Chapter 4: Frequency Domain Systems Analysis

1 - Introduction

The Frequency Domain offers a powerful alternative perspective. Instead of applying transient signals.

The fundamental principle is that when a linear time-invariant (LTI) system is excited by a sinusoid, its steady-state output is also a sinusoid of the same frequency, but with a different amplitude and phase.

- **Amplitude Ratio (Gain):** The output amplitude is scaled by a factor, $|G(j\omega)|$.
- **Phase Shift:** The output waveform is shifted in time, represented by an angle, $\angle G(j\omega)$.

By sweeping the input frequency ω from 0 to $+\infty$, we can map out the system's behavior across the entire frequency spectrum. This is the essence of Frequency Response.

The primary tool for this is the Transfer Function, G(s). We obtain the frequency response by substituting $s = j\omega$ (where j is the imaginary unit) to get $G(j\omega)$. This complex-valued function contains all the amplitude and phase information.

2 - Graphical Representation of Transfer Functions

2.1 Bode Plot/ Bode Diagram:

A Bode Plot is a graphical representation of a linear, time-invariant (LTI) system's frequency response. It's actually a pair of plots:

- **Magnitude Plot:** Shows how the amplitude of the output signal changes with frequency, compared to the input.
- **Phase Plot:** Shows how the phase shift (time delay) of the output signal changes with frequency.

The key feature of Bode plots is that they use a logarithmic scale for frequency, which allows us to represent a very wide range of frequencies and easily approximate the plot using straight lines. This is known as the "Asymptotic Bode Plot" or "Bode Approximation."

Bode plots are crucial in control systems for several reasons:

- **Stability Analysis:** We can determine the stability of a closed-loop system from its open-loop Bode plot using concepts like Gain Margin and Phase Margin.
- **Frequency Domain Design:** They allow us to design compensators (like PID controllers) to shape the system's response to meet specific performance criteria (e.g., faster response, reduced overshoot, better disturbance rejection).
- **Experimental Data:** It's easy to obtain a Bode plot experimentally by sweeping the frequency of an input signal and measuring the output, which helps in creating models of real-world systems.

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Simplicity: The straight-line approximation makes it very easy to sketch and understand a system's behavior without complex calculations.

The Components of a Bode Plot

1. Magnitude Plot

- **Y-Axis:** Magnitude in **Decibels (dB)**. The conversion is:
- $|G(j\omega)|_{dB} = 20\log_{10}|G(j\omega)|$
- **X-Axis:** Frequency (ω) on a **logarithmic scale** (base 10).

2. Phase Plot

- **Y-Axis:** Phase angle in **degrees (°)**.
- **X-Axis:** Frequency (ω) on the same **logarithmic scale** as the magnitude plot.

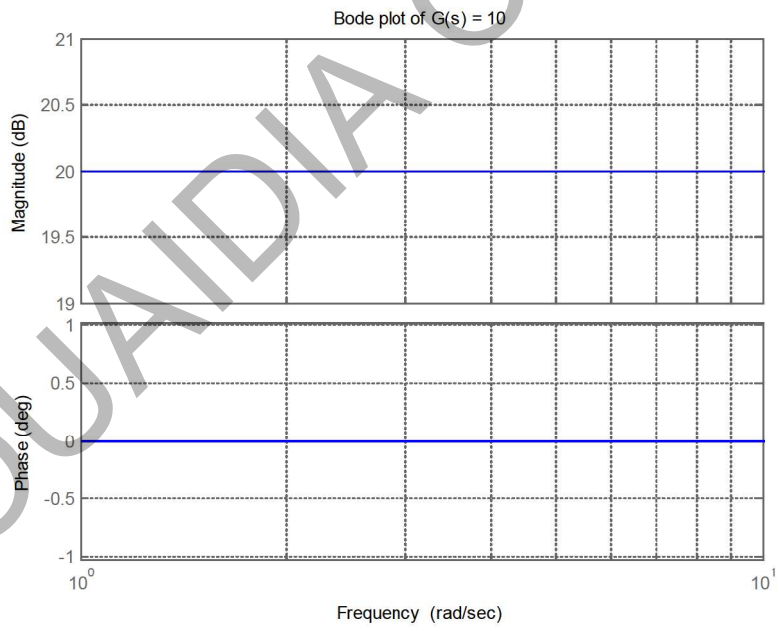
Building Bode Plots

Any transfer function can be broken down into a product of fundamental components.

1. Constant Gain (K)

- **Magnitude:** A horizontal line at $20\log_{10}K$ dB.

Phase: $0°$ if $K > 0$; $-180°$ if $K < 0$.

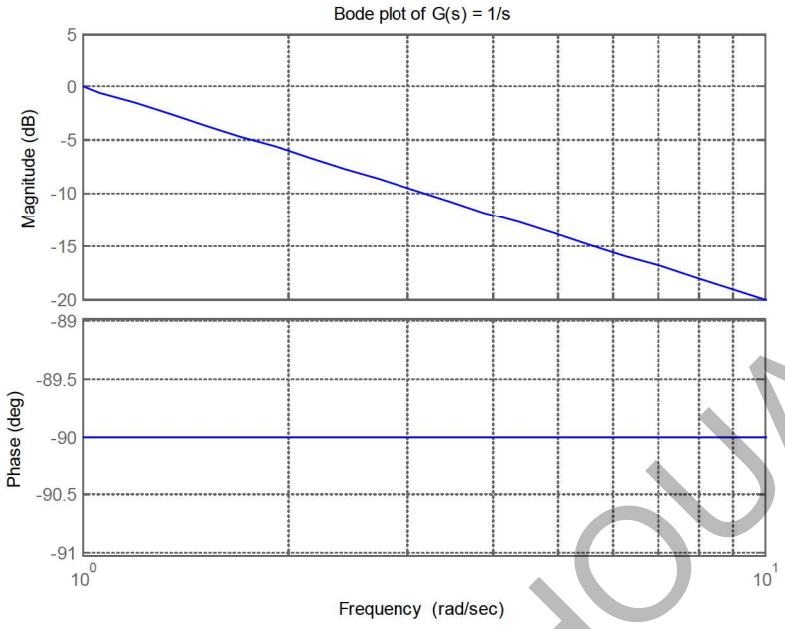


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2. Integrator ($1/s$) or Differentiator (s)

- **Integrator ($1/s$):**
  - **Magnitude:** A straight line with a slope of **-20 dB/decade**. It crosses 0 dB at $\omega = 1$.
  - **Phase:** Constant **-90°**.
- **Differentiator (s):**

- **Magnitudc:** A straight line with a slope of **+20 dB/dccadc**. It crosses 0 dB at $\omega = 1$.
- **Phase:** Constant **+90°**.

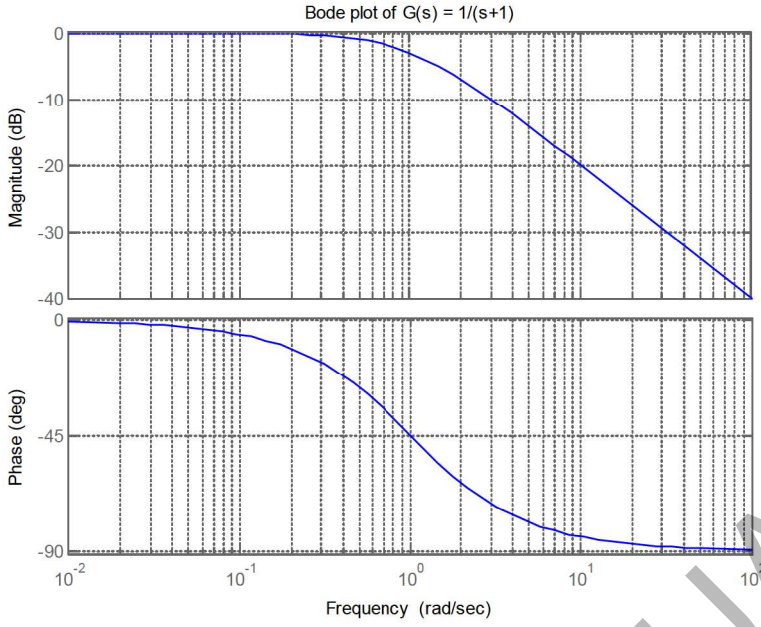


3. First Order Term $(1 + s\tau)$ *or* $1/(1 + s\tau)$

- **First Order Lag** $(1/(1 + s\tau))$**:**
  - **Magnitude:**
    - For low frequencies ($\omega << 1/\tau$): ~0 dB (a flat line).
    - For high frequencies ($\omega >> 1/\tau$): a line with a slope of **-20 dB/decade**.
    - The **corner frequency** (or break frequency) is at $\omega_c = 1/\tau$. At this point, the magnitude is -3 dB.
  - **Phase:**
    - At low frequencies: 0°.
    - At high frequencies: $-90°$.
    - At the corner frequency: $-45°$.
    - It's an arctangent curve, but it's often approximated by a straight line from $0.1\omega_c$ to $10\omega_c$.
- **First Order Lead** $(1 + s\tau)$**:** It's the mirror image of the lag.
  - **Magnitude Slope: +20 dB/decade** after the corner frequency.

Phase Shift: $0°$ to $+90°$.

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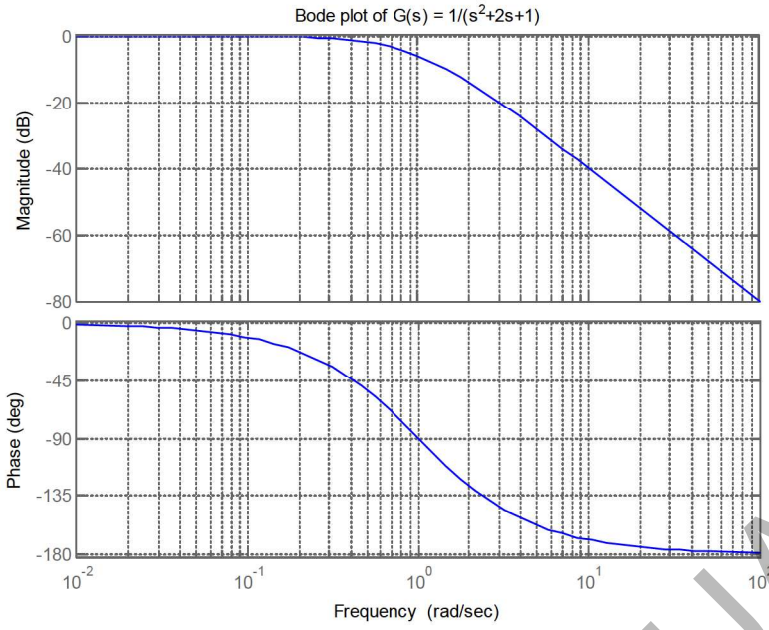
4. Second Order Term ($\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$)

This represents underdamped systems (like a mass-spring-damper).

- **Corner Frequency:** At the natural frequency ω_n.
- **Magnitude:**
  - The slope changes by **-40 dB/decade** after ω_n.
  - The actual magnitude at ω_n depends on the damping ratio (ζ): $|G(j\omega_n)|_{dB} = -20\log_{10}(2\zeta)$. This creates a resonant peak if $\zeta < 0.707$.
- **Phase:**
  - Shifts from $0°$ to $-180°$.
  - At ω_n, the phase is $-90°$.

The steepness of the phase transition depends on ζ; lower damping means a sharper transition.

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Example:

Sketch the Bode plot for: $G(s) = \frac{1000(s+2)}{(s+1)(s+10)}$

Step 1: Identify Poles, Zeros, and DC Gain

- Zero at s = –2 → corner frequency $\omega_z = 2$ rad/s
- Pole at s = –1 → corner frequency $\omega_{p1} = 1$ rad/s
- Pole at s = –10 → corner frequency $\omega_{p2} = 10$ rad/s

DC (low-frequency) gain:

G(0) = 1000(2) / (1 × 10) = 200

$M_{dB} = 20 \log_{10}(200) = 46.02$ dB

Step 2: Determine Slopes of the Asymptotic Magnitude Plot

Each zero contributes +20 dB/dec.

Each pole contributes –20 dB/dec.

Order the corner frequencies: 1, 2, 10

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• For $\omega << 1$: slope − 0 dB/dec

• After pole at 1 rad/s: slope = –20 dB/dec

• After zero at 2 rad/s: slope becomes 0 dB/dec

• After pole at 10 rad/s: slope becomes –20 dB/dec

Step 3: Asymptotic Magnitude Sketch Steps

• Start at 46.02 dB for frequencies below 1 rad/s

• Stay flat (0 dB/dec) until $\omega = 1$

• From $\omega = 1$ to 2: slope –20 dB/dec

• From $\omega = 2$ to 10: slope becomes flat again (0 dB/dec)

• After $\omega = 10$: slope becomes –20 dB/dec indefinitely

Step 4: Useful Exact Magnitude Values for Plotting

ω (rad/s)	**Magnitude**	**Magnitude (dB)**
0	200	46.02
1	157.4	43.95
2	124.0	41.87
10	71.77	37.12

These points correct the asymptotic lines near the break frequencies.

Step 5: Phase of $G(j\omega)$

Total phase:

$\varphi(\omega) = \arctan(\omega / 2) – \arctan(\omega / 1) – \arctan(\omega / 10)$

• Zero adds +90°

• Each pole subtracts –90°

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Step 6: Useful Exact Phase Values

At $\omega = 0.1$: $\varphi \approx -3.4°$

At $\omega = 1$: $\varphi \approx -24.1°$

At $\omega = 2$: $\varphi \approx -29.7°$

At $\omega = 10$: $\varphi \approx -50.6°$

At $\omega \to +\infty$: $\varphi \to -90°$

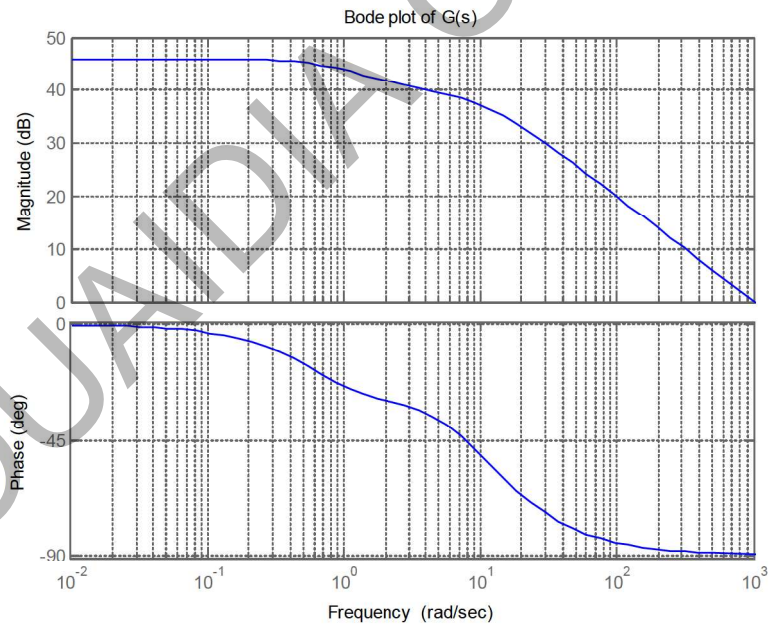
Step7: Final Bode Sketch Summary

Magnitude:

- Start 46.02 dB
- Breaks at 1, 2, 10 rad/s
- Slopes: $0 \to -20 \to 0 \to -20$ dB/dec

Phase:

- Starts near 0°
- Drops to about –30° around $\omega = 2$
- Approaches –90° for large ω .



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2.2 Nyquist Plot:

The primary objective of a Nyquist plot is to teach how to determine the stability of a closed-loop system from the open-loop transfer function, even if the system has poles in the right-half plane.

The Nyquist criterion uses the open-loop frequency response G(jω) to determine closed-loop stability. It does this by mapping the Nyquist Contour (a path that encloses the entire right-half s-plane) through the open-loop transfer function G(s) and counting the resulting encirclements around a critical point.

A **Nyquist Plot** is a graphical representation of the frequency response $G(j\omega)$ on the complex plane.

- **X-axis:** Real part of $G(j\omega)$ [Re$G(j\omega)$]
- **Y-axis:** Imaginary part of $G(j\omega)$ [Im$G(j\omega)$]
- **Parameter:** Frequency ω, which varies from 0 to $+\infty$.

It's essentially a polar plot of $G(j\omega)$, where the magnitude $|G(j\omega)|$ is the distance from the origin, and the phase $\angle G(j\omega)$ is the angle from the positive real axis.

Example:

Draw a Simple Nyquist Plot (The "Sketch" Method)

$G(s) = \frac{1}{s+1}$.

1. **Write the Frequency Response:** Substitute $s = j\omega$.

$$G(j\omega) = \frac{1}{j\omega + 1}$$

2. **Find Real and Imaginary Parts:** Multiply numerator and denominator by the complex conjugate.

$$G(j\omega) = \frac{1}{j\omega + 1} \cdot \frac{1 - j\omega}{1 - j\omega} = \frac{1 - j\omega}{1 + \omega^2} = \frac{1}{1 + \omega^2} - j\frac{\omega}{1 + \omega^2}$$

So,

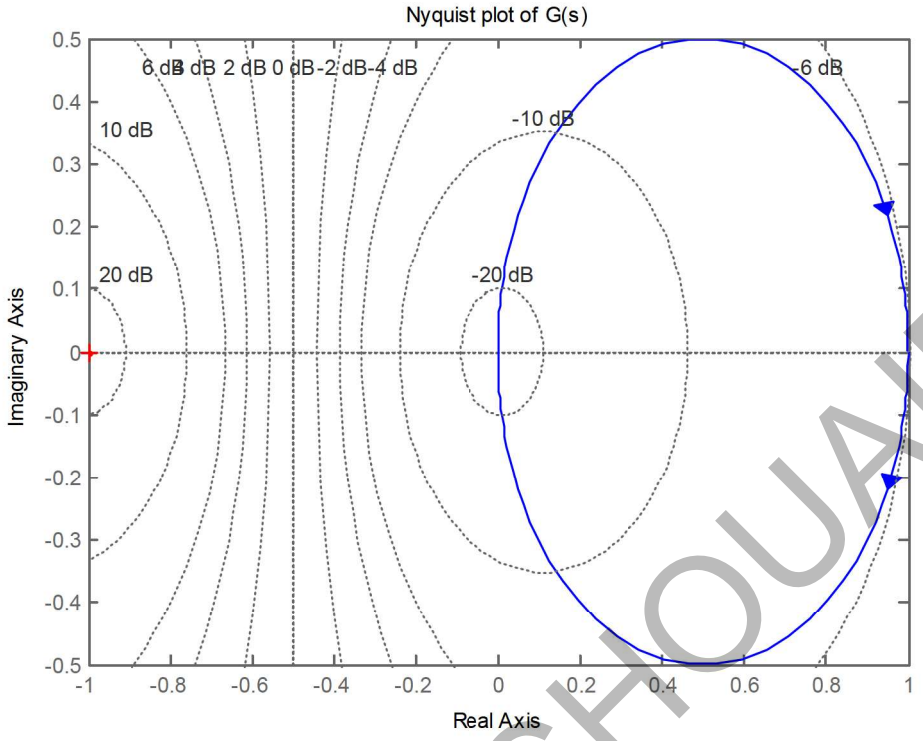
$$\text{Re}[G] = \frac{1}{1+\omega^2},$$

$$\text{Im}[G] = -\frac{\omega}{1+\omega^2}.$$

3. **Identify Key Frequency Points:**
   - **When $\omega = 0$:** Re $= 1$, Im $= 0$. The plot starts at (1, 0).
   - **When $\omega \to \infty$:** Re $\to 0$, Im $\to 0$. The plot ends at the origin (0, 0).
   - **Phase at $\omega \to \infty$:** The phase $\angle G(j\omega)$ approaches -90° (-π/2 rad). This tells us the curve approaches the origin from the direction of the negative imaginary axis.

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Sketch the Curve: Plot the start and end points. Knowing the phase, you can sketch a smooth curve from (1, 0) to (0, 0) that lies in the 4th quadrant (negative imaginary part).



2.3 Black-Nichols Chart:

The **Nichols chart** is a graphical representation of a system's **open-loop frequency response** in terms of **gain (in dB)** versus **phase (in degrees)**. It combines Bode plot information into a single plot, making it easier to assess **stability margins** and **closed-loop performance**.

- **Horizontal axis:** Phase angle (degrees)
- **Vertical axis:** Gain (dB)
- **Curves:** Contours of **constant closed-loop gain** (iso-gain) and **constant closed-loop phase** (iso-phase).

The purpose of the **Nichols chart** is for:

1. **Stability Analysis:** You can quickly determine **gain margin** and **phase margin**.
2. **Controller Design:** Helps in shaping open-loop response to meet **performance specifications**.
3. **Single-Plot Insight:** Combines magnitude and phase in one chart (unlike the Bode plot, which has two separate plots).

$G(s)$ is an open-loop transfer function in a unity feedback system. The closed-loop transfer function is:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

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Calculating the magnitude and phase of $T(j\omega)$ across all frequencies can be tedious. The Nichols Chart provides a clever **graphical method** to find the closed-loop frequency response $T(j\omega)$ directly from the open-loop frequency response $G(j\omega)$.

The key insight is that for a given open-loop point $G(j\omega)$, the corresponding closed-loop magnitude M and phase α are:

$$M = \left|\frac{G(j\omega)}{1+G(j\omega)}\right|, \quad \alpha = \angle\frac{G(j\omega)}{1+G(j\omega)}$$

It turns out that on the open-loop Nyquist plot, contours of constant M (magnitude) and constant α (phase) form specific patterns. The Nichols Chart is a plot of these contours, overlaid on a grid of open-loop gain (in dB) vs. open-loop phase (in degrees).

Example:

Suppose we have a system:

$$G(s) = \frac{1}{s(s+1)}$$

Open-Loop Bode Plot (Sketch):

- At low frequency: High gain, phase = -90°.
- At high frequency: Low gain, phase -> -180°.
- The phase curve will go from -90° down to -180°.

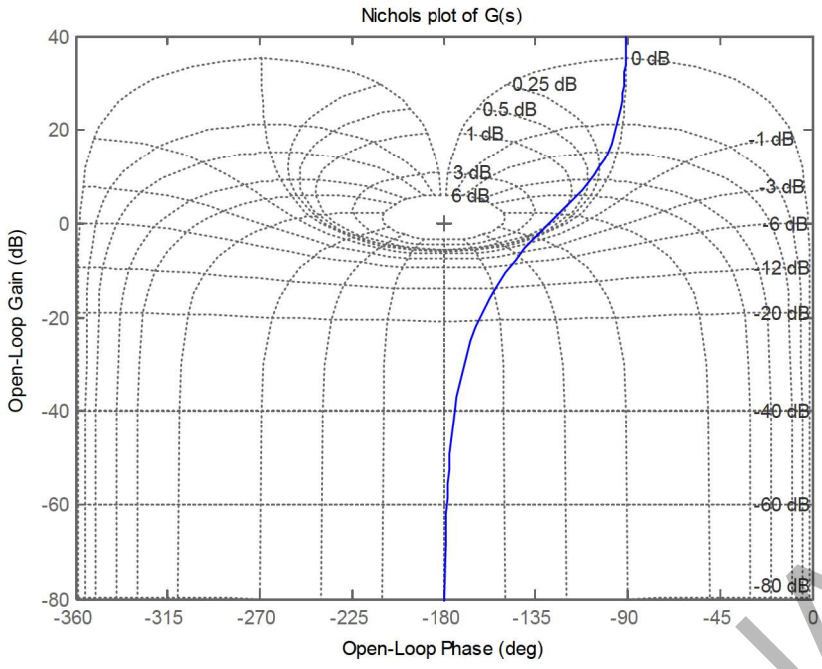
Plotting on Nichols Chart:

- The curve will start in the top-left quadrant (high gain, phase near -90°).
- It will curve downward and to the right as gain decreases and phase lags increase, eventually approaching the bottom of the chart (low gain) near -180° phase.

Reading the Chart:

- Find the point where your Nichols Plot is **tangent** to the highest Constant-M contour. This gives you the **closed-loop resonant peak,** M_r. This is a key measure of relative stability.
- The frequency at which this tangency occurs is the **resonant frequency,** ω_r.
- Find the -3 dB Constant-M contour. The frequency at which your plot crosses this contour is the **closed-loop bandwidth,** ω_{BW}.

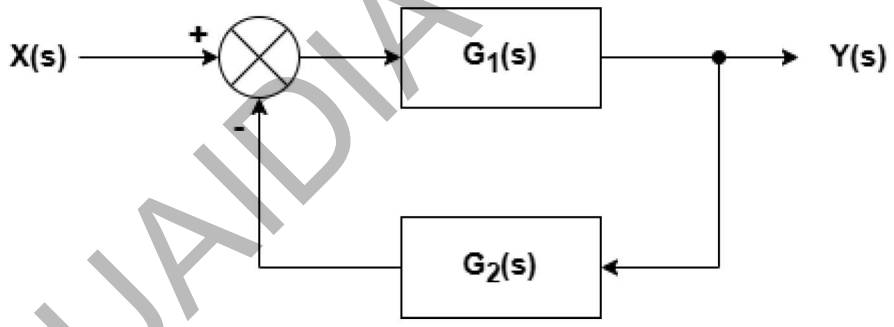
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3 - Analysis and Stability Criteria

3.1 Bode plot stability

In classical control, the stability of a feedback system can be evaluated using the **Bode plot** of the **open-loop transfer function**:



$$L(s) = G_1(s).G_2(s)$$

We use two main concepts:

1. Gain Margin (GM)

- Defined at the **phase crossover frequency**:

$$\omega_{pc}: \angle L(j\omega_{pc}) = -180°$$

- At that frequency, read the gain:

$$|L(j\omega_{pc})|$$

- Then:

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$$\text{GM} = \frac{1}{|L(j\omega_{\text{pc}})|} \quad \text{or in dB} \quad \text{GM(dB)} = -20\log_{10}|L(j\omega_{\text{pc}})|$$

Stable if GM > 0 dB

2. Phase Margin (PM)

- Defined at the **gain crossover frequency**:

$$\omega_{\text{gc}}: |L(j\omega_{\text{gc}})| = 1 \ (0 \text{ dB})$$

- At that frequency, read the phase:

$$\angle L(j\omega_{\text{gc}})$$

- Phase margin:

$$\text{PM} = 180^\circ + \angle L(j\omega_{\text{gc}})$$

Stable if PM > 0°

Stability Decision

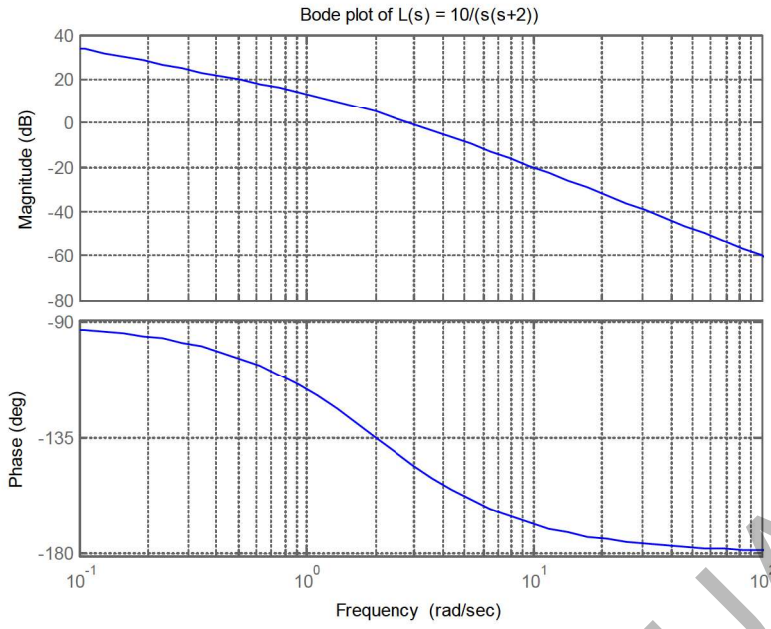
Criterion	Stable?
PM > 0°	✓ Yes
GM > 0 dB	✓ Yes
Both margins small	Marginal
PM < 0° or GM < 0 dB	✗ Unstable

. **Example**

Consider the open-loop transfer function:

$$L(s) = \frac{10}{s(s+2)}$$

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STEP 1 - **Bode Magnitude and Phase**

Magnitude slope:

- At low freq: -20 dB/dec (due to 1/s)
- After 2 rad/s pole: -40 dB/dec

Phase:

- Pole at 0 → -90°
- Pole at 2 → shifts from 0° to -90° around ω = 2 rad/s
- Final → -180°

STEP 2 - **Find Gain Crossover Frequency** (|L| = 1)

For this system, the magnitude curve crosses **0 dB at about:**

$$\omega_{gc} \approx 1.5 \text{ rad/s}$$

STEP 3 - **Read Phase at Gain Crossover**

At ω = 1.5 rad/s:

- Integrator: -90°
- Pole at 2 rad/s contributes ≈ -40° (approx from Bode rules)

So total phase:

$$\angle L(j\omega_{gc}) \approx -130^\circ$$

STEP 4 - **Compute Phase Margin**

$$\text{PM} = 180^\circ - 130^\circ = 50^\circ$$

Phase Margin = +50° → STABLE

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STEP 5 - **Check Gain Margin**

Phase reaches -180° at about:

$$\omega_{pc} \approx 5 \text{ rad/s}$$

At that frequency, the magnitude is roughly:

$$|L| \approx -14 \text{ dB}$$

$$\text{GM(dB)} = -(-14) = +14 \text{ dB}$$

Gain Margin = +14 dB → STABLE

Final Stability Result

Both margins are positive:

- **PM ≈ +50°**
- **GM ≈ +14 dB**

The closed-loop system is stable.

3.2 Nyquist Test Stability

The Nyquist stability criterion is used in control systems to determine the closed-loop stability of a system from its open-loop transfer function ***$G_1(s).G_2(s)$***.

It is especially useful when the characteristic equation is difficult to solve directly or when the system has a time delay.

Let:

- P = number of **open-loop poles in the right half-plane (RHP)**.
- Z = number of **closed-loop poles in RHP** (what we want to know).
- N = number of **clockwise encirclements** of the point -1 by the Nyquist plot.

Then:

$$Z = N + P$$

The system is **stable** when:

$$Z = 0$$

Steps of the Nyquist Test:

1. **Find** $G(s)H(s)$.
2. Count RHP poles → this is P.
3. **Draw or compute** the Nyquist plot of $G(j\omega)H(j\omega)$.
4. **Determine** the number of clockwise encirclements of -1 → this is N.
5. Use the Nyquist equation: $Z = N + P$.
6. If $Z = 0$, the system is **closed-loop stable**.

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Example:

Given open-loop transfer function:

$$G(s)H(s) = \frac{10}{s(s+2)(s+3)}$$

Step 1 - Count RIGHT HALF-PLANE Poles P

Poles at:

$$s = 0, -2, -3$$

All in the left half-plane except 0 (on the imaginary axis).

➞ No RIGHT HALF-PLANE poles → $P = 0$
(Zeros on imaginary axis do *not* count as RIGHT HALF-PLANE.)

Step 2 - Nyquist Plot Facts

At low frequency:

$$G(j0) \to \infty \text{ (vertical asymptote due to pole at 0)}$$

At high frequency:

$$|G(j\omega)| \approx \frac{10}{\omega^3} \to 0$$

Plot approaches the origin.

Phase shift:

Total phase = sum of phases:

- From $1/s$: $-90°$
- From $1/(s+2)$: about $-90°$ at high ω
- From $1/(s+3)$: about $-90°$

Total phase ≈ $-270°$.

Thus Nyquist curve goes through the negative real axis, but:

Check the real-axis crossing.

Step 3 - Find where the Nyquist plot crosses the real axis

Set the imaginary part of $G(j\omega) = 0$.

Let:

$$G(j\omega) = \frac{10}{j\omega(j\omega+2)(j\omega+3)}$$

This is messy, but here is the important result:

When evaluating real-axis crossing numerically:

At ω = 1 rad/s:

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$$G(j1) = -0.357 + j0.476$$

At ω ≈ 2.8 rad/s:

$$G(j2.8) \approx -0.27 + j0$$

So the Nyquist plot crosses the real axis at about: -0.27

This is to the **right** of the critical point -1.

➔ The Nyquist plot does NOT encircle -1.

$$N = 0$$

Step 4 - Apply Nyquist Criterion

- $P = 0$
- $N = 0$

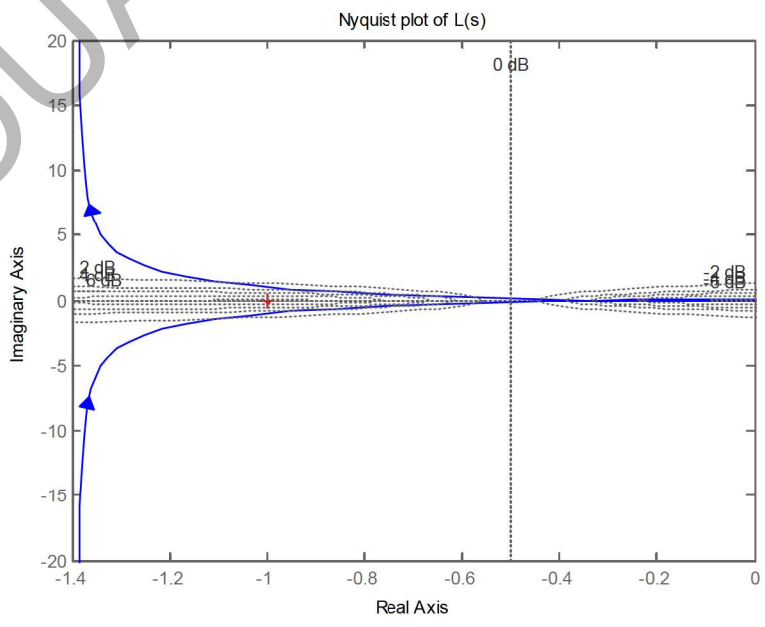
$$Z = N + P = 0$$

Conclusion: The closed-loop system is stable.

The Nyquist plot shows no encirclement of the point -1, so the system has **no right-half-plane closed-loop poles**.

Summary:

Quantity	Value
RIGHT HALF-PLANE open-loop poles P	0
Encirclements of -1 N	0
RIGHT HALF-PLANE closed-loop poles $Z = N + P$	0
Stability	Stable



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3.3 Black-Nickols Plot

The Nichols Chart provides an excellent visualization of the classic stability margins.

The Critical Point

On the Nyquist plot, the critical point for stability is $(-1,0)$. What is this point on the Nichols Chart?

- $G(j\omega) = -1$ corresponds to:
  - Gain: $|G| = 1$ which is **0 dB**
  - Phase: $\angle G = -180°$

Therefore, **the critical point on the Nichols Chart is (Phase = -180°, Gain = 0 dB)**.

Gain Margin

- **Definition:** The amount of gain increase allowed before the system becomes unstable.
- **How to Find it:**
  a. Find the point on your Nichols Plot where the **phase is 180°**. Note the gain g (in dB) at this point.
  b. The Gain Margin is $GM = -g$ (in dB). (If the gain is negative, you add gain; if it's positive, you have to reduce gain.)
  - *Example:* If at -180° the gain is -6 dB, then $GM = 6$ dB. You can multiply the gain by $10^{6/20} \approx 2$ before the system becomes unstable.

Phase Margin

- **Definition:** The amount of additional phase lag allowed at the gain crossover frequency before the system becomes unstable.
- **How to Find it:**
  c. Find the point on your Nichols Plot where the **gain is 0 dB**. Note the phase ϕ (in degrees) at this point.
  d. The Phase Margin is $PM = 180° + \phi$.
  - *Example:* If at 0 dB gain the phase is -150°, then $PM = 30°$.

Visualization: The distance of your Nichols Plot from the critical point (0 dB, -180°) directly indicates stability. A plot that passes close to this point has low stability margins.

Closed-Loop Performance & Design with Nichols Chart

From your Nichols Plot, you can directly read:

- **Resonant Peak (M_r):** The highest closed-loop magnitude. Lower M_r (closer to 0 dB) means better damping and less overshoot in the time domain.
- **Resonant Frequency (ω_r):** The frequency at which M_r Occurs. Related to the speed of the time response.

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- **Bandwidth (ω_{BW}):** The frequency at which the closed-loop gain drops to -3 dB. A higher bandwidth means a faster response.

The Power of the Chart: Compensator Design

The Nichols Chart is exceptionally powerful for designing lead/lag compensators.

- **Lead Compensator (Adds Phase):** You want to "bend" the Nichols Plot away from the critical point. A lead compensator provides a phase boost near the gain crossover frequency. On the chart, you can see exactly how much you need to "lift" the curve vertically (in the phase axis) to achieve a desired Phase Margin or reduce the resonant peak. M_r.
- **Lag Compensator (Attenuates Gain at High Freq):** You want to "push down" the Nichols Plot at high frequencies without affecting the phase at low frequencies. This is useful when you have adequate PM but need to reduce gain at high frequency for noise rejection. On the chart, you can see how much to lower the high-frequency part of the curve to stay clear of the critical point.

3.4 Stability Analysis using the Routh-Hurwitz Criterion

In control systems engineering, stability is the single most important requirement for any system. A stable system produces a bounded output for a bounded input. An unstable system can oscillate uncontrollably or diverge to infinity, leading to catastrophic failure.

For a Linear Time-Invariant (LTI) system, the stability is determined entirely by the location of the roots (poles) of its **characteristic equation**.

- **The Characteristic Equation** is the denominator of the closed-loop transfer function set equal to zero.
    - For a transfer function $G(s) = \frac{N(s)}{D(s)}$, the characteristic equation is $D(s) = 0$.

The Fundamental Rule: A system is stable **if and only if** all the roots of its characteristic equation have **negative real parts** (i.e., they are all in the left-half of the s-plane).

The Problem: For high-order systems (e.g., 4th order and above), finding the roots of the characteristic equation analytically becomes very difficult and computationally intensive. This is where the **Routh-Hurwitz Criterion** comes in.

The Routh-Hurwitz criterion:

The Routh-Hurwitz criterion is a **necessary and sufficient** mathematical test that determines whether all the roots of a polynomial have negative real parts, **without having to calculate the roots themselves**.

The meaning is:

- The number of poles in the Right-Half Plane (RHP - unstable poles).
- The number of poles on the imaginary axis (marginally stable).
- It does **not** give the exact location of the poles, only their region in the s-plane.

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The Routh-Hurwitz Algorithm:

Let the characteristic equation be of the form:

$$a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$$

where all coefficients a_i are real, and $a_n > 0$.

Step 1: Necessary Condition (Quick Check): A necessary (but not sufficient) condition for stability is that **all coefficients** $a_0, a_1, \ldots, a_n$ Must be present and **positive**.

- If any coefficient is missing (zero) or negative, the system is **definitely unstable**. **Stop here.**
- If they are all positive, you must proceed to Step 2.

Step 2: Construct the Routh Array: The Routh array is structured as follows:

Row	s^n	s^{n-1}	s^{n-2}	s^{n-3}	...
s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	...
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	...
s^{n-2}	b_1	b_2	b_3	b_4	...
s^{n-3}	c_1	c_2	c_3	c_4	...
...	...	...	...	...	...
s^0	...	...	...	...	...

The elements b_1, b_2, c_1, c_2, etc., are calculated systematically:

$$b_1 = \frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} = \frac{-(a_n \cdot a_{n-3} - a_{n-1} \cdot a_{n-2})}{a_{n-1}}$$

$$b_2 = \frac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}} = \frac{-(a_n \cdot a_{n-5} - a_{n-1} \cdot a_{n-4})}{a_{n-1}}$$

$$c_1 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1}$$

$$c_2 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1}$$

And so on. The pattern continues until the last row (s^0) is completed. Each new row is calculated from the two rows immediately above it.

Step 3: The Stability Criterion The system is stable **if and only if** all the elements in the **first column** of the Routh array are **positive**.

- The number of sign changes in the first column is equal to the **number of poles in the Right-Half Plane (RHP)**.

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Examples:

Example 1: Stable System Characteristic Equation: $s^3 + 6s^2 + 11s + 6 = 0$

Step 1: All coefficients > 0. ➔ Proceed.

Step 2: Build Routh Array.

Row	Calculation
s^3	1 | 11
s^2	6 | 6
s^1	$b_1 = \frac{-(1*6-6*11)}{6} = 10$
s^0	$c_1 = \frac{-(6*6-10*0)}{10} = 6$

Step 3: First column is [1, 6, 10, 6]. All positive.

Conclusion: The system is stable. No RHP poles.

Example 2: Unstable System (Special Case 1)

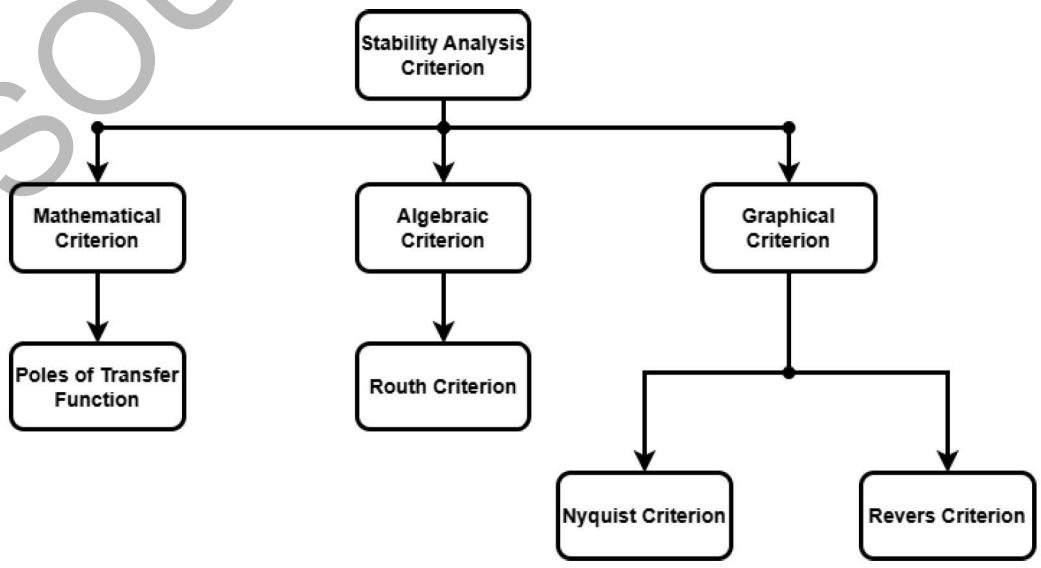
Characteristic Equation: $s^3 + s^2 + 2s + 8 = 0$

Step 1: All coefficients > 0. ➔ Proceed.

Step 2: Build Routh Array.

Row	Calculation
s^3	1 | 2
s^2	1 | 8
s^1	$b_1 = \frac{-(1*8-1*2)}{1} = -6$
s^0	$c_1 = \frac{-(1*8-(-6)*0)}{-6} = 8$

Step 3: First column is [1, 1, -6, 8]. There are **two sign changes** (1 → -6, and -6 → 8).
Conclusion: The system is unstable with **2 poles in the RHP**.



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