Exo 1 : (Intégrales simples)

1 -

* $\displaystyle\int_{-1}^{1} \frac{2}{x^2+2\sqrt{3}x+3}\,dx = 2\int_{-1}^{1} \frac{1}{(x+\sqrt{3})^2}\,dx$

$= 2\left[\frac{-1}{x+\sqrt{3}}\right]_{-1}^{+1} = 2$

* $\displaystyle\int_{2}^{e} \frac{2}{x^2-1}\,dx = \int_{2}^{e} \frac{2}{x^2-1}\,dx$

$= \displaystyle\int_{2}^{e}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$

$= \ln(x-1) - \ln(x+1)\Big]_{2}^{e}$

$= \ln\left(\frac{e-1}{e+1}\right)$

* $\displaystyle\int_{1}^{2} \frac{1}{x^3+1}\,dx = \int_{1}^{2} \frac{1}{(x+1)(x^2-x+1)}\,dx$

$$\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{Ax^2-Ax+A+Bx^2+Bx+Cx+C}{(x+1)(x^2-x+1)}$$

$$= \frac{(A+B)x^2+(-A+B+C)x+A+C}{(x+1)(x^2-x+1)}$$

Par identif. $\begin{cases} A+B=0 & \text{— } 1 \\ -A+B+C=0 & \text{— } 2 \\ A+C=1 & \text{— } 3 \end{cases}$

$\begin{cases} A=\frac{1}{3} \\ B=-\frac{1}{3} \\ C=\frac{2}{3} \end{cases}$

$$\int_1^2 \frac{1}{x^3+1}\,dx = \frac{1}{3}\int_1^2 \frac{1}{x+1}\,dx - \frac{1}{3}\int_1^2 \frac{x-2}{x^2-x+1}\,dx$$

$$= \frac{1}{3} I_1 - \frac{1}{3} I_2$$

$$I_1 = \int_1^2 \frac{1}{x+1}\,dx = \left[\ln(x+1)\right]_1^2 = \ln\frac{3}{2}$$

$$I_2 = \int_1^2 \frac{x-2}{x^2-x+1}\,dx = \int_1^2 \frac{x-2}{x^2-x+1}\,dx$$

$$= \int_1^2 \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}\,dx.$$

on pose $x = t + \frac{1}{2} \Rightarrow t = x - \frac{1}{2}$

$$I_2 = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{t-\frac{3}{2}}{t^2+\frac{3}{4}}\,dt$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{2t}{t^2+\frac{3}{4}}\,dt - \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\frac{3}{2}}{t^2+\frac{3}{4}}\,dt$$

$$= \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) - \frac{3}{2}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$= \left[\frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) - \frac{3}{2}\times\frac{2}{\sqrt{3}}\arctan\left(\frac{2t}{\sqrt{3}}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \left[\frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) - \sqrt{3}\arctan\left(\frac{2t}{\sqrt{3}}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{2}\ln 3 + \sqrt{3}\left(\arctan(\sqrt{3}) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$\int_1^2 \frac{1}{x^3+1}\,dx = \frac{1}{3} I_1 - \frac{1}{3} I_2$$

2)
$$I_1 = \int_0^{\frac{\pi}{2}} (2x+3)\sin(2x)\,dx$$

$u(x) = 2x+3 \Rightarrow u'(x) = 2$

$v'(x) = \sin(2x) \Rightarrow v(x) = -\frac{1}{2}\cos(2x)$

$$I_1 = -\frac{1}{2}(2x+3)\cos(2x)\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(2x)\,dx.$$

$$= -\frac{1}{2}(2x+3)\cos(2x) + \frac{1}{2}\sin(2x)\Big]_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{2}(\pi+3)(-1)\right) - \left(-\frac{1}{2}(3)(1)\right)$$

$$= \frac{1}{2}(\pi+6)$$

$$= \frac{\pi}{2} + 3$$

$$I_2 = \int_2^e \ln(x^2-1)\,dx$$

$u(x) = \ln(x^2-1) \Rightarrow u'(x) = \frac{2x}{x^2-1}$

$v'(x) = 1 \Rightarrow v(x) = x$

$$I_2 = x\ln(x^2-1)\Big]_2^e - \int_2^e \frac{2x^2}{x^2-1}\,dx.$$

$$= x\ln(x^2-1)\Big]_2^e - \int_2^e \left(2 + \frac{2}{x^2-1}\right)dx$$

$$= x\ln(x^2-1) - 2x\Big]_2^e - \int_2^e \frac{2}{x^2-1}\,dx$$

$$= e\ln(e^2-1) - 2e - 2\ln(3) + 4 - \int_2^e \frac{2}{x^2-1}\,dx$$

3 Voir la question → Q1

$$I_3 = \int_1^e \ln^2(x)\,dx$$

$u(x) = \ln^2(x) \Rightarrow u'(x) = 2\frac{\ln x}{x}$

$v'(x) = 1 \Rightarrow v(x) = x$.

$$I_3 = x\ln^2(x) - \int_1^e 2\ln x\,dx.$$

$$= x\ln^2(x) - 2(x\ln x - x)\Big]_1^e$$

$$= \Big[(x\ln x)(\ln x - 2) + 2x\Big]_1^e$$

$$= e - 2$$

3) $I_1 = \int_e^{e^2} \frac{2}{x\sqrt{\ln x}}\,dx$ / $t = \ln x \Rightarrow x = e^t$

$dx = e^t\,dt$

$dx = x\,dt$

$$I_1 = \int_1^2 \frac{2}{\cancel{x}\sqrt{t}}\,\cancel{x}\,dt$$

$$= 2\int_1^2 \frac{1}{\sqrt{t}}\,dt = 2\Big[2\sqrt{t}\Big]_1^2$$

$$= 4(\sqrt{2} - 1)$$

$$I_2 = \int_0^{\ln(2)} \frac{e^{3x}}{e^{2x}+1}\, dx$$

on pose $t = e^x$

$$\frac{dt}{dx} = e^x = t$$

$$dx = \frac{1}{t}\, dt$$

$$I_2 = \int_1^2 \frac{t^3}{t^2+1} \times \frac{1}{t}\, dt$$

$$= \int_1^2 \frac{t^2}{t^2+1}\, dt$$

$$= \int_1^2 \left(1 - \frac{1}{t^2+1}\right) dt$$

$$= \Big[t - \arctan(t) \Big]_1^2$$

$$= 2 - \arctan(2) - 1 + \arctan(1)$$

$$= 1 - \arctan(2) + \frac{\pi}{4}.$$