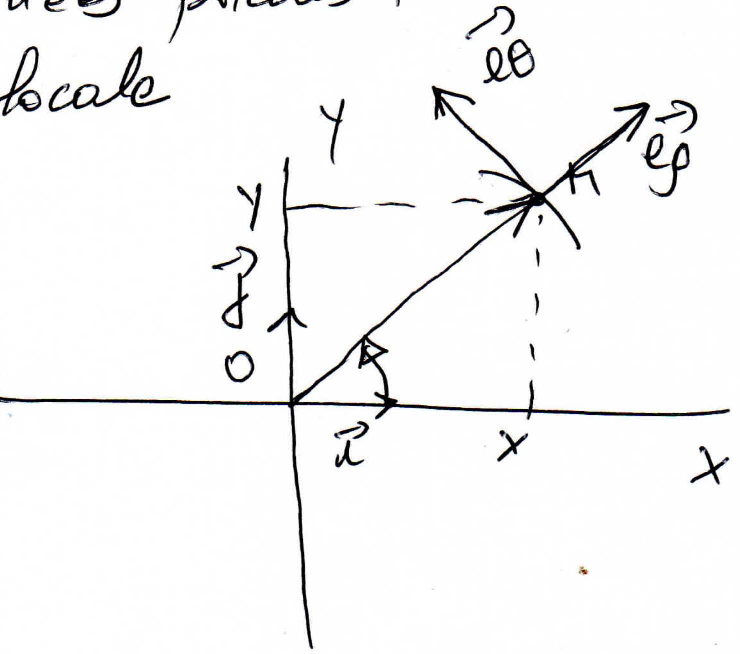
Etude du mvt dans différents systèmes

Système de coordonnées polaires

(ρ, θ) = coordonnées polaires.

$(\vec{e_\rho}, \vec{e_\theta})$ = base locale



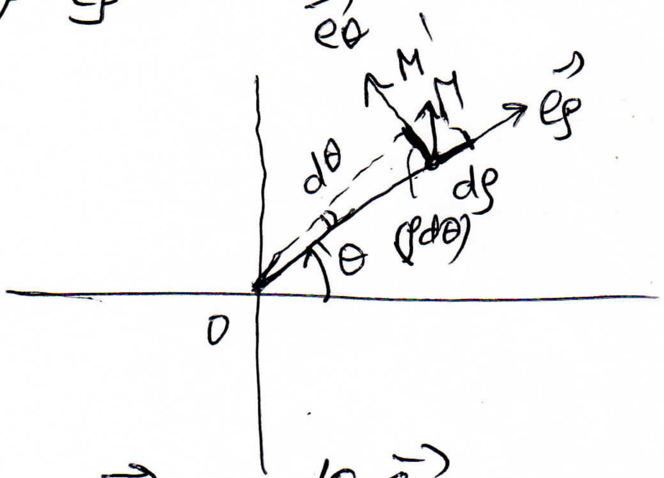
Repérage : $\begin{cases} OM = \rho \\ \theta = (\vec{Ox}, \vec{OM}) \end{cases}$

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$$

* Vecteur - position :

$$\vec{OM}(t) = \vec{r}(t) = \rho\, \vec{e_\rho}$$

* Vecteur déplacement :



$$\vec{MM'} = d\vec{r}(t) =$$

$$d\vec{r} = d\rho \cdot \vec{e_\rho} + \rho\, d\theta\, \vec{e_\theta}$$

* vecteur vitesse :

a) $\vec{v} = \dfrac{\text{déplacement}}{\text{temps}} = \dfrac{d\rho \cdot \vec{e_\rho} + \rho\, d\theta\, \vec{e_\theta}}{dt}$

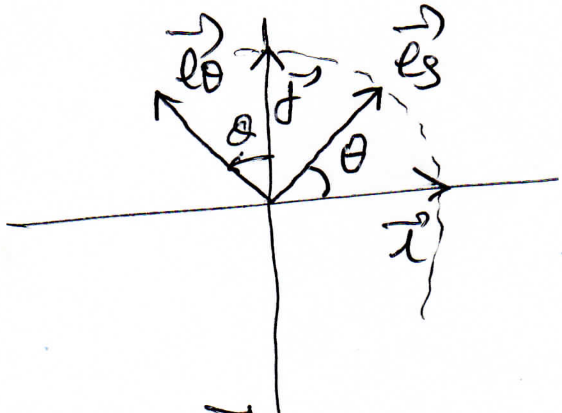
$$= \frac{d\rho}{dt} \vec{e_\rho} + \rho \frac{d\theta}{dt} \vec{e_\theta} = \dot{\rho}\, \vec{e_\rho} + \rho\, \dot{\theta}\, \vec{e_\theta}$$

b) $\vec{v} = \frac{d}{dt}$ (Vecteur-position).

$$\vec{r} = \rho\,\vec{e_\rho}$$

$$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\rho \cdot \vec{e_\rho}) = \frac{d\rho}{dt}\vec{e_\rho} + \rho\,\frac{d\vec{e_\rho}}{dt}.$$

<u>Rappels:</u>



$$\vec{e_\rho} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}.$$

$$\vec{e_\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\frac{d\vec{e_\rho}}{d\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} = \vec{e_\theta}$$

$$\frac{d\vec{e_\theta}}{d\theta} = -\cos\theta\,\vec{i} - \sin\theta\,\vec{j} = -(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) = -\vec{e_\rho}$$

$$\text{Donc} \begin{cases} \frac{d\vec{e_\rho}}{d\theta} = \vec{e_\theta} \\ \frac{d\vec{e_\theta}}{d\theta} = -\vec{e_\rho} \end{cases}$$

$$\vec{v} = \frac{d\rho}{dt}\vec{e_\rho} + \rho\,\frac{d\vec{e_\rho}}{dt} = \frac{d\rho}{dt}\vec{e_\rho} + \rho\,\frac{d\vec{e_\rho}}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= \frac{d\rho}{dt}\vec{e_\rho} + \rho\,\frac{d\theta}{dt}(\vec{e_\theta})$$

$$\Rightarrow \boxed{\vec{v} = \dot{\rho}\,\vec{e_\rho} + \rho\,\dot{\theta}\,\vec{e_\theta}}$$

* Vecteur - accélération :

$$\vec{a} = \frac{d\vec{v}(t)}{dt}$$

$$= \frac{d}{dt}\left(\dot{\rho}\,\vec{e_\rho} + \rho\,\dot{\theta}\,\vec{e_\theta}\right)$$

$$= \frac{d\dot{\rho}}{dt}\,\vec{e_\rho} + \dot{\rho}\,\frac{d\vec{e_\rho}}{dt} + \frac{d\rho}{dt}\left(\dot{\theta}\,\vec{e_\theta}\right) + \rho\,\frac{d}{dt}\left(\dot{\theta}\,\vec{e_\theta}\right)$$

$$= \frac{d\dot{\rho}}{dt}\,\vec{e_\rho} + \dot{\rho}\,\frac{d\vec{e_\rho}}{dt} + \dot{\theta}\,\frac{d\rho}{dt}\,\vec{e_\theta} + \rho\left(\frac{d\dot{\theta}}{dt}\,\vec{e_\theta} + \dot{\theta}\,\frac{d\vec{e_\theta}}{dt}\right)$$

$$= \frac{d\dot{\rho}}{dt}\,\vec{e_\rho} + \dot{\rho}\,\frac{d\vec{e_\rho}}{dt} + \dot{\theta}\,\dot{\rho}\,\vec{e_\theta} + \rho\,\ddot{\theta}\,\vec{e_\theta} + \rho\,\dot{\theta}\,\frac{d\vec{e_\theta}}{d\theta}\,\frac{d\theta}{dt}$$

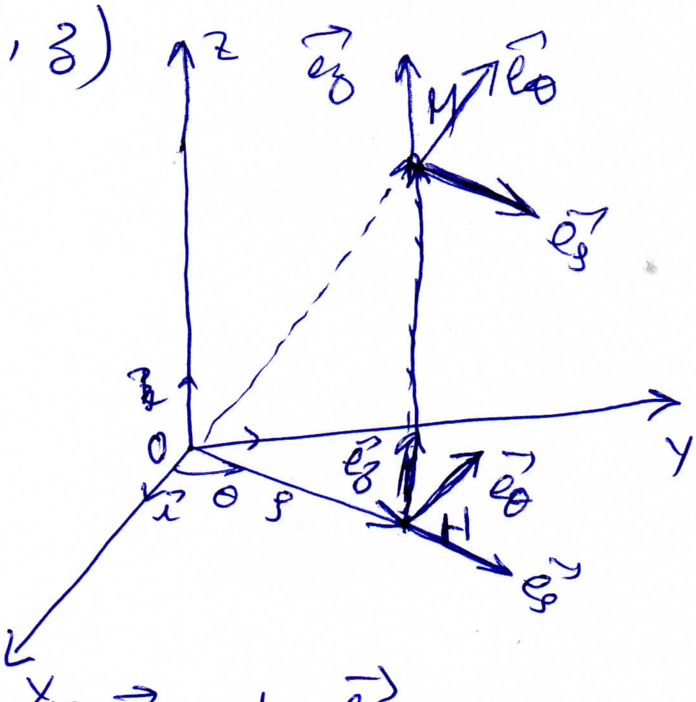
$$= \ddot{\rho}\,\vec{e_\rho} + \dot{\rho}\,\frac{d\vec{e_\rho}}{d\theta}\cdot\frac{d\theta}{dt} + \dot{\theta}\,\dot{\rho}\,\vec{e_\theta} + \rho\,\ddot{\theta}\,\vec{e_\theta} + \rho\,\dot{\theta}^2\left(-\vec{e_\rho}\right)$$

$$\vec{a} = \ddot{\rho}\,\vec{e_\rho} + \dot{\rho}\,\dot{\theta}\,\vec{e_\theta} + \dot{\theta}\,\dot{\rho}\,\vec{e_\theta} + \rho\,\ddot{\theta}\,\vec{e_\theta} + \rho\,\dot{\theta}^2\left(-\vec{e_\rho}\right)$$

$$\boxed{\vec{a} = \left(\ddot{\rho} - \rho\,\dot{\theta}^2\right)\vec{e_\rho} + \left(\rho\,\ddot{\theta} + 2\dot{\rho}\,\dot{\theta}\right)\vec{e_\theta}}$$

Etude du mvt en coordonnées cylindriques.

* coord. cylindriques : (ρ, θ, z)
* base locale $(\vec{e_\rho}, \vec{e_\theta}, \vec{e_z})$



* Vecteur-position

$$\vec{OM}(t) = \vec{r}(t) = \vec{OH} + \vec{HM}$$

$$\vec{OM}(t) = \rho\,\vec{e_\rho} + z\,\vec{e_z}$$

$$\Rightarrow \boxed{\vec{r}(t) = \rho\,\vec{e_\rho} + z\,\vec{e_z}}$$

* Vecteur-deplacement :

$$\vec{MM'} = \vec{OM'} - \vec{OM} \underset{d\vec{r}}{=} d\rho\cdot\vec{e_\rho} + \rho\cdot d\theta\,\vec{e_\theta} + dz\cdot\vec{e_z}.$$

* Vecteur vitesse :

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d\rho\,\vec{e_\rho} + \rho\,d\theta\,\vec{e_\theta} + dz\cdot\vec{e_z}}{dt}$$

$$\vec{v}(t) = \frac{d\rho}{dt}\vec{e_\rho} + \rho\frac{d\theta}{dt}\vec{e_\theta} + \frac{dz}{dt}\vec{e_z}$$

$$\boxed{\vec{v}(t) = \dot{\rho}\,\vec{e_\rho} + \rho\,\dot{\theta}\,\vec{e_\theta} + \dot{z}\,\vec{e_z}}$$

* Vecteur-accélération

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\left(\dot{\rho}\,\vec{e_\rho} + \rho\,\dot{\theta}\,\vec{e_\theta}\right) + \frac{d}{dt}\left(\dot{z}\,\vec{e_z}\right)$$

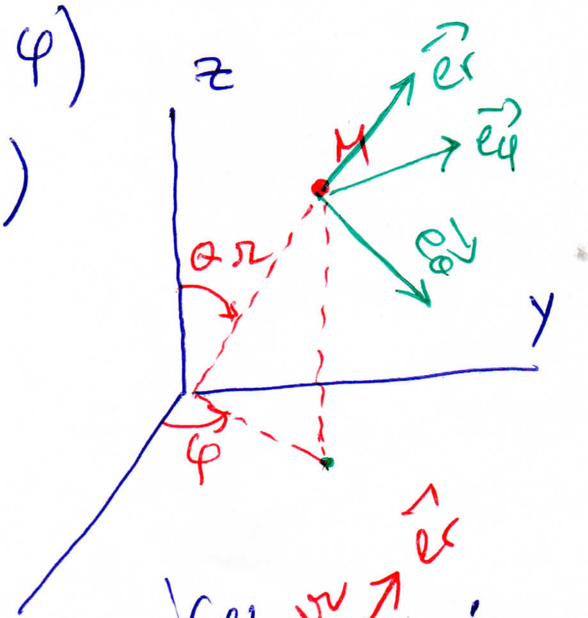
$$= \ddot{\rho}\,\vec{e_\rho} + \dot{\rho}\frac{d\vec{e_\rho}}{dt} + \frac{d\rho}{dt}\left(\dot{\theta}\,\vec{e_\theta}\right) + \rho\frac{d}{dt}\left(\dot{\theta}\,\vec{e_\theta}\right) + \ddot{z}\,\vec{e_z}$$

$$= \ddot{\rho}\,\vec{e_\rho} + \dot{\rho}\dot{\theta}\,\vec{e_\theta} + \dot{\rho}\dot{\theta}\,\vec{e_\theta} + \rho\ddot{\theta}\,\vec{e_\theta} - \rho\dot{\theta}^2\,\vec{e_\rho} + \ddot{z}\,\vec{e_z}$$

$$\boxed{\vec{a}(t) = (\ddot{\rho} - \rho\dot{\theta}^2)\,\vec{e_\rho} + (2\dot{\rho}\dot{\theta} + \ddot{\theta}\rho)\,\vec{e_\theta} + \ddot{z}\,\vec{e_z}}$$

Etude du mvt en coordonnées sphériques

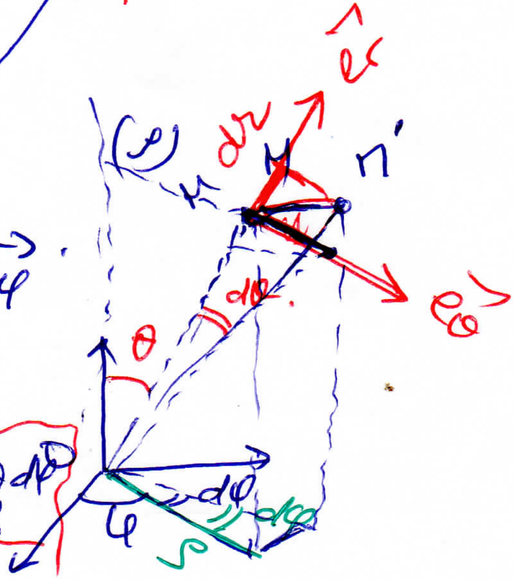
* coord. sphériques : (r, θ, φ)
* Base locale : $(\vec{e_r}, \vec{e_\theta}, \vec{e_\varphi})$



Vecteur-position :

$$\vec{OM}(t) = \vec{r}(t) = r\,\vec{e_r}$$

Vecteur-déplacement



$$\vec{MM'} = d\vec{r}(t) = dr.\vec{e_r} + r\,d\theta\,\vec{e_\theta} + \rho\,d\varphi\,\vec{e_\varphi}$$

or $\rho = r\sin\theta$

$$\Rightarrow \boxed{d\vec{r}(t) = dr\,\vec{e_r} + r\,d\theta\,\vec{e_\theta} + r\sin\theta\,d\varphi\,\vec{e_\varphi}}$$

Vecteur-vitesse

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dr.\vec{e_r} + r\,d\theta\,\vec{e_\theta} + r\sin\theta\,d\varphi\,\vec{e_\varphi}}{dt}$$

$$\vec{v}(t) = \frac{dr}{dt}\vec{e_r} + r\frac{d\theta}{dt}\vec{e_\theta} + r\sin\theta\frac{d\varphi}{dt}\vec{e_\varphi}$$

$$\boxed{\vec{v}(t) = \dot{r}\,\vec{e_r} + r\dot{\theta}\,\vec{e_\theta} + r\sin\theta\,\dot{\varphi}\,\vec{e_\varphi}}$$

Vecteur accélération

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\left(\dot{r}\vec{e_r} + r\dot{\theta}\vec{e_\theta} + r\sin\theta\,\dot{\varphi}\,\vec{e_\varphi}\right)$$

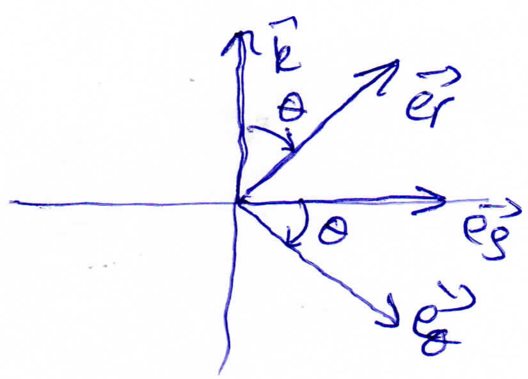
$$\vec{a}(t) = \frac{d\dot{r}}{dt}\,\vec{e_r} + \dot{r}\,\frac{d\vec{e_r}}{dt} + \frac{dr}{dt}\cdot\dot{\theta}\,\vec{e_\theta} + r\,\frac{d}{dt}(\dot{\theta}\,\vec{e_\theta}) + \frac{dr}{dt}\left[\sin\theta\,\dot{\varphi}\,\vec{e_\varphi}\right] + r\,\frac{d}{dt}\left[\sin\theta\,\dot{\varphi}\,\vec{e_\varphi}\right].$$

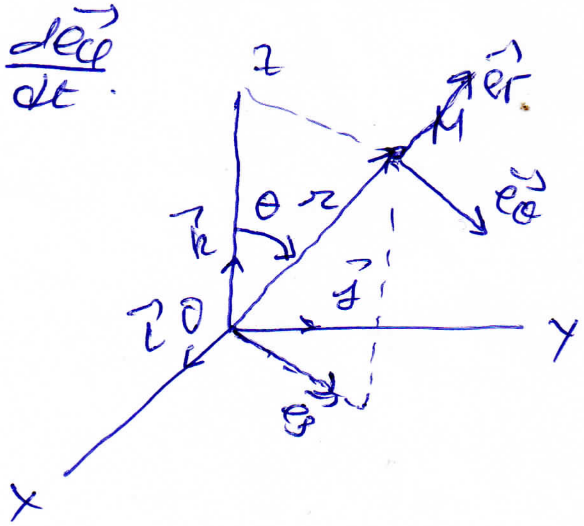
$$= \ddot{r}\,\vec{e_r} + \dot{r}\,\frac{d\vec{e_r}}{dt} + \dot{r}\dot{\theta}\,\vec{e_\theta} + r\,\frac{d\dot{\theta}}{dt}\,\vec{e_\theta} + r\dot{\theta}\,\frac{d\vec{e_\theta}}{dt} + \dot{r}\sin\theta\,\dot{\varphi}\,\vec{e_\varphi} + r\dot{\theta}\cos\theta\,\dot{\varphi}\,\vec{e_\varphi} + r\sin\theta\,\frac{d}{dt}(\dot{\varphi}\,\vec{e_\varphi})$$

$$= \ddot{r}\,\vec{e_r} + \dot{r}\,\frac{d\vec{e_r}}{dt} + \dot{r}\dot{\theta}\,\vec{e_\theta} + r\ddot{\theta}\,\vec{e_\theta} + r\dot{\theta}\,\frac{d\vec{e_\theta}}{dt} + \dot{r}\dot{\varphi}\sin\theta\,\vec{e_\varphi} + r\dot{\theta}\dot{\varphi}\cos\theta\,\vec{e_\varphi} + r\ddot{\varphi}\sin\theta\,\vec{e_\varphi} + r\dot{\varphi}\sin\theta\,\frac{d\vec{e_\varphi}}{dt}.$$

Calculons les dérivés des vecteurs unitaires :

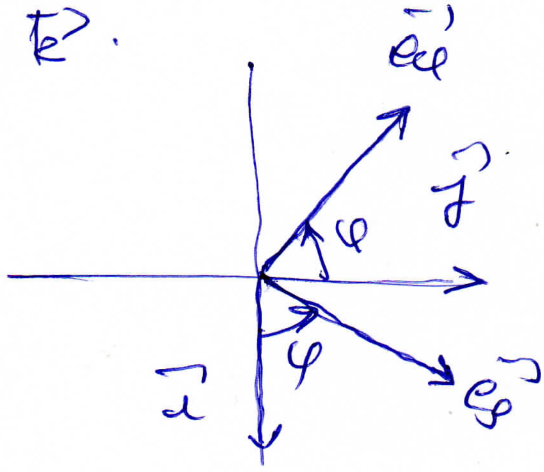
$$\frac{d\vec{e_r}}{dt},\quad \frac{d\vec{e_\theta}}{dt} \quad \text{et} \quad \frac{d\vec{e_\varphi}}{dt}.$$





On a :

$$\begin{cases} \vec{e_r} = \cancel{\cos}\,\sin\theta\,\vec{e_\rho} + \cos\theta\,\vec{k} \\ \vec{e_\theta} = \cos\theta\,\vec{e_\rho} \end{cases}$$
[redacted] $\sin\theta\,\vec{k}$.



D'autre part :

$$\vec{e_\rho} = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$$

$$\vec{e_\varphi} = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$$

$$\frac{d\vec{e_\rho}}{dt} = \frac{d\vec{e_\rho}}{d\varphi} \cdot \frac{d\varphi}{dt}$$

$$= \dot{\varphi}\left(-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}\right) = \dot{\varphi}\,\vec{e_\varphi}$$

$$\frac{d\vec{e_\varphi}}{dt} = \frac{d\vec{e_\varphi}}{d\varphi} \cdot \frac{d\varphi}{dt} = \dot{\varphi}\left(-\cos\varphi\,\vec{i} - \sin\varphi\,\vec{j}\right) = -\dot{\varphi}\,\vec{e_\rho}.$$

$$\frac{d\vec{e_r}}{dt} = \dot{\theta}\cos\theta\,\vec{e_\rho} + \sin\theta\,\frac{d\vec{e_\rho}}{dt} - \dot{\theta}\sin\theta\,\vec{k}$$

$$= \dot{\theta}\cos\theta\,\vec{e_\rho} + \sin\theta\,(\dot{\varphi}\,\vec{e_\varphi}) - \dot{\theta}\sin\theta\,\vec{k}.$$

$$= \dot{\theta}\left(\cos\theta\,\vec{e_\rho} - \sin\theta\,\vec{k}\right) + \dot{\varphi}\sin\theta\,e_\varphi$$

$$\boxed{\frac{d\vec{e_r}}{dt} = \dot{\theta}\,\vec{e_\theta} + \dot{\varphi}\sin\theta\,e_\varphi}$$

$$\frac{d\vec{e_\theta}}{dt} = -\dot{\theta}\sin\theta\,e_\rho + \cos\theta\,\frac{d\vec{e_\rho}}{dt} - \dot{\theta}\cos\theta\,\vec{k}$$

$$= -\dot{\theta}\sin\theta\,\vec{e_\rho} + \cos\theta\,.\,\dot{\varphi}\,\vec{e_\varphi} - \dot{\theta}\cos\theta\,\vec{k}$$

$$= \dot{\theta}\left(-\sin\theta\,\vec{e_\rho} - \cos\theta\,\vec{k}\right) + \dot{\varphi}\cos\theta\,\vec{e_\varphi}$$

$$\boxed{\frac{d\vec{e_\theta}}{dt} = -\dot{\theta}\,\vec{e_r} + \dot{\varphi}\cos\theta\,e_\varphi}$$

$$\boxed{\frac{d\vec{e_\varphi}}{dt} = -\dot{\varphi}\,\vec{e_\rho} = -\dot{\varphi}\left(\sin\theta\,e_r + \cos\theta\,e_\theta\right)}.$$

En remplaçant dans $\vec{a}(t)$, on obtient :

$$\vec{a} = \ddot{r}\,\vec{e_r} + \dot{r}\left[\underline{\dot{\theta}\,\vec{e_\theta} + \dot{\varphi}\sin\theta\,\vec{e_\varphi}}\right] + \dot{r}\dot{\theta}\,\vec{e_\theta} + r\ddot{\theta}\,\vec{e_\theta}$$

$$+ r\dot{\theta}\left[\underline{-\dot{\theta}\,\vec{e_r} + \dot{\varphi}\cos\theta\,\vec{e_\varphi}}\right] + \dot{r}\dot{\varphi}\sin\theta\,\vec{e_\varphi}$$

$$+ r\dot{\theta}\dot{\varphi}\cos\theta\,\vec{e_\varphi} + r\ddot{\varphi}\sin\theta\,\vec{e_\varphi} + r\dot{\varphi}\sin\theta\left[\underline{-\dot{\varphi}\left(\sin\theta\,\vec{e_r} + \cos\theta\,\vec{e_\theta}\right)}\right]$$

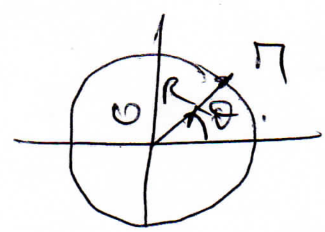
$$\boxed{\begin{aligned}\vec{a} = & \left[\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 \sin^2\theta\right]\vec{e_r} \\ & + \left[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta\right]\vec{e_\theta} \\ & + \left[2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\right]\vec{e_\varphi}\end{aligned}}$$

Exemple: Dans (XOY) un mobile M décrit une traj circulaire de centre O et de rayon R.

Soit $\theta = (Ox, \vec{OM}) = \theta(t)$.

① Les composantes cartésienne des vecteur $\vec{V}(t)$ et $\vec{a}(t)$

② $a_n(t)$ et $a_t(t)$.



On a:

$$\begin{cases} x(t) = R\cos\theta \\ y(t) = R\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}(t) = -R\dfrac{d\theta}{dt}\sin\theta = -R\dot{\theta}\sin\theta \\ \dot{y}(t) = R\dfrac{d\theta}{dt}\cos\theta = R\dot{\theta}\cos\theta \end{cases}$$

$$\begin{cases} \ddot{x}(t) = -R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta \\ \ddot{y}(t) = R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta \end{cases}$$

② $a_t = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\sqrt{R^2\dot{\theta}^2\sin^2\theta + R^2\dot{\theta}^2\cos^2\theta}\right) = R\ddot{\theta} = R\alpha$.

$a_n = \dfrac{v^2}{R} = \dfrac{R^2\dot{\theta}^2}{R} = R\dot{\theta}^2 = R\omega^2$.