c_1, c_2 \in \mathbb{R}.$$

Find a particular solution

We look for a particular solution of the form

$$y_p(x) = Q(x)e^x,$$

where $Q(x)$ is a polynomial of degree 2 :

$$y_p(x) = (ax^2 + bx + c)e^x.$$

$$Then, \quad y_p'(x) = (2ax + b)e^x + (ax^2 + bx + c)e^x = (ax^2 + (2a + b)x + (b + c))e^x,$$

$$y_p''(x) = 2ae^x + 2(2ax + b)e^x + (ax^2 + bx + c)e^x = (ax^2 + (4a + b)x + (2a + 2b + c))e^x.$$

Substituting y_p, y_p', y_p'' into the equation, we get

$$((a - 1)x^2 + (b - 4a)x + (c + b + 2a - 1))e^x = 0.$$

Equating coefficients to zero, we find

$$\begin{cases} a - 1 = 0, \\ b - 4a = 0, \\ c + b + 2a - 1 = 0, \end{cases} \qquad \Rightarrow \quad a = 1,\ b = 4,\ c = 7.$$

$$the\ \ particular\ \ solution\ \ y_p(x) = (x^2 + 4x + 7)e^x.$$

General solution of the nonhomogeneous equation

$$Finally\ \ y(x) = y_h + y_p = (c_1x + c_2)e^{2x} + (x^2 + 4x + 7)e^x, \quad c_1, c_2 \in \mathbb{R}.$$

5.4 Partial differential equations

In this section, we will extend our study to the case where the unknown function depends on several variables. In this case, we say that we are dealing with **partial differential equations (PDE)**. We will limit ourselves to the study and solution of **first- and second-order equations**.

Let u be a function depending on n independent variables $x_1, x_2, \dots, x_n$.

Definition 5.4.1. A **partial differential equation (PDE)** is any relation between an unknown function u, the variables $x_1, x_2, \dots, x_n$, and its partial derivatives, whose general form is

$$F\Big(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial^n u}{\partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_k^{m_k}}\Big) = g(x_1, x_2, \dots, x_n, u),$$

where $1 \leq k \leq n$ and $m_1 + m_2 + \cdots + m_k = n$.

Definition 5.4.2. The **order** of a partial differential equation (PDE) is the order of the highest partial derivative it contains.

Definition 5.4.3. A PDE is called **linear** if it is linear in the unknown function u and all its partial derivatives.

Definition 5.4.4. A PDE is called **homogeneous** if the function $g(x_1, x_2, \ldots, x_n, u) = 0$ in its general form.

Definition 5.4.5. A PDE is called **quasi-linear** if it is linear with respect to the highest-order partial derivatives.

Remark 5.4.1. Let $u(x, y)$ be a function of two variables. Then the first and second-order partial derivatives can be written as :

$$u'_x = \frac{\partial u}{\partial x}, \quad u'_y = \frac{\partial u}{\partial y}$$

$$u''_{x^2} = \frac{\partial^2 u}{\partial x^2}, \quad u''_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u''_{y^2} = \frac{\partial^2 u}{\partial y^2}$$

u'_x is the derivative of u_x with respect to x.
u'_y is the derivative of u_y with respect to y.
u''_{x^2} is the derivative of u'_x with respect to x.
u''_{y^2} is the derivative of u'_y with respect to y.
u''_{yx} is the derivative of u'_y with respect to x.

Theorem 5.4.6 (Schwarz's Theorem)**.** *Let $u(x, y)$ be a function whose mixed partial derivatives are continuous. Then the mixed derivatives are equal :*

$$u''_{xy} = u''_{yx}.$$

Example 5.4.1. Compute partial derivatives :

$$u(x, y) = x^2 + xy^2 - y$$

First-order derivatives :

$$u'_x = \frac{\partial u}{\partial x} = 2x + y^2, \quad u'_y = \frac{\partial u}{\partial y} = 2xy - 1$$

Second-order derivatives :

$$u''_{x^2} = \frac{\partial^2 u}{\partial x^2} = 2, \quad u''_{xy} = \frac{\partial^2 u}{\partial x \partial y} = 2y$$

$$u''_{y^2} = \frac{\partial^2 u}{\partial y^2} = 2x, \quad u''_{yx} = \frac{\partial^2 u}{\partial y \partial x} = 2y$$

5.4.1 Chain rule for a composite function

Let $F(x, y) = f(u, v)$, where u and v are functions of x and y that admit partial derivatives. If f admits partial derivatives, then F also admits partial derivatives, and :

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v},$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v}.$$

Example 5.4.2. Let

$$f(u,v) = u^2 + uv + v, \quad u(x,y) = 2x - y, \quad v(x,y) = x + y,$$

and define the composite function

$$F(x,y) = f(u(x,y), v(x,y)).$$

Step 1 : Compute partial derivatives of f with respect to u and v

$$\frac{\partial f}{\partial u} = 2u + v, \quad \frac{\partial f}{\partial v} = u + 1$$

Step 2 : Compute partial derivatives of u and v with respect to x and y

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 1$$

Step 3 : Apply the chain rule

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} = 2(2u+v) + 1(u+1)$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v} = -1(2u+v) + 1(u+1)$$

5.4.2 First-order partial differential equation (PDE)

We consider the following first-order differential equation :

$$M(x,y) + N(x,y)y' = 0.$$

This can be written in differential form as :

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

If there exists a function $U : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial U}{\partial x}(x,y) = M(x,y), \quad \frac{\partial U}{\partial y}(x,y) = N(x,y),$$

$$\textit{then} \;\; U(x,y) = \text{constant}.$$

Example 5.4.3. Solve the differential equations

1. $y' = -\dfrac{1+2xy}{x^2}$

   Consider the differential equation in differential form :

$$(1+2xy)\,dx + x^2\,dy = 0$$

$$We\ \ identify\ \ :\ \ M(x,y) = 1+2xy, \quad N(x,y) = x^2$$

$$we\ \ have\ \ \frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x$$

   hence $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, is exact.

   Find the potential function $U(x,y)$

$$\frac{\partial U}{\partial x} = 1+2xy, \quad \frac{\partial U}{\partial y} = x^2$$

   Integrate $M(x,y)$ with respect to x :

$$U(x,y) = \int (1+2xy)\,dx = x + x^2y + k(y)$$

   Determine $k(y)$

$$\frac{\partial U}{\partial y} = x^2 + k'(y) = x^2 \quad \Longrightarrow \quad k'(y) = 0 \quad \Longrightarrow \quad k(y) = C$$

   general solution of the differential equation

$$U(x,y) = x + x^2y = C \quad \Longrightarrow \quad y = \frac{C-x}{x^2}$$

2. $y' = -\dfrac{1+2xy^3}{3x^2y^2}$

$$y' = -\frac{1+2xy^3}{3x^2y^2} \quad \Longrightarrow \quad (1+2xy^3)\,dx + 3x^2y^2\,dy = 0$$

$$we\ \ have\ \ M(x,y) = 1+2xy^3, \quad N(x,y) = 3x^2y^2$$

$$\frac{\partial M}{\partial y} = 6xy^2, \quad \frac{\partial N}{\partial x} = 6xy^2$$

   $\Rightarrow$ Equation is exact.

   Find potential function $U(x,y)$

$$\frac{\partial U}{\partial x} = 1+2xy^3 \quad \Rightarrow \quad U(x,y) = x + x^2y^3 + k(y)$$

$$\frac{\partial U}{\partial y} = 3x^2y^2 + k'(y) = 3x^2y^2 \quad \Rightarrow \quad k(y) = C$$

$$U(x,y) = x + x^2y^3 = C \quad \Rightarrow \quad y = \sqrt[3]{\frac{C-x}{x^2}}$$

$C = 0$,

$$y(x) = -x^{-1/3}, \quad y'(x) = \frac{1}{3}x^{-4/3}$$

Some types of first-order partial differential equations

1. Transport equation

$$\frac{\partial u}{\partial t}(t,x) + c\,\frac{\partial u}{\partial x}(t,x) = 0$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and c is a constant.

2. Shock wave equation

$$\frac{\partial u}{\partial t}(t,x) + u(t,x)\,\frac{\partial u}{\partial x}(t,x) = 0$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

3. Burgers' equation

$$a(u)\,\frac{\partial u}{\partial t}(t,x) + \frac{\partial u}{\partial x}(t,x) = 0$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

5.4.3 Second-order partial differential equations

Definition 5.4.7. A **second-order partial differential equation** (PDE) is any equation of the form

Definition 5.4.8. A **second-order partial differential equation** (PDE) is any equation of the form

$$F\Big(x_1, x_2, \ldots, x_n,\, u,\, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_{n-1} \partial x_n}\Big) = 0$$

or, equivalently, in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_1,\ldots,x_n,u)\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x_1,\ldots,x_n,u)\,\frac{\partial u}{\partial x_i} = f(x_1,\ldots,x_n,u),$$

where $u = u(x_1,\ldots,x_n)$ is the unknown function, and a_{ij}, b_i, f are given functions.

Restriction to two independent variables :
In this study, we focus on linear and quasi-linear second-order PDE with two independent variables (x, y) :

i) **Linear second-order PDE :**

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}+p(x,y)\frac{\partial u}{\partial x}+q(x,y)\frac{\partial u}{\partial y}+g(x,y)u=f(x,y) \quad (5.3)$$

ii) **Quasi-linear second-order PDE :**

$$a(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x^2}+b(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y,u,u_x,u_y)\frac{\partial^2 u}{\partial y^2}=R(x,y,u,u_x,u_y) \quad (5.4)$$

where $u_x = \dfrac{\partial u}{\partial x}$ and $u_y = \dfrac{\partial u}{\partial y}$.

These are the general forms of linear and quasi-linear second-order PDE in two variables.

Classification of second-order PDE

The type of the second-order PDE (Equation 5.3) and (Equation 5.4) depends on their **discriminant** :

$$\Delta = b^2 - ac$$

where a, b, and c are the coefficients of the second-order derivatives.

1. **Elliptic :** $\Delta < 0$ Example: Laplace equation
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$
2. **Parabolic :** $\Delta = 0$ Example: Heat equation
$$\frac{\partial u}{\partial t}-k\frac{\partial^2 u}{\partial x^2}=0$$
3. **Hyperbolic :** $\Delta > 0$ Example : Wave equation
$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$$

Example 5.4.4. Consider the following partial differential equations :

1. $u_{xx} + u_{yy} + 4u_{xy} = u$

$$or\ \ equivalently\ \ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+4\frac{\partial^2 u}{\partial x\partial y}=u$$

Identify coefficients :

$$a = 1, \quad 2b = 4 \implies b = 2, \quad c = 1$$

Compute the discriminant :

$$\Delta = b^2 - ac = 2^2 - (1)(1) = 4 - 1 = 3$$

Classification :

Since $\Delta = 3 > 0$ for all $(x, y) \in \mathbb{R}^2$, the PDE is **hyperbolic**.

2. $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} = 0$

Compare with the general second-order PDE in two variables :

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + \cdots = 0$$

$$We\ \ have\ \ :\ \ a = x, \quad b = 0, \quad c = y$$

Compute the discriminant :

$$\Delta = b^2 - ac = 0^2 - (x)(y) = -xy$$

Classification :

1. If $xy > 0$, then $\Delta < 0 \implies$ **elliptic** PDE
2. If $xy = 0$, then $\Delta = 0 \implies$ **parabolic** PDE
3. If $xy < 0$, then $\Delta > 0 \implies$ **hyperbolic** PDE

Hence, the type of the PDE depends on the values of x and y.

Main equations in Physics

1. Laplace Equation It has the form :

$$\frac{\partial^2 u}{\partial x^2} + c^2\frac{\partial^2 u}{\partial y^2} = 0$$

For $c \neq 0$, this is an **elliptic** equation.

2. Poisson Equation

$$\nabla^2 u = f(x, y)$$

This is also an **elliptic** equation.

3. Heat Equation (Diffusion Equation) It has the form :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\frac{\partial u}{\partial t}$$

This is a **parabolic** equation. The term $\frac{\partial u}{\partial t}$ is called the *diffusion term*.

4. Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \quad \text{or equivalently, } \nabla^2 u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

This is a **hyperbolic** equation. The term $\dfrac{\partial^2 u}{\partial t^2}$ is called the *propagation term.*

5. Tricomi Equation

$$\frac{\partial^2 u}{\partial y^2} = y\frac{\partial^2 u}{\partial x^2}$$

The Tricomi equation is of mixed type : it is **elliptic** for $y > 0$ and **hyperbolic** for $y < 0$.

5.4.4 Standard form of second-order PDE (Method of Characteristics)

Theorem 5.4.9. *(Characteristic Curves) Consider the second-order PDE :*

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + \cdots = 0$$

The characteristic curves are determined as follows :

1. *If $a(x,y) \neq 0$, the characteristic curves are the solutions of*

$$a(x,y)\left(\frac{dy}{dx}\right)^2 + b(x,y)\frac{dy}{dx} + c(x,y) = 0$$

2. *If $a(x,y) = 0$ and $c(x,y) \neq 0$, the characteristic curves are the solutions of*

$$c(x,y)\left(\frac{dy}{dx}\right)^2 + b(x,y)\frac{dy}{dx} + a(x,y) = 0$$

3. *If $a(x,y) = 0$ and $c(x,y) = 0$, the characteristic curves are straight lines given by*

$$x = k_1, \quad y = k_2$$

   where k_1 and k_2 are constants.

Theorem 5.4.10 (Hyperbolic Equations and Characteristic Variables)**.**
If the second-order PDE

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + \cdots = 0$$

*is **hyperbolic**, we can use the characteristic curves as a change of variables :*

$$\phi_1(x,y) = k_1, \quad \phi_2(x,y) = k_2$$

Define new variables :

$$x_1 = \phi_1(x, y), \quad x_2 = \phi_2(x, y)$$

Then the PDE transforms into :

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = G\Big(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\Big)$$

Furthermore, if we define :

$$y_1 = \phi_1(x, y) + \phi_2(x, y), \quad y_2 = \phi_1(x, y) - \phi_2(x, y)$$

then the PDE can be written as :

$$\frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} = H\Big(y_1, y_2, u, \frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}\Big)$$

where G and H are functions determined by the change of variables and the lower-order terms of the original PDE.

Theorem 5.4.11 (Elliptic Equations and Characteristic Variables)**.**
If the second-order PDE

$$a(x, y)\frac{\partial^2 u}{\partial x^2} + 2b(x, y)\frac{\partial^2 u}{\partial x \partial y} + c(x, y)\frac{\partial^2 u}{\partial y^2} + \cdots = 0$$

*is of **elliptic type**, we can use the characteristic variables as a change of variables :*

$$\phi_1(x, y) + i\psi_1(x, y) = constant_1, \quad \phi_2(x, y) + i\psi_2(x, y) = constant_2.$$

Introduce the new variables :

$$x_1 = \phi_1(x, y) + i\psi_1(x, y), \quad x_2 = \phi_2(x, y) + i\psi_2(x, y).$$

Then, in the transformed PDE, the mixed derivative term disappears, simplifying the equation :

$$\frac{\partial^2 u}{\partial x_1 \partial x_2}.$$

That is, the mixed derivative term vanishes.

This change of variables is particularly useful for reducing elliptic PDEs to canonical form.

Example 5.4.5. Solution of the Cauchy Problem for the Wave Equation by the Method of Characteristics. Consider the wave equation :

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0$$

with initial conditions :

$$\begin{aligned} u(x,0) &= \phi_1(x), \\ \frac{\partial u}{\partial t}(x,0) &= \phi_2(x), \end{aligned}$$

where a is a constant.

Determine the type and characteristic curves

The discriminant of the PDE is :

$$\Delta = (0)^2 - (-a^2) = 4a^2 > 0$$

so the PDE is **hyperbolic**. The characteristic curves satisfy :

$$\left(\frac{dx}{dt}\right)^2 - a^2 = 0$$

or equivalently :

$$(dx - a\,dt)(dx + a\,dt) = 0$$

Hence, the characteristic curves are :

$$x - at = c_1, \quad x + at = c_2.$$

Change of variables

Introduce new variables along characteristics :

$$z = x - at, \quad y = x + at.$$

Then the wave equation reduces to :

$$\frac{\partial^2 u}{\partial z \partial y} = 0.$$

General solution along characteristics

Integrating, we obtain :

$$u(z,y) = F(z) + G(y),$$

or equivalently, in original variables :

$$u(x,t) = F(x - at) + G(x + at).$$

Determine F and G from initial conditions

$$u(x,0) = F(x) + G(x) = \phi_1(x), \qquad \frac{\partial u}{\partial t}(x,0) = -aF'(x) + aG'(x) = \phi_2(x),$$

Integrating the second equation :

$$-F(x) + G(x) = \frac{1}{a}\int_0^x \phi_2(\theta)\,d\theta.$$

Adding and subtracting gives :

$$\begin{aligned} G(x) &= \frac{1}{2}\phi_1(x) + \frac{1}{2a}\int_0^x \phi_2(\theta)\,d\theta, \\ F(x) &= \frac{1}{2}\phi_1(x) - \frac{1}{2a}\int_0^x \phi_2(\theta)\,d\theta. \end{aligned}$$

Substituting back, the solution of the Cauchy problem is :

$$u(x,t) = \frac{1}{2}\left[\phi_1(x - at) + \phi_1(x + at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \phi_2(\theta)\,d\theta.$$

This is the classical d'Alembert formula for the vibrating string.

5.4.5 Separation of Variables

Consider the following second-order partial differential equation with constant coefficients:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial u}{\partial x} + \gamma\frac{\partial u}{\partial y} + \lambda u = 0. \tag{5.3}$$

Note that the mixed derivative term does not appear in this PDE.
The method of separation of variables consists in looking for a solution of the form

$$u(x,y) = f(x)\,g(y). \tag{5.4}$$

Substituting (5.4) into (5.3), we obtain

$$af''(x)g(y) + bf(x)g''(y) + \beta f'(x)g(y) + \gamma f(x)g'(y) + \lambda f(x)g(y) = 0.$$

Assuming $f(x)g(y) \neq 0$, we divide by $f(x)g(y)$ to obtain

$$a\frac{f''(x)}{f(x)} + b\frac{g''(y)}{g(y)} + \beta\frac{f'(x)}{f(x)} + \gamma\frac{g'(y)}{g(y)} + \lambda = 0.$$

Since the left-hand side is the sum of a function of x and a function of y, each side must be equal to a constant. Let this constant be $c \in \mathbb{R}$. We obtain the two ordinary differential equations

$$a\frac{f''(x)}{f(x)} + \beta\frac{f'(x)}{f(x)} + c = 0, \quad b\frac{g''(y)}{g(y)} + \gamma\frac{g'(y)}{g(y)} + \lambda - c = 0.$$

Multiplying each equation by $f(x)$ or $g(y)$, we obtain the classical form :

$$\begin{cases} af''(x) + \beta f'(x) + cf(x) = 0, \\ bg''(y) + \gamma g'(y) + (\lambda - c)g(y) = 0. \end{cases}$$

The two resulting ordinary differential equation. are solved separately. By substituting the explicit expressions of $f(x)$ and $g(y)$ into the ansatz $u(x, y) = f(x)g(y)$, we obtain the solution of the original PDE.

Example 5.4.6. Solve the following wave equation using separation of variables :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < l,\ t > 0, \\ u(0, t) = u(l, t) = 0, & t \geq 0, \\ u(x, 0) = \phi_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = \phi_2(x), & 0 \leq x \leq l, \end{cases}$$

where a is a constant.

Assume a separated solution :

$$u(x, t) = f(x)g(t).$$

Substituting into the wave equation gives

$$f(x)g''(t) = a^2 f''(x)g(t),$$

which can be written as

$$\frac{g''(t)}{a^2 g(t)} = \frac{f''(x)}{f(x)} = -\omega^2,$$

where ω is a separation constant.

Solve the spatial equation : The spatial ODE is

$$f''(x) + \omega^2 f(x) = 0.$$

Its general solution is

$$f(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x).$$

Applying the boundary conditions $f(0) = 0$ and $f(l) = 0$ gives $c_1 = 0$ and $\omega = \dfrac{k\pi}{l}$ with $k = 1, 2, 3, \ldots$. Therefore,

$$f_k(x) = \sin\frac{k\pi}{l}x.$$

Solve the temporal equation : The temporal ODE is

$$g''(t) + (a\omega)^2 g(t) = 0,$$

with solution

$$g_k(t) = a_k \cos \frac{ka\pi}{l} t + b_k \sin \frac{ka\pi}{l} t.$$

Form the general solution : Combining the spatial and temporal parts, the solution is

$$u(x,t) = \sum_{k=1}^{\infty} \sin \frac{k\pi}{l} x \left(a_k \cos \frac{ka\pi}{l} t + b_k \sin \frac{ka\pi}{l} t \right).$$

Determine coefficients from initial conditions From $u(x,0) = \phi_1(x)$:

$$\sum_{k=1}^{\infty} a_k \sin \frac{k\pi}{l} x = \phi_1(x) \quad \Rightarrow \quad a_k = \frac{2}{l} \int_0^l \phi_1(x) \sin \frac{k\pi}{l} x \, dx.$$

From $\dfrac{\partial u}{\partial t}(x,0) = \phi_2(x)$:

$$\sum_{k=1}^{\infty} b_k \frac{ka\pi}{l} \sin \frac{k\pi}{l} x = \phi_2(x) \quad \Rightarrow \quad b_k = \frac{2}{l} \int_0^l \phi_2(x) \sin \frac{k\pi}{l} x \, dx.$$

Thus, the solution of the wave equation is

$$u(x,t) = \sum_{k=1}^{\infty} \sin \frac{k\pi}{l} x \left(a_k \cos \frac{ka\pi}{l} t + b_k \sin \frac{ka\pi}{l} t \right)$$

5.5 Special functions

5.5.1 Euler functions

Definition 5.5.1 (Euler integral of the first kind or Beta function)**.** The *Euler integral of the first kind*, also called the *Beta function*, is a function of two parameters x and y defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt,$$

or equivalently,

$$B(x,y) = 2 \int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} \, d\theta.$$

Definition 5.5.2 (Euler integral of the second kind or Gamma function)**.** The *Euler integral of the second kind*, also called the *Gamma function*, is a function of a parameter x defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt,$$

or equivalently,

$$\Gamma(x) = 2 \int_0^{\infty} t^{2x-1} e^{-t^2} \, dt.$$

Properties of Beta and Gamma functions

Theorem 5.5.3. *For all positive x and y, the Beta and Gamma functions satisfy*

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 5.5.4. *For all $x > 0$, the Gamma function satisfies the recurrence relation*

$$\Gamma(x+1) = x\,\Gamma(x).$$

Theorem 5.5.5. *For all $0 < x < 1$, the Gamma function satisfies the reflection formula:*

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

Chapter 6

Series

6.1 Numerical Series

Definition 6.1.1. Let (u_n) be a sequence of real numbers. We define

$$S_n = u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n = \sum_{k=0}^{n} u_k.$$

S_n is called the *partial sum of order* n.

Definition 6.1.2. Let $(u_n)_{n\geq 0}$ be a sequence of real numbers. The *numerical series* with general term u_n is defined as :

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots + u_{n-1} + u_n + u_{n+1} + \cdots$$

The *partial sum* of order n is

$$S_n = u_0 + u_1 + u_2 + \cdots + u_n,$$

and the sum of the series (if it exists) is

$$S = \lim_{n\to\infty} S_n.$$

Here, S_n is called the *partial sum of order* n.

Remark 6.1.1. Some commonly used notations :

- $\sum_{n=0}^{\infty} u_n$, $\sum u_n$, or $\sum_{n\geq 0} u_n$: denotes the numerical series.
- $(S_n)_{n\geq 0}$: the sequence of partial sums of order n.
- u_n: the general term of the series.
- S: the sum of the series, if it exists.

6.1.1 The nature of a series

Consider the series $\sum_{n=0}^{\infty} u_n$

with partial sums $(S_n)_{n\geq 0}$. The series is said to be *convergent* if the sequence (S_n) converges. Otherwise, it is *divergent.* More precisely :

1. If $(S_n)_{n\geq 0}$ converges, then
$$\sum_{n=0}^{\infty} u_n \text{ converges.}$$
2. If $(S_n)_{n\geq 0}$ diverges, then
$$\sum_{n=0}^{\infty} u_n \text{ diverges.}$$

Example 6.1.1. Calculate the sum of the following numerical series :

(a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Let $u_n = \frac{1}{n(n+1)}$, $n \geq 1$. We can write u_n in the following form (partial fraction decomposition) :
$$u_n = \frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}.$$
Solving for a and b :
$$(a+b)n + a = 1 \implies a+b=0,\ a=1 \implies b=-1.$$
$$Thus,\quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},\quad \forall n \geq 1.$$
The partial sum is
$$S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}.$$
Taking the limit as $n \to \infty$:
$$S = \lim_{n\to\infty} S_n = 1.$$
$$Therefore,\quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \ is\ convergent.$$

(b) $\sum_{n=0}^{\infty} (-1)^n$

Let $u_n = (-1)^n$, $n \geq 0$. The partial sums are :
$$\begin{aligned}
S_0 &= (-1)^0 = 1,\\
S_1 &= (-1)^0 + (-1)^1 = 1-1 = 0,\\
S_2 &= (-1)^0 + (-1)^1 + (-1)^2 = 1-1+1 = 1,\\
S_3 &= (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 = 1-1+1-1 = 0,\\
&\vdots
\end{aligned}$$

The partial sums alternate between 0 and 1. Therefore, the sequence (S_n) does not converge, and the series is divergent with no sum.

Proposition 6.1.3. *Let $\sum u_n$ and $\sum v_n$ be numerical series, and let $\alpha \in \mathbb{R}$. Then :*

1. *The series $\sum u_n$ and $\sum \alpha u_n$ are of the same nature (both converge or both diverge).*
2. *If $\sum u_n$ and $\sum v_n$ converge, then the series $\sum(u_n + v_n)$ also converges.*
3. *If $\sum u_n$ converges and $\sum v_n$ diverges, then the series $\sum(u_n + v_n)$ diverges.*
4. *If both series $\sum u_n$ and $\sum v_n$ diverge, then nothing can be concluded about the series $\sum(u_n + v_n)$.*

Necessary condition for the convergence of series

Theorem 6.1.4. *If $\sum_{n=0}^{\infty} u_n$ is convergent, then*

$$\lim_{n\to\infty} u_n = 0.$$

Corollary 6.1.5. *If $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=0}^{\infty} u_n$ is divergent.*

Example 6.1.2. Study the nature (convergence or divergence) of the following numerical series:

$$\sum_{n=0}^{\infty} \frac{3n+1}{2n+5}.$$

$$Let \quad u_n = \frac{3n+1}{2n+5}.$$

Compute the limit of the general term as $n \to \infty$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2} \neq 0.$$

Since the limit of the general term is not zero, we conclude, by the necessary condition for convergence, that the series $\sum_{n=0}^{\infty} u_n$ diverges.

6.1.2 Convergence criteria for series with positive terms

In the study of the nature of a series, it is not always possible to calculate its sum. However, we can determine its nature using other techniques. For this, we need additional tools.

Series with positive terms

Definition 6.1.6. Let $\sum_{n=0}^{\infty} u_n$ be a series such that a series $\displaystyle\sum_{n=0}^{\infty} u_n$ is said to have **positive terms** if

$$u_n \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

A). Geometric series

We now study the geometric series defined as follows :

$$\sum_{n=0}^{\infty} q^n, \quad q \in \mathbb{R}^+.$$

$$Then, \quad \begin{cases} \text{Converges if } 0 \leq q < 1, \\ \text{Diverges if } q \geq 1. \end{cases}$$

Example 6.1.3. Study the convergence of $\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n$.

Here $q = \frac{1}{2}$. Since $0 \leq q < 1$, the series converges.

B). Integral test

Theorem 6.1.7. *Let $f : [1, +\infty) \to \mathbb{R}^+$ be a positive, continuous, and decreasing function.*

$$u_n = f(n), \quad \forall n \geq 1.$$

Then, the series

$$\sum_{n=1}^{\infty} u_n \quad \textit{and the improper integral} \quad \int_1^{\infty} f(x)\, dx$$

are of the same nature. Moreover :

1. *If $\int_1^{\infty} f(x)\, dx$ converges, then $\sum_{n=1}^{\infty} u_n$ converges.*
2. *If $\int_1^{\infty} f(x)\, dx$ diverges, then $\sum_{n=1}^{\infty} u_n$ diverges.*

Example 6.1.4. Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
We can apply the integral test.

$$Let \quad f(x) = \frac{1}{x}, \quad x \in [1, +\infty).$$

This function is positive, continuous, and decreasing. Consider the improper integral :

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\, dx = \lim_{t \to \infty} \ln x \Big|_1^t = \lim_{t \to \infty} (\ln t - \ln 1) = +\infty.$$

Since the integral diverges, by the integral test, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad also \quad diverges.$$

C). Riemann Series

A Riemann series is any numerical series of the form

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad p \in \mathbb{R}.$$

We observe that

$$u_n = \begin{cases} 0, & p < 0, \\ 1, & p = 0, \\ +\infty, & p > 0. \end{cases}$$

Thus the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges for $p \leq 0$. For $p > 0$, we apply the integral test.

First case : $p = 1$. This is the harmonic series, which diverges.

Second case : $p \neq 0$. Let $f(x) = \dfrac{1}{x^p}$ on $[1, +\infty[$. This function is continuous, positive, and decreasing. Moreover,

$$\int_1^{+\infty} \frac{1}{x^p}\, dx \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p < 1. \end{cases}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p < 1. \end{cases}$$

Conclusion :

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

D). Comparison test for series

Theorem 6.1.8. *Let $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ be two series with positive terms :*

$$u_n \geq 0, \quad v_n \geq 0 \quad \forall n \in \mathbb{N}.$$

Suppose that $0 \leq u_n \leq v_n \quad \forall n \in \mathbb{N}$. *Then :*

1. *If $\sum_{n=0}^{\infty} v_n$ converges, then $\sum_{n=0}^{\infty} u_n$ also converges.*
2. *If $\sum_{n=0}^{\infty} u_n$ diverges, then $\sum_{n=0}^{\infty} v_n$ also diverges.*

Example 6.1.5. Study the convergence of the following series

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^n}$

For $n \geq 2$, we have

$$n^n \geq 2^n \quad \Rightarrow \quad \frac{1}{n^n} \leq \frac{1}{2^n}.$$

Since the series

$$\sum_{n=2}^{\infty} \frac{1}{2^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$$

converges (geometric series with ratio $0 < q = \frac{1}{2} < 1$), by the Comparison Test, the series

$$\sum_{n=2}^{\infty} \frac{1}{n^n} \quad \textit{also converges.}$$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

For $n \geq 2$, we know that $\ln n \leq n$, so

$$\frac{\ln n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

Since the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

converges (Riemann series with $p = 2 > 1$), by the Comparison Test, the series

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^3} \quad \textit{also converges.}$$

E). Equivalence criterion

Theorem 6.1.9. *Let $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ be two series with strictly positive terms :*

$$u_n > 0, \quad v_n > 0 \quad \forall n \in \mathbb{N}.$$

Suppose that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1.$$

Then the two series $\sum u_n$ and $\sum v_n$ are of the same nature, i.e. :

1. *If $\sum_{n=0}^{\infty} v_n$ converges, then $\sum_{n=0}^{\infty} u_n$ also converges.*
2. *If $\sum_{n=0}^{\infty} v_n$ diverges, then $\sum_{n=0}^{\infty} u_n$ also diverges.*

Example 6.1.6. Study the convergence of the following series

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2}{5n^4+n}$

$$Set \quad u_n = \frac{3n^2+2}{5n^4+n}, \qquad v_n = \frac{3n^2}{5n^4} = \frac{3}{5} \cdot \frac{1}{n^2}.$$

$$We \;\; have \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{3n^2+2}{5n^4+n} \cdot \frac{5n^2}{3} = 1.$$

Since $\sum 1/n^2$ converges (Riemann series with $p = 2 > 1$), by the Equivalence Criterion, the series

$$\sum_{n=1}^{\infty} \frac{3n^2+2}{5n^4+n} \quad also \quad converges.$$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2+n}$

$$Set \quad u_n = \frac{\ln n}{n^2+n}, \qquad v_n = \frac{\ln n}{n^2}.$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\ln n}{n^2+n} \cdot \frac{n^2}{\ln n} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1.$$

Since $\sum(\ln n)/n^2$ converges (comparison with $1/n^{3/2}$ or using integral test), by the Equivalence Criterion, the series

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2+n} \quad also \quad converges.$$

F). Cauchy root test

Theorem 6.1.10. *Let $\sum_{n=0}^{\infty} u_n$ be a series with positive terms ($u_n \geq 0$), and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \ell \quad (\ell \textit{ finite or infinite}).$$

Then :

1. *If $\ell < 1$, the series $\sum_{n=0}^{\infty} u_n$ converges.*
2. *If $\ell > 1$, the series $\sum_{n=0}^{\infty} u_n$ diverges.*
3. *If $\ell = 1$, the test is inconclusive.*

Example 6.1.7. Study the convergence of the series

$$\sum_{n=0}^{\infty} \left(\frac{4n+3}{7n+5}\right)^n$$

$$Let \quad u_n = \left(\frac{4n+3}{7n+5}\right)^n.$$

Apply the Cauchy Root Test :

$$\sqrt[n]{u_n} = \frac{4n+3}{7n+5}.$$

Compute the limit :

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \frac{4n+3}{7n+5} = \frac{4}{7} < 1.$$

Since the limit is less than 1, by the Cauchy Root Test, the series

$$\sum_{n=0}^{\infty} \left(\frac{4n+3}{7n+5}\right)^n \quad \textit{converges.}$$

G). D'Alembert's ratio test

Theorem 6.1.11. *Let $\sum_{n=0}^{\infty} u_n$ be a series with strictly positive terms ($u_n > 0$), and suppose that*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell \quad (\ell \textit{ finite or infinite}).$$

Then :

1. *If $\ell < 1$, the series $\sum_{n=0}^{\infty} u_n$ converges.*
2. *If $\ell > 1$, the series $\sum_{n=0}^{\infty} u_n$ diverges.*
3. *If $\ell = 1$, nothing can be concluded about the series.*

Example 6.1.8. Study the convergence of :

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

$$\textit{Let} \quad u_n = \frac{3^n}{n!}.$$

Compute the ratio :

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}/(n+1)!}{3^n/n!} = \frac{3}{n+1}.$$

Taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1.$$

Hence, by D'Alembert's Ratio Test, the series converges.

2. $\displaystyle\sum_{n=1}^{\infty} 2^n$

$$\textit{Let} \quad u_n = 2^n.$$

Compute the ratio :

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}}{2^n} = 2.$$

$$\textit{Since} \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 2 > 1,$$

by D'Alembert's Ratio Test, the series diverges.

6.1.3 Series with terms of arbitrary sign

Definition 6.1.12. A numerical series $\sum u_n$ is said to be *with terms of arbitrary sign* when the sequence (u_n) contains infinitely many positive terms and infinitely many negative terms. In other words, the terms of the series change sign.

Definition 6.1.13. A series $\sum_{n=0}^{+\infty} u_n$ is said to be *absolutely convergent* if the series $\sum_{n=0}^{+\infty} |u_n|$ converges.

Theorem 6.1.14. *Every absolutely convergent series is convergent.*

Example 6.1.9. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

Compute the series of absolute values :

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a Riemann series with $p = 2 > 1$, so it converges. Hence, the original series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

is *absolutely convergent*, and therefore convergent.

6.1.4 Alternating series

Definition 6.1.15. A series $\displaystyle\sum_{n=0}^{\infty} u_n$ is called *alternating* if

$$u_n = (-1)^n b_n, \quad b_n \geq 0 \text{ for all } n.$$

Leibniz criterion

Theorem 6.1.16. *Let $\sum_{n=0}^{\infty} u_n$ be an alternating series, i.e.,*

$$u_n = (-1)^n b_n \quad \textit{with } b_n \geq 0.$$

If the sequence (b_n) satisfies :

1. $\lim_{n\to\infty} b_n = 0$,
2. *(b_n) is decreasing for all sufficiently large n,*

*then the series $\sum_{n=0}^{\infty} u_n$ **converges**.*

Example 6.1.10. Study the convergence of the alternating series

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Let $u_n = \dfrac{(-1)^n}{n} = (-1)^n b_n$, where $b_n = \dfrac{1}{n} \geq 0$.
Check the conditions of the Leibniz criterion :

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

(ii) The sequence $(b_n) = \left(\frac{1}{n}\right)$ is decreasing since

$$\frac{1}{n+1} < \frac{1}{n}, \qquad n \geq 1.$$

Hence, by Leibniz's criterion, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad converges$$

$$Moreover, \; \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Therefore, the series is conditionally convergent.

6.2 Sequences and series of functions

Definition 6.2.1. A *sequence of functions* is any sequence whose general term is a function depending on a parameter n. We denote it by $(f_n)_{n\in\mathbb{N}}$.

6.2.1 Types of onvergence

A). pointwise convergence.

Definition 6.2.2. Let $f_n : I \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. We say that (f_n) *converges pointwise* (or simply) to a function f on I if, for every $x \in I$, the sequence of real numbers $(f_n(x))_{n\in\mathbb{N}}$ converges to a finite limit, denoted by $f(x)$:

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in I.$$

Example 6.2.1. Let $f_n : [0,1] \to \mathbb{R}$ be defined by :

$$f_n(x) = x^n$$

Fix $x \in [0,1]$.

1. If $0 \leq x < 1$, then $\lim_{n\to\infty} x^n = 0$
2. If $x = 1$, then $\lim_{n\to\infty} 1^n = 1$

Define the limit function f.

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

The sequence (f_n) **converges pointwise** to f on $[0,1]$.

B). Absolute convergence

Definition 6.2.3. Let $f_n : I \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. We say that (f_n) is **absolutely convergent** if the sequence formed by the absolute values of its terms is convergent, that is :

$$\sum_{n=0}^{\infty} |f_n(x)| \text{ converges.}$$

Example 6.2.2. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by :

$$f_n(x) = \frac{(-1)^n x^n}{n}$$

Consider the series of absolute values

$$\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

- For $x \in [0,1)$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges (it is a power series with ratio $x < 1$).
- For $x = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series).

Then the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent for $0 \leq x < 1$.

It is conditionally convergent at $x = 1$.

C). Uniform convergence

Definition 6.2.4. Let $f_n : I \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. We say that (f_n) **converges uniformly** to f on I if

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Example 6.2.3. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by :

$$f_n(x) = \frac{x}{1+nx}$$

Find the pointwise limit

For a fixed $x \in [0,1]$:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx} = 0$$

So the pointwise limit function is :

$$f(x) = 0$$

Compute the supremum :

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \frac{x}{1+nx}$$

- The function $g(x) = \dfrac{x}{1+nx}$ is increasing on $[0,1]$
- So the supremum is at $x = 1$:

$$\sup_{x\in[0,1]} |f_n(x) - 0| = \frac{1}{1+n} \to 0 \quad \text{as } n \to \infty$$

then (f_n) **converges uniformly** to $f(x) = 0$ on $[0,1]$.

Theorem 6.2.5. *Let $f_n : I \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions that converges pointwise to f. If $\dfrac{\partial f_n}{\partial x}$ is bounded (uniformly in n), then the convergence is uniform.*

Theorem 6.2.6. *Every uniformly convergent sequence of functions is pointwise convergent. The converse is false; that is, a pointwise convergent sequence need not converge uniformly.*

6.2.2 Properties of Sequences of Functions

1. If $f_n \to f$ and $g_n \to g$, then $\alpha f_n + \beta g_n \to \alpha f + \beta g$.
2. If $f_n \to f$ and $g_n \to g$ uniformly, then $f_n g_n \to fg$ uniformly.
3. If $f_n \to f$, $g_n \to g$ uniformly and $g_n, g \neq 0$, then $\dfrac{f_n}{g_n} \to \dfrac{f}{g}$ uniformly.
4. If f_n is continuous and $f_n \to f$ uniformly, then f is continuous.
5. If $f_n \to f$ uniformly on $[a, b]$, then $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.
6. If $f_n' \to g$ uniformly and $f_n(x_0)$ converges, then $f_n \to f$ uniformly and $f' = g$.

6.2.3 Series of functions

Definition 6.2.7. Let $(f_n)_{n\geq 0}$ be a sequence of functions $f_n : I \subset \mathbb{R} \to \mathbb{R}$. The *series of functions* is the infinite sum of the terms f_n, and is denoted by

$$\sum_{n=0}^{+\infty} f_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x) + \cdots$$

Definition 6.2.8. Let $f_n : I \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. The series

$$\sum_{n=0}^{+\infty} f_n(x)$$

is said to *converge pointwise* on I if the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ converges pointwise on I.

Definition 6.2.9. Let $f_n : I \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. The series

$$\sum_{n=0}^{+\infty} f_n(x)$$

is said to *converge absolutely* on I if the series

$$\sum_{n=0}^{+\infty} |f_n(x)|$$

converges pointwise on I.

Definition 6.2.10. Let $f_n : I \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. The series

$$\sum_{n=0}^{+\infty} f_n(x)$$

is said to *converge uniformly* on I if the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ converges uniformly on I.

Definition 6.2.11. A series of functions

$$\sum_{n=0}^{+\infty} f_n(x)$$

is said to be *majorable* (bounded above) on a certain domain if there exists a positive and convergent numerical series which dominates it. That is, there exists a sequence (v_n) with $v_n \geq 0$ and

$$\sum_{n=0}^{+\infty} v_n \quad \text{convergent},$$

such that for all $n \in \mathbb{N}$,

$$|f_n(x)| \leq v_n.$$

Definition 6.2.12. Every majorable series is said to be *normally convergent.*

Theorem 6.2.13. *Every normally convergent series is uniformly convergent.*

Every normally convergent series is absolutely convergent.

Theorem 6.2.14. *Suppose that*

$$\sum_{n=0}^{+\infty} f_n(x)$$

is majorable on I, with sum $s(x)$, and let $S_n(x)$ be the sum of its first n terms. Then :

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ \forall x \in I, \quad |s(x) - S_n(x)| < \varepsilon.$$

Properties of Series of Functions

Let $\sum_{n=0}^{+\infty} f_n(x)$ be a series of functions defined on a set I.

1. If the series converges uniformly on I, then

$$\lim_{x\to a}\left(\sum_{n=0}^{+\infty} f_n(x)\right) = \sum_{n=0}^{+\infty}\left(\lim_{x\to a} f_n(x)\right).$$

2. If the series converges uniformly on I, then

$$\int_I \left(\sum_{n=0}^{+\infty} f_n(x)\right) dx = \sum_{n=0}^{+\infty} \int_I f_n(x)\, dx.$$

3. If each f_n is differentiable on I, if the series of derivatives

$$\sum_{n=0}^{+\infty} f_n'(x)$$

converges uniformly on I, and if f_0 is differentiable, then

$$\left(\sum_{n=0}^{+\infty} f_n(x)\right)' = \sum_{n=0}^{+\infty} f_n'(x).$$

4. If the series is normally convergent, then
   - it is uniformly convergent;
   - it is absolutely convergent;
   - termwise integration is always allowed;
   - termwise differentiation is valid under mild hypotheses.

6.3 power series

Definition 6.3.1. A *power series* is any series that can be written in the form :

$$\sum_{n=0}^{\infty} a_n x^n,$$

where $(a_n)_{n\geq 0}$ is a numerical sequence (real or complex numbers).

Theorem 6.3.2 (Abel's Theorem)**.** *Consider the power series*

$$\sum_{n=0}^{\infty} a_n x^n.$$

Let $\rho > 0$ be a strictly positive number.

1. *If the sequence of functions* $(|a_n|\rho^n)_{n\in\mathbb{N}}$ converges*, then the series*

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for all x such that $|x| < \rho$.

2. *If the sequence of functions* $(|a_n|\rho^n)_{n\in\mathbb{N}}$ diverges*, then the series*

$$\sum_{n=0}^{\infty} a_n x^n$$

diverges for all x such that $|x| > \rho$.

Definition 6.3.3 (Domain of convergence)**.** The *domain of convergence* of a power series is the set of points for which the series converges *absolutely*.

By Abel's theorem, this domain is always an interval *centered at the origin*, which can be written as

$$]-R, R[, \quad \text{with } R \in [0, +\infty].$$

Here, R is called the *radius of convergence* of the series.

6.3.1 Determination of the radius of convergence

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

D'Alembert's Ratio Test

The radius of convergence R can be determined using the ratio test :

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided the limit exists.

- If $|x| < R$, the series converges absolutely.

- If $|x| > R$, the series diverges.

- If $|x| = R$, convergence must be checked separately.

Example 6.3.1. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$.

To find the radius of convergence, we apply D'Alembert's ratio test :

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}} = \lim_{n\to\infty} \frac{n!(n+1)^{n+1}}{(n+1)!n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = e.$$

Thus, the radius of convergence is $R = e$.
Therefore, the series converges absolutely for $|x| < e$, diverges for $|x| > e$, and the endpoints $x = \pm e$ must be checked separately.

Remark 6.3.1. The radius of convergence R of a power series can be computed as the limit of the ratio of successive terms as $n \to \infty$:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

More generally, for any fixed integer $k \geq 1$, one can also consider the ratio of terms in a subsequence :

$$R = \lim_{n\to\infty} \left| \frac{a_{n+k}}{a_{n+k+1}} \right|.$$

The key point is that as n tends to infinity, the specific shift k does not change the limit, and hence the radius of convergence remains the same.

Example 6.3.2. Consider the power series : $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$.

To find the radius of convergence, we apply D'Alembert's ratio test :

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{n^2}{2^n}}{\frac{(n+1)^2}{2^{n+1}}} = \lim_{n\to\infty} \frac{n^2 \cdot 2^{n+1}}{(n+1)^2 \cdot 2^n} = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2}.$$

Simplifying the limit :

$$R = \lim_{n\to\infty} 2 \left(\frac{n}{n+1} \right)^2 = 2.$$

hence :
- The series converges absolutely for $|x| < 2$. - The series diverges for $|x| > 2$. - Convergence at the endpoints $x = \pm 2$ must be checked separately

Cauchy's root test

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

The radius of convergence R is given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

- If $|x| < R$, the series converges absolutely.
- If $|x| > R$, the series diverges.
- If $|x| = R$, convergence must be checked separately.

Example 6.3.3. Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{3n+1}{4^n} \right) x^n.$$

Using Cauchy's root test :

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{3n+1}{4^n}}} = \frac{1}{\lim_{n\to\infty} \frac{\sqrt[n]{3n+1}}{\sqrt[n]{4^n}}} = \frac{1}{\frac{1}{4}} = 4.$$

The series converges absolutely for $|x| < 4$ and diverges for $|x| > 4$. The endpoints $x = \pm 4$ must be checked separately.

Remark 6.3.2. The radius of convergence R can also be expressed as the reciprocal of the limit of the n-th root of the terms of the series as $n \to \infty$. More precisely, for a power series $\sum a_n x^n$,

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}.$$

More generally, one can apply this to subsequences of the series. For example :

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[2n+1]{|a_{2n+1}|}} \quad \text{or} \quad R = \frac{1}{\lim_{n\to\infty} \sqrt[n^2]{|a_{n^2}|}}.$$

The idea is that as $n \to \infty$, the choice of a subsequence does not affect the value of R.

6.3.2 Properties of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$, and let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$.

1. Continuity : f is continuous on $(-R, R)$.
2. Termwise differentiation : $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, for $|x| < R$.
3. Termwise integration : $\displaystyle\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$, for $|x| < R$.
4. Uniform convergence : the series converges uniformly on every $[-r, r]$ with $0 < r < R$.

6.3.3 Applications of power series

1. Function approximation (Computing limits)

Power series allow us to represent functions as an infinite sum of powers of x.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad |x| < R.$$

Example : The Maclaurin series of a function $f(x)$ is given by :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

where $f^{(n)}(0)$ is the n-th derivative of f evaluated at $x = 0$.

$if\ \ f(x) = e^x$:
Hence, the Maclaurin series of e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

1. Compute : $\lim_{x\to 0} \dfrac{e^x - 1}{\sin x}$.

$$e^x - 1 = x + \frac{x^2}{2} + o(x^2), \qquad \sin x = x - \frac{x^3}{6} + o(x^3).$$

$$Then, \quad \frac{e^x - 1}{\sin x} = \frac{x + \frac{x^2}{2} + o(x^2)}{x - \frac{x^3}{6} + o(x^3)} = \frac{x\left(1 + \frac{x}{2} + o(x)\right)}{x\left(1 - \frac{x^2}{6} + o(x^2)\right)}.$$

Therefore,

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x} = 1.$$

2.Differentiation and integration

We know that every continuous function has an antiderivative, but some functions do not have a primitive that can be expressed in a closed form. Nevertheless, such integrals can be computed using power series expansions. Within their radius of convergence, power series may be differentiated and integrated term by term.

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Example : Compute $\int_0^1 e^{-x^2}\,dx$.

We use the Maclaurin series :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \Longrightarrow \qquad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx.$$

By term-by-term integration (valid because the series is uniformly convergent on $[0,1]$):

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx.$$

$$Now, \quad \int_0^1 x^{2n}\,dx = \frac{1}{2n+1}.$$

$$Therefore, \quad \int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.$$

3. Solving differential equations

If we are asked to integrate a certain ordinary differential equation (ODE), whose solution cannot be reduced to a quadrature, we can always find an approximate solution by assuming that the solution can be expressed as a power series

Assume the solution is a power series :

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Example : Solve $y' = y$ using series :

Assume a solution in the form of a power series :

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \Longrightarrow \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Substitute into the differential equation $y' = y$:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n.$$

Change the index in the first sum : let $m = n - 1$:

$$\sum_{m=0}^{\infty} (m+1)a_{m+1}x^m = \sum_{n=0}^{\infty} a_n x^n.$$

Equate coefficients of x^n :

$$(m+1)a_{m+1} = a_m \quad \Longrightarrow \quad a_{m+1} = \frac{a_m}{m+1}.$$

Starting with $a_0 = 1$, we get

$$a_1 = \frac{1}{1!}, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \ldots$$

Thus, the general term is

$$a_n = \frac{1}{n!}.$$

$$\textit{Hence,} \;\; y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x.$$

4. Analytic continuation

Power series can extend the domain of a function beyond its original definition.
Example : $\ln(1+x)$ is defined by the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad |x| < 1,$$

but using analytic continuation, it can be extended beyond $|x| < 1$ (except $x = -1$).

5. Numerical computation

Power series are used to compute approximations of functions.
Example : Approximate $\sin(x)$ near $x = 0$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

For $x = 0.1$, truncating at x^3 gives $\sin(0.1) \approx 0.1 - 0.0001667 \approx 0.0998333$.

6.4 Fourier series

Definition 6.4.1. A function f is called **periodic** with period p if it satisfies :

$$\forall x \in D, \; x + p \in D : \quad f(x+p) = f(x),$$

where D is the domain of f, and p is the smallest positive number satisfying this property.

Definition 6.4.2. A function f is said to be ***piecewise monotone*** on the interval $[a, b]$ if the interval can be divided into subintervals such that f is monotone on each subinterval.

Definition 6.4.3. A ***trigonometric series*** is any series that can be written as :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where a_0, a_n and b_n are real numbers, called the *coefficients of the series.*

6.4.1 Determination of Fourier coefficients

Consider a 2π-periodic function that can be represented by a trigonometric series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

assuming the series is absolutely convergent. Then it is uniformly convergent and can be integrated term by term.

$$\int_{-\pi}^{\pi} \cos nx \cos kx \, dx = \begin{cases} \pi & \text{if } n = k, \\ 0 & \text{if } n \neq k, \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Moreover :

$$\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos nx \sin kx \, dx = 0 \quad \forall n, k.$$

Example 6.4.1. Determine the Fourier series of the following functions defined on $[-\pi, \pi]$

1. $f(x) = |x|$

$f(x) = |x|$ is even function, so $b_n = 0$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \pi, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx \, dx = \frac{2}{\pi n^2} \left((-1)^n - 1\right).$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\cos(nx)}{n^2}.$$

2. $f(x) = x^2$

$f(x) = x^2$ is even function, so $b_n = 0$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos nx \, dx = \frac{4(-1)^n}{n^2}.$$

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Theorem 6.4.4. *Every periodic, bounded, and piecewise monotone function can be expanded in a Fourier series. Its sum $s(x)$ equals $f(x)$ at points of continuity. At points of discontinuity, $s(x)$ equals the arithmetic mean of the right-hand and left-hand limits :*

$$s(x) = \frac{f(x^+) + f(x^-)}{2}.$$

Remark 6.4.1. For Fourier series :

- If the function is **even**, its Fourier series contains only cosine terms; all sine coefficients vanish : $b_n = 0$.
- If the function is **odd**, its Fourier series contains only sine terms; all cosine coefficients vanish : $a_0 = a_n = 0$.

6.4.2 Fourier Series of Functions with Period $2l$ ($2l \neq 2\pi$)

Suppose the function $f(x)$ is $2l$-periodic ($l \neq 0$ and $l \neq \pi$). If, in addition, f is piecewise monotone and bounded, then it can be expanded in a Fourier series with Fourier coefficients :

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Thus, the Fourier series of f is given by :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right).$$

Example 6.4.2. Fourier series on $[-l, l]]$
Consider the function $f(x) = x$ defined on $[-l, l]$, with period $2l$.
Since $f(x)$ is an **odd function**, all cosine coefficients vanish : $a_0 = a_n = 0$.
The sine coefficients are :

$$b_n = \frac{1}{l}\int_{-l}^{l} x\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_{0}^{l} x\sin\left(\frac{n\pi x}{l}\right)dx$$

Using integration by parts :

$$\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx = -\frac{l^2}{n\pi}\cos(n\pi) + \frac{l^2}{n\pi} = \frac{l^2}{n\pi}\Big(1-(-1)^n\Big).$$

$$Thus\ :\ b_n = \frac{2}{l}\cdot\frac{l^2}{n\pi}\Big(1-(-1)^n\Big) = \frac{2l}{n\pi}\Big(1-(-1)^n\Big).$$

$$the\ Fourier\ series\ is f(x) = x = \sum_{n=1}^{\infty}\frac{2l}{n\pi}\Big(1-(-1)^n\Big)\sin\left(\frac{n\pi x}{l}\right).$$

Notice that only odd n terms contribute since $1-(-1)^n = 0$ for even n.

6.4.3 Parseval's Equality

Theorem 6.4.5 (Parseval's Theorem)**.** *Let f be a function that can be expanded in a Fourier series with period $2l$. Then we have :*

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right),$$

where a_n and b_n are the Fourier coefficients of f.

Example 6.4.3. Applying Parseval's theorem on $[-\pi, \pi]$
Consider $f(x) = x^2$, which is even, so all sine coefficients vanish : $b_n = 0$.
From the Fourier series of x^2, we have :

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx,$$

$$so \quad a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}.$$

Parseval's theorem states :

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Compute the left-hand side :

$$\frac{1}{\pi}\int_{-\pi}^{\pi}x^4dx = \frac{1}{\pi}\cdot\frac{2\pi^5}{5} = \frac{2\pi^4}{5}.$$

Compute the right-hand side :

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2 = \frac{(2\pi^2/3)^2}{2} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2 = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Equating left-hand side and right-hand side gives the famous result :

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

6.4.4 Fourier series in complex form

If f can be expanded in a Fourier series, we write :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right).$$

Recalling that

$$\cos\frac{n\pi x}{l}=\frac{e^{in\pi x/l}+e^{-in\pi x/l}}{2},\quad \sin\frac{n\pi x}{l}=\frac{e^{in\pi x/l}-e^{-in\pi x/l}}{2i},$$

substituting these into the Fourier series gives :

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}e^{in\pi x/l}+\frac{a_n+ib_n}{2}e^{-in\pi x/l}\right)=\sum_{n=-\infty}^{\infty}c_n e^{in\pi x/l},$$

where the complex Fourier coefficients are

$$c_0=\frac{a_0}{2},\quad c_n=\frac{a_n-ib_n}{2},\quad c_{-n}=\frac{a_n+ib_n}{2}.$$

Or, equivalently, using the integral formula :

$$c_n=\frac{1}{2l}\int_{-l}^{l}f(x)e^{-in\pi x/l}\,dx.$$

Example 6.4.4. Complex Fourier Series on $[-\pi,\pi]$
Consider $f(x)=x$, which is odd.
We know from the real Fourier series :

$$f(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx.$$

Using the complex form :

$$\sin nx=\frac{e^{inx}-e^{-inx}}{2i},$$

$$so\ \ f(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\cdot\frac{e^{inx}-e^{-inx}}{2i}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{in}\left(e^{inx}-e^{-inx}\right).$$

We can combine positive and negative terms as :

$$f(x)=\sum_{n=-\infty,\,n\neq 0}^{\infty}c_n e^{inx},$$

$$where\ \ c_n=\frac{(-1)^{n+1}}{in}\quad\text{for } n\neq 0,\quad c_0=0.$$

Exercise 6.4.1. Consider the 2π-periodic function f defined on $[-\pi,\pi]$ by

$$f(x)=|\sin x|.$$

1. Show that $f(x)$ can be expanded in a Fourier series.
2. Compute the Fourier coefficients of $f(x)$.

3. Deduce the following sum :
$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}.$$

Solution :

1. The function $f(x) = |\sin x|$ is 2π-periodic, piecewise continuous, and even. Therefore, it can be represented by a Fourier series with only cosine terms :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

2. The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right],$$

where the coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Since $f(x)$ is even, all sine coefficients vanish : $b_n = 0$.
We only need to compute a_0 and a_n :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{-2}{\pi}\left[\cos x\right]_0^{\pi} = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx.$$

Using the identity :

$$\sin x \cos nx = \frac{1}{2}\left[\sin((n+1)x) + \sin((1-n)x)\right],$$

$$we \quad get \quad a_n = \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x)\,dx + \frac{1}{\pi}\int_0^{\pi} \sin((1-n)x)\,dx.$$

$$\text{For } n=1, \qquad a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = -\frac{1}{2\pi}\Big[\cos(2x)\Big]_0^{\pi} = 0.$$

- For $n \neq 1$, after evaluating the integrals :

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{4}{\pi(1-n^2)} & \text{if } n \text{ is even.} \end{cases}$$

Hence, setting $n = 2k$, we have

$$a_{2k} = \frac{4}{\pi(1-4k^2)}.$$

Thus, the Fourier series becomes :

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos 2nx.$$

3. The function $f(x)$ is continuous at $x_0 = 0$, so

$$f(0) = 0 = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos(2n \cdot 0) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)}.$$

$$Rewriting, \quad we \quad get \quad : \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

Chapter 7

Fourier Transform

The Fourier transform is a powerful tool for solving differential equations, as it allows us to reformulate them into simpler, more manageable problems. It is defined as an extension of the Fourier series to non-periodic functions, which corresponds to assuming that the function has an infinite period

Definition 7.0.1. The space L^1 is the set of integrable functions, that is, functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty.$$

The space L^2 is the set of square-integrable functions, that is, functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx < \infty.$$

Example 7.0.1. Consider the function

$$f(x) = e^{-|x|}.$$

This function is piecewise continuous on $\mathbb{R}$, and we have

$$\int_{-\infty}^{+\infty} |f(x)|\, dx = \int_{-\infty}^{+\infty} e^{-|x|}\, dx = 2 < \infty, \quad f \in L^1(\mathbb{R}).$$

Definition 7.0.2. (Dirac delta function)

The **Dirac delta function** (or **Dirac impulse**) is defined as

$$\delta_a(t) = \begin{cases} 1, & t = a, \\ 0, & t \neq a. \end{cases}$$

7.1 Fourier transform

Definition 7.1.1. Let $f \in L^1(\mathbb{R})$. The Fourier transform $\mathcal{F} : \mathbb{R} \to \mathbb{C}$ of the function $f(t)$ is defined by

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt,$$

where $\omega \in \mathbb{R}$.

Example 7.1.1. Consider the function

$$f(t) = e^{-a|t|}, \quad a > 0.$$

This function is piecewise continuous and decays exponentially as $|t| \to \infty$, so it is absolutely integrable :

$$\int_{-\infty}^{+\infty} |f(t)|\, dt = \int_{-\infty}^{+\infty} e^{-a|t|}\, dt = \frac{2}{a} < \infty.$$

Its Fourier transform is defined as

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-i\omega t}\, dt.$$

Since $e^{-a|t|}$ is an even function, we split the integral :

$$\hat{f}(\omega) = \int_{-\infty}^{0} e^{at} e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt.$$

Using the formula $\int_0^\infty e^{-(a+i\omega)t} dt = \frac{1}{a+i\omega}$, we get

$$\hat{f}(\omega) = \frac{2a}{a^2 + \omega^2}.$$

Hence, $f \in L^1(\mathbb{R})$ and its Fourier transform is well-defined.

7.1.1 Specific cases of fourier transform

i) **Even functions :**

If f is an even function, the Fourier transform simplifies to

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = 2 \int_0^\infty f(t) \cos(2\pi\omega t)\, dt.$$

ii) **Odd functions :**

If f is an odd function, the Fourier transform simplifies to

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = -2i \int_0^\infty f(t) \sin(2\pi\omega t)\, dt.$$

Example 7.1.2. Consider the following function :

1.**Even function :** Cosine function

$$f(t) = \cos(2\pi t).$$

It is even. Its Fourier transform is

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \cos(2\pi t) e^{-i2\pi\omega t}\, dt = \delta(\omega - 1) + \delta(\omega + 1),$$

where δ is the Dirac delta function.

2.**Odd function :** Sine function

$$f(t) = \sin(2\pi t).$$

It is odd. Its Fourier transform is

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \sin(2\pi t) e^{-i2\pi\omega t}\, dt = i\Big(\delta(\omega + 1) - \delta(\omega - 1)\Big).$$

7.1.2 Inverse Fourier transform

The inverse Fourier transform allows us to recover a function $f(t)$ from its Fourier transform $\hat{f}(\omega)$.

Definition 7.1.2. Let $f : \mathbb{R} \to \mathbb{C}$. The inverse Fourier transform of f is defined by

$$f(t) = \mathcal{F}^{-1}(\hat{f})(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega,$$

where $\hat{f}(\omega) = \mathcal{F}(f)(\omega)$ is the Fourier transform of f.

Example 7.1.3. Consider the following function :
1. Let $\hat{f}(\omega) = e^{-\omega^2/4}$.
Start with the Fourier transform :

$$\hat{f}(\omega) = e^{-\omega^2/4}.$$

Write the inverse Fourier transform formula :

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega.$$

Substitute $\hat{f}(\omega)$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\omega^2/4} e^{i\omega t}\, d\omega.$$

Recognize the Gaussian integral :

$$\int_{-\infty}^{+\infty} e^{-a\omega^2 + b\omega}\, d\omega = \sqrt{\frac{\pi}{a}}\, e^{b^2/(4a)}, \quad a > 0,$$

with $a = \frac{1}{4}, b = it$.
Compute the integral :

$$f(t) = \frac{1}{2\pi} \sqrt{\frac{\pi}{1/4}}\, e^{-t^2} = \frac{1}{\sqrt{\pi}} e^{-t^2}.$$

7.1.3 Fourier transform table

We will present the Fourier transform table of some usual functions

Function $f(t)$	**Fourier Transform** $F(\omega)$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$e^{-a\|t\|},\ a>0$	$\dfrac{2a}{a^2+\omega^2}$
$e^{-at}u(t),\ a>0$	$\dfrac{1}{a+i\omega}$
$\cos(\omega_0 t)$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(\omega_0 t)$	$i\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$
$f'(t)$	$i\omega F(\omega)$
$f''(t)$	$(i\omega)^2F(\omega)$
$tf(t)$	$i\dfrac{d}{d\omega}F(\omega)$
$t^2f(t)$	$i^2\dfrac{d^2}{d\omega^2}F(\omega)$
$f(t-t_0)$	$e^{-i\omega t_0}F(\omega)$
$f(kt)$	$\dfrac{1}{\|k\|}F\left(\dfrac{\omega}{k}\right)$
$(f*g)(t)$	$F(\omega)G(\omega)$
Even function $f(t)$	$2\int_0^\infty f(t)\cos(\omega t)dt$
Odd function $f(t)$	$2i\int_0^\infty f(t)\sin(\omega t)dt$

7.2 Properties of the Fourier Transform

Let $f,g\in L^1(\mathbb{R})$, $F=\mathcal{F}[f]$, $G=\mathcal{F}[g]$.

1. Linearity: $\mathcal{F}[\alpha f+\mu g]=\alpha F+\mu G$.
2. Inverse linearity: $\mathcal{F}^{-1}[\alpha F+\mu G]=\alpha f+\mu g$.
3. Derivatives: $\mathcal{F}[f^{(n)}](\omega)=(i\omega)^nF(\omega)$.
4. Integration: $\mathcal{F}\Big[\int_{-\infty}^t f(s)\,ds\Big]=\dfrac{F(\omega)}{i\omega}$.
5. Conjugation: $\mathcal{F}[\overline{f}]=\overline{F}$.
6. Time shift: $\mathcal{F}[f(t-t_0)]=e^{-i\omega t_0}F(\omega)$.
7. Modulation: $\mathcal{F}[e^{i\omega_0 t}f(t)]=F(\omega-\omega_0)$.
8. Convolution: $\mathcal{F}[f*g]=FG,\quad (f*g)(x)=\int_{-\infty}^{\infty}f(x-t)g(t)\,dt$.

Theorem 7.2.1. *(Parseval's Theorem)*
Let $f(t)\in L^2(\mathbb{R})$ with Fourier transform $F(\omega)$. Then

$$\int_{-\infty}^{+\infty}|f(t)|^2\,dt=\int_{-\infty}^{+\infty}|F(\omega)|^2\,d\omega.$$

Theorem 7.2.2. *(Plancherel's Theorem)*
Let $f,g\in S$ (Schwartz space). Then

$$\int_{-\infty}^{+\infty}f(t)g(t)\,dt=\int_{-\infty}^{+\infty}F(\omega)G(\omega)\,d\omega.$$

7.3 Solving differential equations using Fourier transform

The Fourier transform allows one to solve a linear differential equation explicitly by transforming it into a simpler equation.

7.3.1 Second order differential equation

Consider the second-order differential equation :

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t).$$

Solution procedure :

1. Take the Fourier transform of both sides :

$$\mathcal{F}\{f(t)\} = F(\omega), \quad \mathcal{F}\{y(t)\} = Y(\omega).$$

2. Apply Fourier transform properties for derivatives :

$$\mathcal{F}\{y'(t)\} = i\omega Y(\omega), \quad \mathcal{F}\{y''(t)\} = (i\omega)^2 Y(\omega).$$

3. Substitute into the differential equation :

$$a_2(i\omega)^2 Y(\omega) + a_1(i\omega)Y(\omega) + a_0 Y(\omega) = F(\omega),$$

which simplifies to

$$\Big(a_2(i\omega)^2 + a_1 i\omega + a_0\Big)Y(\omega) = F(\omega).$$

4. Solve for $Y(\omega)$:

$$Y(\omega) = \frac{F(\omega)}{a_2(i\omega)^2 + a_1 i\omega + a_0}.$$

5. Let$G(\omega) = \frac{1}{a_2(i\omega)^2 + a_1 i\omega + a_0}$,
then

$$Y(\omega) = G(\omega) \cdot F(\omega),$$

i.e., a product of two Fourier transforms.

6. Take the inverse Fourier transform :

$$y(t) = \mathcal{F}^{-1}[Y(\omega)] = \mathcal{F}^{-1}[G(\omega)F(\omega)] = g(t) * f(t),$$

where $*$ denotes convolution and $g(t) = \mathcal{F}^{-1}[G(\omega)]$ is the impulse response of the system.

Example 7.3.1. Solving a differential equation using Fourier transform

$$-y''(t) + y(t) = e^{-t^2}, \quad t \in \mathbb{R}.$$

1. The Fourier transform to every term of the differential equation :

$$\mathcal{F}\Big[-y''(t)+y(t)\Big]=\mathcal{F}[e^{-t^2}].$$

2. Use the Fourier transform property for derivatives :

$$-(i\omega)^2Y(\omega)+Y(\omega)=\mathcal{F}[e^{-t^2}],$$

which simplifies to

$$(\omega^2+1)Y(\omega)=\mathcal{F}[e^{-t^2}].$$

3. Solve for $Y(\omega)$:

$$Y(\omega)=\frac{\mathcal{F}[e^{-t^2}]}{\omega^2+1}.$$

4. Recognize that

$$\frac{1}{\omega^2+1}=\mathcal{F}\left[\frac{1}{2}e^{-|t|}\right],$$

$$so\ \ Y(\omega)=\mathcal{F}\left[\frac{1}{2}e^{-|t|}\right]\cdot\mathcal{F}[e^{-t^2}].$$

5. Take the inverse Fourier transform to obtain the solution :

$$y(t)=\frac{1}{2}e^{-|t|}*e^{-t^2},$$

where $*$ denotes convolution.

Remark 7.3.1. The solution is expressed as a convolution of the impulse response $\frac{1}{2}e^{-|t|}$ with the forcing function e^{-t^2}.

Chapter 8

Laplace Transform

The Laplace transform is a powerful mathematical tool used in engineering and physics to simplify complex calculations. It is particularly useful for solving differential equations, as it transforms them into algebraic equations that are easier to solve.
In this chapter, we will focus on applying the Laplace transform to solve a variety of differential equations.

8.1 Laplace Transform

Definition 8.1.1. Let $f : [0, +\infty[\to \mathbb{R}$ be a function defined for $t \geq 0$. The **Laplace transform** of $f(t)$ is the function $F(s)$ defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt,$$

where $s \in \mathbb{R}$.

Example 8.1.1. Calculate the following Laplace transform of the functions
1. $f(t) = t^2$.

$$By\ definition,\ \ \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} = \int_0^{+\infty} e^{-st} t^2\, dt.$$

We use integration by parts. Let us apply the formula :

$$\int_0^{+\infty} t^n e^{-st}\, dt$$

Integration by parts for t^2 :

$$\int_0^{+\infty} t^2 e^{-st}\, dt = \left[-\frac{1}{s} t^2 e^{-st}\right]_0^{+\infty} + \frac{2}{s}\int_0^{+\infty} t e^{-st}\, dt.$$

Compute $\int_0^{+\infty} te^{-st}\, dt$ similarly :

$$\int_0^{+\infty} te^{-st}\, dt = \left[-\frac{1}{s} te^{-st}\right]_0^{+\infty} + \frac{1}{s}\int_0^{+\infty} e^{-st}\, dt = \frac{1}{s^2}.$$

$$\textit{Substitute back} \int_0^{+\infty} t^2 e^{-st}\, dt = 0 + \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}.$$

Thus, the Laplace transform is

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad s > 0.$$

2. $f(t) = 1$.

$$\mathcal{L}\{1\} = \int_0^{+\infty} e^{-st}\cdot 1\, dt.$$

Compute the integral :

$$\int_0^{+\infty} e^{-st}\, dt = \left[-\frac{1}{s}e^{-st}\right]_0^{+\infty} = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}, \quad s > 0.$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

3. $f(t) = e^{at}$.

$$\mathcal{L}\{e^{at}\} = \int_0^{+\infty} e^{-st}e^{at}\, dt = \int_0^{+\infty} e^{-(s-a)t}\, dt.$$

Compute the integral :

$$\int_0^{+\infty} e^{-(s-a)t}\, dt = \left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^{+\infty} = 0 - \left(-\frac{1}{s-a}\right) = \frac{1}{s-a}, \quad s > a.$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

.

8.1.1 Table of Laplace transform

We will present the Laplace transform table of some usual functions

Function $f(t)$	$\mathcal{L}\{f(t)\}(s)$
$U(t)$	$\frac{1}{s}$
$e^{at}U(t)$	$\frac{1}{s-a}$
$t^nU(t),\ n\in\mathbb{N}$	$\frac{n!}{s^{n+1}}$
$t^ne^{at}U(t)$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(kt)\,U(t)$	$\frac{k}{s^2+k^2}$
$\cos(kt)\,U(t)$	$\frac{s}{s^2+k^2}$
$e^{at}\sin(kt)\,U(t)$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos(kt)\,U(t)$	$\frac{s-a}{(s-a)^2+k^2}$
$U(t-a)\,f(t-a),\ a>0$	$e^{-as}\,F(s)$
$f(t)e^{at}$	$F(s-a)$
$f(at),\ a>0$	$\frac{1}{a}\,F\left(\frac{s}{a}\right)$
$f'(t)$	$sF(s)-f(0^+)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0^+)-\cdots-f^{(n-1)}(0^+)$
$\int_0^t f(u)\,du$	$\frac{1}{s}F(s)$

Example 8.1.2. Calculate the Laplace transform of following functions :

1. $f(t)=\cos(4t)$.

Using the Laplace transform table :

$$\mathcal{L}\{\cos(4t)\}=\frac{s}{s^2+16}.$$

2. $f(t)=e^{2t}$.

Using the Laplace transform table, we obtain

$$\mathcal{L}\{e^{2t}\}=\frac{1}{s-2},\qquad s>2.$$

8.2 Properties of the Laplace Transform

- Linearity: $\mathcal{L}\{\alpha f+\beta g\}=\alpha F+\beta G$.
- Exponential shift: $\mathcal{L}\{e^{at}f(t)\}=F(s-a)$.
- Time shift: $\mathcal{L}\{U(t-a)f(t-a)\}=e^{-as}F(s)$.

- Scaling: $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$.
- Derivatives: $\mathcal{L}\{f^{(n)}\} = s^n F(s) - \cdots - f^{(n-1)}(0^+)$.
- Integral: $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}F(s)$.
- Convolution: $\mathcal{L}\{f * g\} = FG$.

8.3 Inverse Laplace transform

Definition 8.3.1. Let $F(s)$ be a function defined for $s > s_0$ and suppose that there exists a function $f(t)$ such that

$$\mathcal{L}\{f(t)\} = F(s).$$

Then $f(t)$ is called the **inverse Laplace transform** of $F(s)$, denoted by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Remark 8.3.1. In practice, the inverse Laplace transform is usually computed using tables and properties of the Laplace transform.

Example 8.3.1. 1. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\}$. We use the formula $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, therefore

$$\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}.$$

2. Find $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\}$.
Since $\mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$, we obtain

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt).$$

8.4 Solving differential equations using Laplace transform

8.4.1 Second order differential equation

Laplace transform method is found to be effective in solving differential equations. We use it to convert a differential equation into an algebraic equation in the variable s, solve the resulting equation, and then take the inverse Laplace transform.
Consider the second-order differential equation :

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t), \qquad y(0) = y_0, \quad y'(0) = y_1.$$

The solution procedure using the Laplace transform is as follows :

1. Apply the Laplace transform to every term in the differential equation :

$$\mathcal{L}\{f(t)\} = F(s), \qquad \mathcal{L}\{y(t)\} = Y(s),$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0), \qquad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0).$$

2. Substitute the initial conditions $y(0) = y_0$ and $y'(0) = y_1$ to obtain :

$$(a_2s^2 + a_1s + a_0)Y(s) + \Phi(s) = F(s), \qquad \text{where } \Phi(s) = -[a_2(sy_0 + y_1) + a_1y_0].$$

3. Solve for $Y(s)$:

$$Y(s) = \frac{F(s) - \Phi(s)}{a_2s^2 + a_1s + a_0}.$$

4. Finally, find the solution $y(t)$ by taking the inverse Laplace transform :

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

Example 8.4.1. Solve the differential equation.

1. $y''(t) + 4y(t) = \sin(2t), \qquad y(0) = 0, \quad y'(0) = 1.$

Apply the Laplace transform.

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin(2t)\},$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{2}{s^2+4}.$$

Substitute the initial conditions.

$$s^2Y(s) - 1 + 4Y(s) = \frac{2}{s^2+4},$$

$$Y(s)(s^2+4) = \frac{2}{s^2+4} + 1.$$

Solve for $Y(s)$.

$$Y(s) = \frac{2}{(s^2+4)^2} + \frac{1}{s^2+4}.$$

Take the inverse Laplace transform.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)^2}\right\} = \frac{1}{2}\sin(2t) + \frac{1}{2}t\sin(2t).$$

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