Ex n° 1 . Corrigé de série de TD N° 3 . Stat II

$\boxed{1}$ La loi de probabilité

$X = \{0, 1, 2\}$.

$P(X=0) = \frac{C_3^2}{C_5^2} = \frac{3}{10}$ ; $P(X=1) = \frac{C_2^1 C_3^1}{C_5^2} = \frac{6}{10}$ , $P(X=2) = \frac{C_2^2}{C_5^2} = \frac{1}{10}$

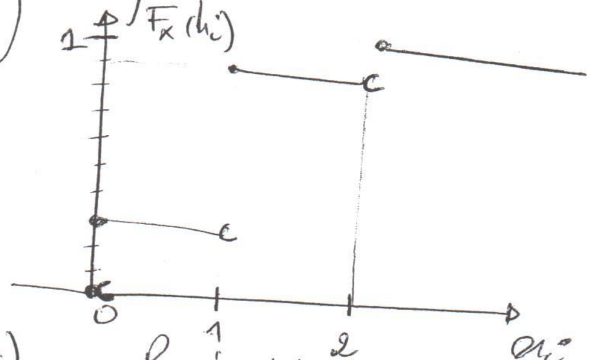
| $x_i$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X=x_i)$ | 3/10 | 6/10 | 1/10 |

$\boxed{2}$ • $F_X(x_i) = P(X \leq x_i)$.

$F_X(0) = P(X \leq 0) = P(X=0) = \frac{3}{10}$ ; $F_X(1) = F_X(0) + P(X=1) = \frac{3}{10} + \frac{6}{10} = \frac{9}{10}$ ;

$F_X(2) = F_X(1) + P(X=2) = \frac{9}{10} + \frac{1}{10} = 1$.

• Représentation graphique :




Représentation graphique de $F_X$

$\boxed{3}$ • $E(X) = \sum_{i=1}^{3} x_i \, P(X=x_i)$

$= 0 \cdot \frac{3}{10} + 1 \cdot \frac{6}{10} + 2 \times \frac{1}{10} = \frac{6}{10} + \frac{2}{10} = \frac{8}{10} = \frac{4}{5}$

$E(X) = \frac{4}{5}$

• $V(X) = E(X^2) - E^2(X)$ ; $E(X^2) = \sum_{i=1}^{3} x_i^2 \, P(X=x_i)$

$= 0^2 \cdot \frac{3}{10} + 1^2 \cdot \frac{6}{10} + 2^2 \times \frac{1}{10} = \frac{6}{10} + \frac{4}{10}$

$= 1$.

$V(X) = 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}$

$V(X) = \frac{9}{25}$

$\boxed{4}$ $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X \leq 0) = 1 - F_X(0) = 1 - 3/10 = 7/10$

Ex2 :

B : Il y'a un banc de poissons sur sa zone.

S : le sonar détecte la présence de poissons.

[1]- $p(\overline{B}) = 1 - p(B) = 0{,}3$, $p(\overline{S}) = 1 - p(S) = 0{,}425$ ;

$p(B \cap \overline{S}) = p(B) - p(B \cap S) = 0{,}14$ ;

$p(\overline{B} \cap S) = p(S) - p(B \cap S) = 0{,}015$ ;

[2]- $X$ la v.a donnant le gain algébrique pour une sortie en mer.

a- La loi de probabilité de $X$.

$X = \{-400, -150, 200\}$.

- $p(X = -400) = p(\overline{B} \cap S) = 0{,}015$ ; (2^ème cas)
- $p(X = -150) = p(\overline{S}) = 0{,}425$ ; (3^ème cas).
- $p(X = 2000) = p(B \cap S) = 0{,}56$ ; (1^er cas)

| $x_i$ | $-400$ | $-150$ | $2000$ |
|---|---|---|---|
| $p(X = x_i)$ | $0{,}015$ | $0{,}425$ | $0{,}56$ |
| $x_i\, p(X = x_i)$ | $-6$ | $-63{,}75$ | $1120$ |

[3]

$$E(X) = \sum_{i=1}^{3} x_i\, p(X = x_i)$$

$$= -6 + 63{,}75 + 1120 = 1050{,}25 \text{ euros.}$$

Ex 03

1) pour quelle valeur de "a", f est densité de prob

f est densité de probabilité ssi :

~~f est positive~~ $\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{+\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{1} f(x)\,dx}_{0} + \int_{1}^{7} f(x)\,dx + \underbrace{\int_{7}^{+\infty} f(x)\,dx}_{0}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 0 + \int_{1}^{7} (-ax^2 + 8ax - 7a)\,dx + 0 = a\int (-x^2 + 8x - 7)\,dx = 1$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = a\left[-\frac{x^3}{3} + 4x^2 - 7x\right]_1^7 = 1$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = a\left[\left(-\frac{7^3}{3} + 4\times7^2 - 7\times7\right) - \left(\frac{1^3}{3} + 4 - 7\right)\right] = 1$$

Donc on aura $36a = 1 \Rightarrow \boxed{a = \frac{1}{36}}$ $f(x) = \begin{cases} +\frac{1}{36}(-x^2 + 8x - 7 \\ 0 \text{ sinon} \end{cases}$

2) La fonction de répartition de X :

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$

si $x < 1$ alors $\int_{-\infty}^{x} f_X(t)\,dt = 0$.

si $1 \le x \le 7$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{1} f_X(t)\,dt + \int_{1}^{x} f(t)\,dt = 0 + \int_{1}^{x} \frac{1}{36}(-x^2 + 8x - 7)\,dt$$

$$F_X(x) = 0 + \frac{1}{36}\int_{1}^{x} (-t^2 + 8t - 7)\,dt = \frac{1}{36}\left[-\frac{t^3}{3} + 4t^2 - 7t\right]_1^x$$

$$F_X(x) = \frac{1}{36}\left[\left(-\frac{x^3}{3} + 4x^2 - 7x\right) - \left(-\frac{1^3}{3} + 4 - 7\right)\right] = -\frac{x^3}{108} + \frac{x^2}{9} - \frac{7x}{36} + \frac{5}{54}$$

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Si $x>7$, $F_X(x)=\int_{-\infty}^{x} f_X(t)\,dt=\int_{-\infty}^{1} f_X(t)\,dt+\int_{1}^{7} f_X(t)\,dt+\int_{7}^{x} f(t)\,dt$

$F_X(x)=0+\int_1^7 \frac{1}{36}(-t^2+8t-7)\,dt+0=\frac{1}{36}\left[-\frac{t^3}{3}+4t^2-7t\right]_1^7=1$

Donc :

$$F_X(x)=\begin{cases}0 & \text{si } x<1\\ \frac{-x^3}{108}+\frac{x^2}{9}-\frac{7x}{36}+\frac{5}{54} & \text{si } 1\leq x\leq 7\\ 1 & \text{si } x>7.\end{cases}$$

calcul de $P(X<2)$.

$P(X<2)=F_X(2)=-\frac{2^3}{108}+\frac{2^2}{9}-\frac{14}{36}+\frac{5}{54}=\frac{2}{27}$.

calcul de $P(X>3)$

$P(X>3)=1-P(X\leq 3)=1-F_X(3)$

$P(X>3)=1-\left(-\frac{3^3}{108}+\frac{3^2}{9}-\frac{21}{36}+\frac{5}{54}\right)=1-\frac{7}{27}=\frac{20}{27}$

Calcul de $P(2<X<3)=F_X(3)-F_X(2)=\frac{7}{27}-\frac{2}{27}=\frac{5}{27}$.

4. On considère la v.a. $Y=X-4$.

a. Détermination de la fonction de densité de Pr. de $Y$.

On a $F_Y(x)=\begin{cases}0 & \text{si } x<-3\\ \frac{1}{2}+\frac{x}{4}-\frac{x^3}{108} & \text{si } -3\leq x\leq 3\\ 1 & \text{si } x>3.\end{cases}$

$f_Y(x)=F_Y'(x)=\begin{cases}-\frac{x^2}{36}+\frac{1}{4} & \text{si } -3\leq x\leq 3.\\ 0 & \text{sinon.}\end{cases}$

b- Calcul de la médiane.

La mediane de Y verifie $F_Y(x) = \frac{1}{2}$.

Donc $-\frac{x^3}{108} + \frac{x}{4} + \frac{1}{2} = \frac{1}{2} \Rightarrow -\frac{x^3}{108} + \frac{x}{4} = 0 \Rightarrow \frac{-x^3 + 27x}{108} = 0$

Donc: $-x^3 + 27x = 0$ i.e $x(-x^2+27) = 0$.

Donc $\begin{cases} x = 0 \\ \text{ou} \\ -x^2 + 27 = 0 \end{cases}$ $\begin{cases} x = 0 \\ \text{ou} \\ x^2 = 27 \end{cases}$ $\begin{cases} x = 0 \\ \text{ou} \\ x = \pm\sqrt{27} \cong \pm 5{,}2 \end{cases}$

Or $-3 \leq x \leq +3$ donc $x = 0$ Alors $\boxed{M_e = 0.}$

c) Calcul de l'esperance de Y:

$$E(Y) = \int_{-\infty}^{+\infty} x F_Y(x)\,dx = \int_{-\infty}^{-3} x f_Y(x)\,dx + \int_{-3}^{+3} x f_Y(x)\,dx + \int_{+3}^{+\infty} x f_Y(x)\,dx.$$

$$E(Y) = 0 + \int_{-3}^{3} x\left(\frac{1}{4} - \frac{x^2}{36}\right)dx + 0 = \int_{-3}^{+3}\left(\frac{x}{4} - \frac{x^3}{36}\right)dx.$$

$$E(Y) = \left[\frac{x^2}{8} - \frac{x^4}{144}\right]_{-3}^{3} = \left[\left(\frac{9}{8} - \frac{81}{144}\right) - \left(\frac{9}{8} - \frac{81}{144}\right)\right] = 0.$$

Calcul de la variance de Y: $V(Y) = (E(Y^2) - E(Y)^2]$.

$$E(Y^2) = \int_{-\infty}^{+\infty} x^2 f_Y(x)\,dx = \underbrace{\int_{-\infty}^{-3} x^2 f_Y(x)\,dx}_{=0} + \int_{-3}^{+3} x^2 f_Y(x)\,dx + \underbrace{\int_{+3}^{+\infty} x^2 f_Y(x)\,dx}_{=0}.$$

$$E(Y^2) = 0 + \int_{-3}^{+3} x^2\left(\frac{1}{4} - \frac{x^2}{36}\right)dx + 0 = \int_{-3}^{+3}\left(\frac{x^2}{4} - \frac{x^4}{36}\right)dx.$$

$$E(Y^2) = \left[\frac{x^3}{12} - \frac{x^5}{180}\right]_{-3}^{3} = \left(\left(\frac{27}{12} - \frac{243}{180}\right) - \left(-\frac{27}{12} + \frac{243}{180}\right)\right)$$

$$V(Y) = E(Y^2) - E(Y)^2 = \frac{9}{5} - 0 = \boxed{\frac{9}{5}}$$

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EN DÉduire la médiane E(x) et V(x).

on a $Y = X - 4$ donc $X = Y + 4$.

$M_e(x) = M_e(Y) + 4 = 0 + 4 = 4$

$E(X) = E(Y) + 4 = 0 + 4 = 4$.

$V(X) = V(Y) = \frac{9}{5}$.