Corrigé des exercices supplémentaires de la série 1 (2024-2025)

(Module physique 1).

Exercice S1 :

On a : $[\rho] = ML^{-3}$, $[R] = [l] = L$, $[m] = M$

$$\rho = \frac{m^x}{\pi\, l^y R^2} \Rightarrow [\rho] = \frac{[m]^x}{[\pi]\cdot[l]^y[R]^2} = \frac{M^x}{1 . L^y L^2}$$

$$\Rightarrow [\rho] = M^x L^{-(y+2)} \Rightarrow ML^{-3} = M^x L^{-(y+2)}$$

Par analogie on obtient $\begin{cases} 1 = x \\ -3 = -(y+2) \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$

Donc l'expression de la masse volumique est :

$$\rho = \frac{m}{\pi l R^2} = \frac{m}{V}.$$

Exercice S2 :

$A\begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix}$, $B(6,8,3)$.

① $\vec{AB} = \vec{OB} - \vec{OA} = (6-3)\vec{i} + (8-4)\vec{j} + (3-(-4))\vec{k} = 3\vec{i} + 4\vec{j} + 7\vec{k}.$

② $\|\vec{AB}\| = \sqrt{3^2+4^2+7^2} = \sqrt{74}$ u.

③ Direction de $\vec{AB}$:

Soit $C\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in$ à la droite $(AB) \Rightarrow \vec{AC} \wedge \vec{AB} = \vec{0}.$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x-3) & (y-4) & (z+4) \\ 3 & 4 & 7 \end{vmatrix} = \begin{pmatrix} 7(y-4) - 4(z+4) \\ 3(z+4) - 7(x-3) \\ 4(x-3) - 3(y-4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 7y+28-4z-16=0 \\ 3y+12-7x+21=0 \\ 4x-12-3y+12=0 \end{cases}$$

$$\Rightarrow z-4y+3x+11=0$$

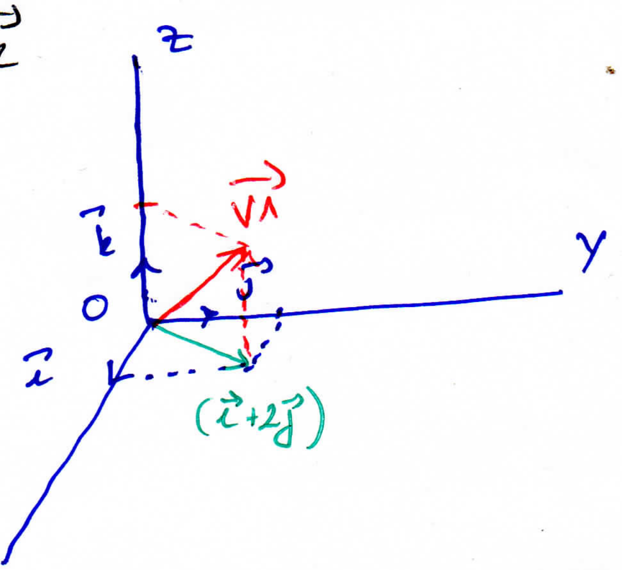
Donc le vecteur $\vec{AB}$ est porté par la droite (direction) d'équation dans l'espace : $z-4y+3x+11=0$

Sens : $\vec{AB}$ est dirigé de A vers B.

Exercice 3 : $\vec{V_1}\begin{pmatrix}1\\2\\2\end{pmatrix}$, $\vec{V_2}\begin{pmatrix}3\\3\sqrt{3}\\3\end{pmatrix}$, $\vec{V_3}\begin{pmatrix}0\\\sqrt{3}\\\sqrt{3}\end{pmatrix}$.

① Représentation de $\vec{V_1}$:

$\vec{V_1}=\vec{i}+2\vec{j}+2\vec{k}$



② Les modules :

$\|\vec{V_1}\|=\sqrt{5}\ u$

$\|\vec{V_2}\|=\sqrt{45}\ u$

$\|\vec{V_3}\|=\sqrt{6}$

③

$$\vec{u}=2\vec{V_1}-3\vec{V_2}+4\vec{V_3}=2(\vec{i}+2\vec{j}+2\vec{k})-3(3\vec{i}+3\sqrt{3}\vec{j}+3\vec{k})+4(\sqrt{3}\vec{j}+\sqrt{3}\vec{k})$$

$$\vec{u}=2\vec{i}+4\vec{j}+4\vec{k}-9\vec{i}-9\sqrt{3}\vec{j}-9\vec{k}+4\sqrt{3}\vec{j}+4\sqrt{3}\vec{k}$$

$$\vec{u}=-7\vec{i}+(4-5\sqrt{3})\vec{j}+(4\sqrt{3}-5)\vec{k}$$

④ Les vecteurs unitaires :

$$\vec{u_1}=\frac{\vec{V_1}}{\|\vec{V_1}\|}=\frac{\vec{i}+2\vec{j}+2\vec{k}}{\sqrt{5}}=\left(\frac{1}{\sqrt{5}}\right)\vec{i}+\left(\frac{2}{\sqrt{5}}\right)\vec{j}+\left(\frac{2}{\sqrt{5}}\right)\vec{k}$$

$$\vec{u_2} = \frac{\vec{V_2}}{\|\vec{V_2}\|} = \frac{3\vec{i} + 3\sqrt{3}\vec{j} + 3\vec{k}}{\sqrt{45}} = \left(\frac{1}{\sqrt{5}}\right)\vec{i} + \sqrt{\frac{3}{5}}\,\vec{j} + \frac{1}{\sqrt{5}}\vec{k}$$

$$\vec{u_3} = \frac{\vec{V_3}}{\|\vec{V_3}\|} = \frac{\sqrt{3}\vec{j} + \sqrt{3}\vec{k}}{\sqrt{6}} = \left(\frac{1}{\sqrt{2}}\right)\vec{j} + \left(\frac{1}{\sqrt{2}}\right)\vec{k}.$$

(5) * $\vec{V_1}\cdot\vec{V_2} = 1\times 3 + 2\times 3\sqrt{3} + 2\times 3 = 9 + 6\sqrt{3}$

* $\vec{V_2}\wedge\vec{V_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3\sqrt{3} & 3 \\ 0 & \sqrt{3} & \sqrt{3} \end{vmatrix} = \begin{pmatrix} 9-3\sqrt{3} \\ -3\sqrt{3} \\ 3\sqrt{3} \end{pmatrix}$

* $\vec{V_1}\cdot(\vec{V_2}\wedge\vec{V_3}) = (9-3\sqrt{3}) + (-6\sqrt{3}) + (6\sqrt{3}) = 3(3-2\sqrt{3})$

* $\vec{V_1}\wedge(\vec{V_2}\wedge\vec{V_3}) = 12\sqrt{3}\,\vec{i} + (18-9\sqrt{3})\vec{j} + (-18+3\sqrt{3})\vec{k}.$

(6) $\cos\theta = \cos(\vec{u_2}, \vec{u_3})$

on a: $\vec{u_1}\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$, $\vec{u_2}\begin{pmatrix} 1/\sqrt{5} \\ \sqrt{3/5} \\ 1/\sqrt{5} \end{pmatrix}$, $\vec{u_3}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

$$\vec{u_2}\cdot\vec{u_3} = \frac{1}{\sqrt{5}}\times 0 + \sqrt{\frac{3}{5}}\times\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}}\times\frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{\sqrt{10}}$$

$$\vec{u_2}\cdot\vec{u_3} = \|\vec{u_2}\|\cdot\|\vec{u_3}\|\cos(\vec{u_2}, \vec{u_3}) = \frac{1+\sqrt{3}}{\sqrt{10}}$$

$$\Rightarrow \cos(\vec{u_2}, \vec{u_3}) = \frac{1+\sqrt{3}}{\sqrt{10}}.$$

$$\vec{u_{23}} = \vec{u_2}\wedge\vec{u_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{5}} & \frac{\sqrt{3}}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} \sqrt{3}-1 \\ -1 \\ 1 \end{pmatrix}$$

$$\|\vec{u_{23}}\| = \frac{1}{\sqrt{10}}\sqrt{(\sqrt{3}-1)^2 + 1 + 1} = \frac{1}{\sqrt{10}}\sqrt{6-2\sqrt{3}}.$$

$$\|\vec{u}_{23}\| = \|\vec{u}_2\| \cdot \|\vec{u}_3\| \sin(\vec{u}_2,\vec{u}_3) = \sin(\vec{u}_2,\vec{u}_3)$$

$$\Rightarrow \boxed{\sin(\vec{u}_2,\vec{u}_3) = \frac{1}{\sqrt{10}}\sqrt{6-2\sqrt{3}}}$$

Exercice S4 :

$A\begin{pmatrix}2\\3\end{pmatrix}$, $B\begin{pmatrix}3\\0\end{pmatrix}$, $C\begin{pmatrix}-2\\x\end{pmatrix}$.

1. $\vec{AB} = \vec{OB} - \vec{OA} = \vec{i} - \vec{j} = (1,-1)$

   $\vec{AC} = \vec{OC} - \vec{OA} = -4\vec{i} + (x-3)\vec{j} = (-4, x-3)$.

2. Les 3 pts sont alignés $\Rightarrow \vec{AB} \parallel \vec{AC}$

$$\Rightarrow \vec{AB} \wedge \vec{AC} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ -4 & x-3 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ x-7 \end{pmatrix} = \vec{0} \Rightarrow \boxed{x = 7}$$

3. $\vec{V}_1 = 3\vec{i} + \alpha\vec{j} + \vec{k}$, $\vec{V}_2 = 4\vec{i} - 2\vec{j} - 2\vec{k}$

$$\vec{V}_1 \perp \vec{V}_2 \Rightarrow \vec{V}_1 \cdot \vec{V}_2 = 0 \Rightarrow 3\times 4 - 2\alpha - 2 = 0$$

$$\Rightarrow \boxed{\alpha = 5}$$

Exercice S5 :

1. $\vec{r}(t) = \left(\frac{1}{2}t^2 - 3t + 2\right)\vec{i} + \left(e^{-\alpha t}\right)\vec{j} + (\cos\omega t)\vec{k}$.

Première dérivée :

$$\frac{d\vec{r}(t)}{dt} = (t-3)\vec{i} - \alpha e^{-\alpha t}\vec{j} - \omega \sin\omega t\,\vec{k}$$

$$\left\|\frac{d\vec{r}(t)}{dt}\right\| = \sqrt{(t-3)^2 + \alpha^2 e^{-2\alpha t} + \omega^2 \sin^2\omega t}$$

Deuxième dérivée :

$$\frac{d^2\vec{r}(t)}{dt^2} = \vec{i} + \alpha^2 e^{-\alpha t}\vec{j} - \omega^2\cos\omega t\,\vec{k}$$

$$\left\|\frac{d^2\vec{r}(t)}{dt^2}\right\| = \sqrt{1 + \alpha^4 e^{-2\alpha t} + \omega^4\cos^2\omega t}.$$

② $\vec{u}(t) = \int \vec{v}(t)\,dt$

$\vec{v}(t) = (-2t+3)\vec{i} + 3t^2\vec{j} + (\sin\omega t + e^{\alpha t})\vec{k}$

et $\vec{u}(0) = \vec{i} + \vec{j} + \vec{k}$.

$$\vec{u}(t) = \int\left[(-2t+3)\vec{i} + (3t^2)\vec{j} + (\sin\omega t + e^{\alpha t})\vec{k}\right]dt$$

$$= (-t^2+3t)\vec{i} + t^3\vec{j} + \left(\frac{-1}{\omega}\cos\omega t + \frac{1}{\alpha}e^{\alpha t}\right)\vec{k} + \vec{C}.$$

À $t=0$, $\vec{u}(0) = \vec{i} + \vec{j} + \vec{k}$

$\Rightarrow \left(-\frac{1}{\omega} + \frac{1}{\alpha}\right)\vec{k} + \vec{C} = \vec{i} + \vec{j} + \vec{k}$

$\Rightarrow \boxed{\vec{C} = \vec{i} + \vec{j} + \left(1 + \frac{1}{\omega} - \frac{1}{\alpha}\right)\vec{k}}$

Donc $\vec{u}(t) = (-t^2+3t+1)\vec{i} + (t^3+1)\vec{j} + \left(\frac{-1}{\omega}\cos\omega t + \frac{1}{\alpha}e^{\alpha t} + 1 + \frac{1}{\omega} - \frac{1}{\alpha}\right)\vec{k}$

Exercice S 6

① $V(x,y,z) = xyz + xy^2z + xyz^2$

$$\overrightarrow{grad}\,V(x,y,z) = \left(\frac{\partial V}{\partial x}\right)\vec{i} + \left(\frac{\partial V}{\partial y}\right)\vec{j} + \left(\frac{\partial V}{\partial z}\right)\vec{k}$$

$$= (yz + y^2z + yz^2)\vec{i} + (xz + 2xyz + xz^2)\vec{j} + (xy + xy^2 + 2xyz)\vec{k}$$

⑤

(2) $\vec{E}(x,y,z) = (xyz)\,\vec{i} + (x+y+z)\,\vec{j} + \frac{xy}{z}\,\vec{k}$

$$\operatorname{div}\vec{E} = \vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\boxed{\operatorname{div}\vec{E} = yz + 1 - \frac{xy}{z^2}}$$

(3) $\vec{E}(x,y,z) = \frac{x}{y}\,\vec{i} + \frac{y}{z}\,\vec{j} + \frac{z}{x}\,\vec{k}$.

$$\vec{\operatorname{rot}}\,\vec{E} = \vec{\nabla}\wedge\vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{x}{y}\right) & \left(\frac{y}{z}\right) & \left(\frac{z}{x}\right) \end{vmatrix}.$$

$$= \begin{pmatrix} \frac{\partial}{\partial y}\left(\frac{z}{x}\right) - \frac{\partial}{\partial z}\left(\frac{y}{z}\right) \\ \frac{\partial}{\partial z}\left(\frac{x}{y}\right) - \frac{\partial}{\partial x}\left(\frac{z}{x}\right) \\ \frac{\partial}{\partial x}\left(\frac{y}{z}\right) - \frac{\partial}{\partial y}\left(\frac{x}{y}\right) \end{pmatrix} = \begin{pmatrix} y/z^2 \\ z/x^2 \\ x/y^2 \end{pmatrix}$$

$$\boxed{\vec{\operatorname{rot}}\,\vec{E} = \left(\frac{y}{z^2}\right)\vec{i} + \left(\frac{z}{x^2}\right)\vec{j} + \left(\frac{x}{y^2}\right)\vec{k}}$$

Fin

(6)