

Course of Algebra 1

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Chapter 3

Binary relations

In mathematics, we often seek to compare two elements of a set or the property that two elements of a set are likely to have.

Definition 3.0.1 Let E and F be two sets, *a relation from E in F is an assertion associating each element of E to an element of F , which are true or false. We denote the relation by $\mathfrak{R}$ a relation denotes some kind of relationship between two objects in a set, which may or may not hold*
Formally, a relation $\mathfrak{R}$ over a set X can be seen as a set of ordered pairs (x, y) of members of X . The relation $\mathfrak{R}$ holds between x , and y if (x, y) is a member of $\mathfrak{R}$.

or else

Definition 3.0.2 Given two sets E and F , *a relation from E to F is a correspondence $\mathfrak{R}$ which links elements of E with elements of F .*

or

Definition 3.0.3 Let E and F be two sets, *a relation $\mathfrak{R}$ from E to F is a subset of $E \times F$, we denote $x\mathfrak{R}y$ if $(x, y) \in \mathfrak{R}$.*

Remark 3.0.1

- The set E is called the *domain* or the set of *departure* of $\mathfrak{R}$.
- The set F is called the *codomain* or the set of *destination* of $\mathfrak{R}$.

- For any element x of E and any element y of F satisfying $\mathfrak{R}$, we say that $x \in E$ is related by R with y , and we write $x\mathfrak{R}y$ otherwise $x\not\mathfrak{R}y$.
- If $E = F$, the relation $\mathfrak{R}$ is called **binary relation** on E .

In the following chapter, we study only the **binary relations on a set**.

Example 3.0.1

1. In $\mathbb{Z}$, we define the relation $\mathfrak{R}_1$ as follows:

$$\forall x, y \in \mathbb{Z}, x \mathfrak{R}_1 y \iff y \text{ multiple of } x.$$

For example, $1 \mathfrak{R}_1 x, \forall x \in \mathbb{Z}$,
 $x \mathfrak{R}_1 0, \forall x \in \mathbb{Z}$,
 $6 \mathfrak{R}_1 12, \quad 4 \not\mathfrak{R}_1 10, \quad \dots, \text{ etc.}$

2. In $\mathbb{Z}$, we define the relation $\mathfrak{R}_2$ as follows:

$$\forall x, y \in \mathbb{Z}, x \mathfrak{R}_2 y \iff x \equiv y[2].$$

For example : $1 \mathfrak{R}_2 1, 1 \mathfrak{R}_2 (-1), 2 \not\mathfrak{R}_2 3 \dots \text{ etc.}$

3. In $\mathbb{R}$, we define the relation $\mathfrak{R}_3$ as follows:

$$\forall x, y \in \mathbb{R}, x \mathfrak{R}_3 y \iff x^2 = y^2.$$

For example : $1 \mathfrak{R}_3 1, 1 \mathfrak{R}_3 (-1), 1 \not\mathfrak{R}_3 3, 2 \mathfrak{R}_3 (-2) \dots \text{ etc.}$

4. In $\mathbb{R}$, we define the relation $\mathfrak{R}_4$ as follows:

$$\forall x, y \in \mathbb{R}, x \mathfrak{R}_4 y \iff x = y.$$

For example : $1 \mathfrak{R}_4 1, 1 \not\mathfrak{R}_4 (-1), \dots \text{ etc.}$

5. In $\mathbb{R}$, we define the relation $\mathfrak{R}_5$ as follows

$$\boxed{\forall x, y \in \mathbb{R}, x \mathfrak{R}_5 y \iff x \leq y, .}$$

For example : $1 \mathfrak{R}_5 1, 1 \cancel{\mathfrak{R}_5} (-1), \dots$ etc.

6. In $\mathbb{R}$, we define the relation $\mathfrak{R}_6$ as follows:

$$\boxed{\forall x, y \in \mathbb{R}, x \mathfrak{R}_6 y \iff x < y.}$$

For example : $5 \cancel{\mathfrak{R}_6} 5, 4 \mathfrak{R}_6 5, \dots$ etc

7. Let $E = \{a, b, c\}$, in $P(E)$, we define the relation $\mathfrak{R}_7$ as follows:

$$\boxed{\forall A, B \in P(E), A \mathfrak{R}_7 B \iff A \cap B \neq \emptyset.}$$

For example : $A \mathfrak{R}_7 B, \emptyset \cancel{\mathfrak{R}_7} A, A \cancel{\mathfrak{R}_7} H, \dots$ etc, with $A = \{a, b\}, B = \{a\}, H = \{c\}$.

8. Let E be any set, in $P(E)$, we define the relation $\mathfrak{R}_8$ as follows:

$$\boxed{\forall A, B \in P(E), A \mathfrak{R}_8 B \iff A \subset B.}$$

For example : $A \mathfrak{R}_8 E, \emptyset \mathfrak{R}_8 A, \forall A \in P(E)$.

9. In a plane (P) , we define the relation $\mathfrak{R}_9$ on all the lines of the plane (P) as follows:

$$\boxed{\forall (\Delta), (\Delta') \in P, (\Delta) \mathfrak{R}_9 (\Delta') \iff (\Delta) \parallel (\Delta').}$$

For example : Let three lines: (Δ_1) equation: $y = x$, (Δ_2) of equation: $y = x + 1$, (Δ_3) of equation: $y = 2x$, we then have:
 $(\Delta_1) \mathfrak{R}_9 (\Delta_2), (\Delta_1) \cancel{\mathfrak{R}_9} (\Delta_3).$

$\cancel{\mathfrak{R}}$

10. In a plane (P) , we define the relation $\mathfrak{R}_{10}$ on all the lines of the plane (P) as follows:

$$\boxed{\forall (\Delta), (\Delta') \in P, (\Delta) \mathfrak{R}_{10} (\Delta') \iff (\Delta) \perp (\Delta').}$$

For example : Consider three lines: (Δ_1) 's equation is: $y = x$, (Δ_2) 's equation is : $y = -x$, (Δ_3) 's equation is : $y = 3x$, we then have: $(\Delta_1) \mathfrak{R}_{10} (\Delta_2), (\Delta_1) \cancel{\mathfrak{R}_{10}} (\Delta_3).$

Definition 3.0.4 (Graph of a relation) Let $\mathfrak{R}$ define a binary relation on a set E , we call *the graph* of the relation $\mathfrak{R}$, denoted $G_{\mathfrak{R}}$, the subset of $E \times E$ defined by

$$G_{\mathfrak{R}} = \{(x, y) \in E \times E, \text{ such that } x \mathfrak{R} y\}.$$

Example 3.0.2 Let $E = \{-2, -1, 0, 1, 2, 3\}$ and we define the relation $\mathfrak{R}$, on E by:

$$\forall x, y \in E, x \mathfrak{R} y \iff x^2 = y^2.$$

therefore the graph of the relation $\mathfrak{R}$ is defined as follows:

$$G_{\mathfrak{R}} = \{(1, 1), (-1, 1), (-1, -1), (1, -1), (2, 2), (-2, 2), (2, -2), (-2, -2), (0, 0), (3, 3)\}.$$

3.1 Properties of binary relations on a set

Let E be a set, $\mathfrak{R}$ a relation defined on E .

Definition 3.1.1 (Reflexivity) The relation $\mathfrak{R}$ is said to be *reflexive* if

$$\forall x \in E, x \mathfrak{R} x.$$

Example 3.1.1 The relations defined previously $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_8, \mathfrak{R}_9$ are all reflexive, but the relations $\mathfrak{R}_6, \mathfrak{R}_7, \mathfrak{R}_{10}$ are not.

Remark 3.1.1 To show that a relation $\mathfrak{R}$ is not reflexive, it is enough to find an element $x_0 \in E$ such that $x_0 \not\mathfrak{R} x_0$.

In the previous example the relation $\mathfrak{R}_7$ is not reflexive because $\emptyset \not\mathfrak{R}_7 \emptyset$, ($\emptyset \cap \emptyset = \emptyset$).

Definition 3.1.2 (Symmetry) The relation $\mathfrak{R}$ is called *symmetric* if

$$\forall x, y \in E, x \mathfrak{R} y \implies y \mathfrak{R} x.$$

Example 3.1.2 the relations defined previously $\mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_7, \mathfrak{R}_9, \mathfrak{R}_{10}$ are all symmetric, on the other hand the relations $\mathfrak{R}_1, \mathfrak{R}_5, \mathfrak{R}_6, \mathfrak{R}_8$ are not.

Remark 3.1.2 To show that a relation $\mathfrak{R}$ is not symmetric, it is enough to find two elements $x_0, y_0 \in E$ such that $x_0 \mathfrak{R} y_0$ and $y_0 \not\mathfrak{R} x_0$.

In the previous example, the relation $\mathfrak{R}_8$ is not symmetric because $\emptyset \mathfrak{R}_8 E$, but $E \not\mathfrak{R}_8 \emptyset$.

Definition 3.1.3 (Antisymmetry) The relation $\mathfrak{R}$ is said to be *antisymmetric* if

$$\forall x, y \in E, x \mathfrak{R} y, \text{ and } y \mathfrak{R} x \implies x = y.$$

Example 3.1.3 The relations defined previously $\mathfrak{R}_2, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_6, \mathfrak{R}_8$ are all antisymmetric, on the other hand the relations $\mathfrak{R}_1, \mathfrak{R}_3, \mathfrak{R}_7, \mathfrak{R}_9, \mathfrak{R}_{10}$ are not.

Remark 3.1.3 To show that a relation $\mathfrak{R}$ is not antisymmetric, it is enough to find two elements $x_0, y_0 \in E$ such that $x_0 \mathfrak{R} y_0$ and $y_0 \mathfrak{R} x_0$, but $x_0 \neq y_0$. In the previous examples, the relation $\mathfrak{R}_3$ is not antisymmetric because $1 \mathfrak{R}_3 (-1)$, and $(-1) \mathfrak{R}_3 1$, but $1 \neq (-1)$.

Definition 3.1.4 (Transitivity) The relation $\mathfrak{R}$ is called *transitive* if

$$\forall x, y, z \in E, x \mathfrak{R} y \text{ and } y \mathfrak{R} z \implies x \mathfrak{R} z.$$

Example 3.1.4 The relations defined previously $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_6, \mathfrak{R}_8, \mathfrak{R}_9$ are all transitive, on the other hand the relations $\mathfrak{R}_7, \mathfrak{R}_{10}$ are not.

Remark 3.1.4 To show that a relation $\mathfrak{R}$ is not transitive, it is enough to find three elements $x_0, y_0, z_0 \in E$ such that $x_0 \mathfrak{R} y_0$ and $y_0 \mathfrak{R} z_0$, but $x_0 \not\mathfrak{R} z_0$. In the previous example, the relation, $\mathfrak{R}_{10}$ is not transitive because if the line (Δ_1) is perpendicular to the line (Δ_2) and the line (Δ_2) is perpendicular to the line (Δ_3) then the line (Δ_1) is not perpendicular to the line (Δ_3) , (but $(\Delta_1) \parallel (\Delta_3)$).

3.2 Equivalence relations

3.2.1 Definitions and examples

Definition 3.2.1 Let $\mathfrak{R}$ be a binary relation on a set E . We say that $\mathfrak{R}$ is *an equivalence relation* if $\mathfrak{R}$ is *reflexive*, *symmetric* and *transitive*.

Example 3.2.1 We define in $\mathbb{R}^*$ the binary relation $\mathfrak{R}$ by:

$$\forall x, y \in \mathbb{R}^*, x \mathfrak{R} y \iff xy > 0.$$

Let us show that $\mathfrak{R}$ is an equivalence relation.

- Let us show that $\mathfrak{R}$ is reflexive. We have

$$R \text{ is reflexive} \iff [\forall x \in \mathbb{R}^*, x \mathfrak{R} x.]$$

It is easy to see that $\forall x \in \mathbb{R}^*, x^2 > 0$, which is equivalent to saying that $x \mathfrak{R} x$. hence $\mathfrak{R}$ is reflexive.

- Let's show that $\mathfrak{R}$ is symmetric. We have by definition

$$R \text{ is symmetric} \iff [\forall x, y \in \mathbb{R}^*, x \mathfrak{R} y \implies y \mathfrak{R} x]$$

we have

$$\begin{aligned} \forall x, y \in \mathbb{R}^* \quad x \mathfrak{R} y &\implies xy > 0 \\ &\implies yx > 0 \\ &\implies y \mathfrak{R} x. \end{aligned}$$

So $\mathfrak{R}$ is symmetric.

- Let us show that $\mathfrak{R}$ is transitive. We have by definition

$$R \text{ is transitive} \iff [\forall x, y, z, \in \mathbb{R}^*, x \mathfrak{R} y \text{ and } y \mathfrak{R} z \implies x \mathfrak{R} z]$$

we have

$$\begin{aligned} \forall x, y, z \in \mathbb{R}^*, \quad x \mathfrak{R} y \text{ and } y \mathfrak{R} z &\implies xy > 0, \text{ and } yz > 0 \\ &\implies xz > 0 \\ (x \text{ has the same sign as } y \text{ and } y, &\text{ has the same sign as } z \text{ then necessarily } z \\ &\text{ has the same sign that } x) \\ &\implies x \mathfrak{R} z. \end{aligned}$$

hence $\mathfrak{R}$ is transitive.

We deduce that $\mathfrak{R}$ is an equivalence relation on

Example 3.2.2 The relations $\mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5$ and $\mathfrak{R}_9$ are all equivalence relations, on the other hand the relations

$\mathfrak{R}_1$ (is not symmetric) „ $\mathfrak{R}_6$ (is not reflexive) , $\mathfrak{R}_7$ (is not is not reflexive) , $\mathfrak{R}_8$ (is not reflexive) , $\mathfrak{R}_{10}$ (is neither reflexive nor transitive), are not.

Exercise 3.2.1   Is the following relation, an equivalence relation on $\mathbb{R}$?

$$\forall x, y \in \mathbb{R}, x \mathcal{R} y \iff xy \leq 0.$$

We can therefore group these elements into "bundles" of elements that are similar, thus defining the notion of equivalence class, to finally construct new sets by "assimilating" similar elements to one and the same element. We then arrive at the notion of **set quotient**.

3.2.2 Quotient set

Definition 3.2.2 Let $\mathcal{R}$ be an equivalence relation in a set E . For each x of E , the set of all elements of E which are related, by $\mathcal{R}$, with x is called **equivalence class of x** denoted by $\dot{x}$ or $\bar{x}$ or $cl(x)$ or C_x .

So, the equivalence class $\dot{x}$ is the subset of E defined by

$$\dot{x} = \bar{x} = \{y \in E, \text{ such that } y \mathcal{R} x\}$$

If $y \in \dot{x}$, y is said to be a representative of the class $\dot{x}$.

The set of equivalence classes is called **quotient set** of E by the relation $\mathcal{R}$ denoted by $E/\mathcal{R}$.

$$E/\mathcal{R} = \{\dot{x} \mid x \in E\}$$

Example 3.2.3 In $\mathbb{Z}$, we define the relation $\mathcal{R}$ by:

$$\forall x, y \in \mathbb{Z}, x \mathcal{R} y \iff x - y = 5k, k \in \mathbb{Z}.$$

$\mathcal{R}$ is an equivalence relation. the class of 0.

$$\begin{aligned} \dot{0} &= \{x \in \mathbb{Z}, \text{ such that } x \mathcal{R} 0\} \\ &= \{x \in \mathbb{Z}, \text{ such that } x - 0 = 5k, k \in \mathbb{Z}\} \\ &= \{5k, k \in \mathbb{Z}\}. \end{aligned}$$

In the same way we determine the other equivalence classes, there are exactly five equivalence classes.

$$\begin{aligned}\dot{0} &= \{5k, k \in \mathbb{Z}\}, \dot{1} = \{5k+1, k \in \mathbb{Z}\}, \dot{2} = \{5k+2, k \in \mathbb{Z}\}, \\ \dot{3} &= \{5k+3, k \in \mathbb{Z}\}, \dot{4} = \{5k+4, k \in \mathbb{Z}\}.\end{aligned}$$

For this relation, we note $x \equiv y[5]$, we read it **x congruo to y modulo 5**. The quotient set is denoted $\mathbb{Z}/5\mathbb{Z}$ instead of $\mathbb{Z}/\mathfrak{R}$ and we then have:

$$\mathbb{Z}/5\mathbb{Z} = \{\dot{0}, \dot{1}, \dot{2}, \dot{3}, \dot{4}\}.$$

Proposition 1 Let $\mathfrak{R}$ be a relation defined on a set E , we have the following properties:

- ❶ Let $a, b \in E$, if $a \in \dot{b}$ then $\dot{a} = \dot{b}$.
- ❷ $\forall a, b \in E$, $\dot{a} = \dot{b} \iff a \mathfrak{R} b$.
- ❸ Let $u, v, x \in E$, if $u, v \in \dot{x}$ then $u \mathfrak{R} v$.
- ❹ $\forall x, y \in E$, we have $\dot{x} = \dot{y}$ or $\dot{x} \cap \dot{y} = \emptyset$.
- ❺ The equivalence classes form a partition of the set E .

$$E = \bigcup_{x \in E} \dot{x}.$$

Proof 3.2.2.1 ✓ If $x \in \dot{a}$, then

$x \mathfrak{R} a$ and we have $a \mathfrak{R} b$,

we deduce, by transitivity, that

$$x \mathfrak{R} b,$$

which implies that

$$x \in \dot{b}.$$

same reasoning to show that $\dot{b} \subset \dot{a}$.

Conclusion

$$\dot{a} = \dot{b}.$$

- ✓ Let us show the direct implication, we assume that $\dot{a} = \dot{b}$ and show that $a\mathfrak{R}b$. it is easy to see that $a \in \dot{a}$ (because $\mathfrak{R}$ is reflexive).

$$\begin{aligned} a \in \dot{a} &\implies a \in \dot{b} \\ &\implies a\mathfrak{R}b. \end{aligned}$$

Reciprocally if $a\mathfrak{R}b$ then $\dot{a} = \dot{b}$. in fact, let $z \in \dot{a}$

$$\begin{aligned} z \in \dot{a} &\implies z\mathfrak{R}a, \\ &\implies z\mathfrak{R}b \text{ (because } a\mathfrak{R}b \text{ and } \mathfrak{R} \text{ is transitive.)} \\ &\implies z \in \dot{b}. \end{aligned}$$

Hence $\dot{a} \subset \dot{b}$ Similarly, we show that $\dot{b} \subset \dot{a}$
Let $z \in \dot{b}$

$$\begin{aligned} z \in \dot{b} &\implies z\mathfrak{R}b, \\ &\implies z\mathfrak{R}a \text{ (because } b\mathfrak{R}a \text{ (}\mathfrak{R} \text{ is symmetric and transitive.)} \\ &\implies z \in \dot{a}. \end{aligned}$$

Hence $\dot{b} \subset \dot{a}$.

- ✓ We have $u, v \in \dot{x}$ then $u\mathfrak{R}x$ and $x\mathfrak{R}v$ hence $u\mathfrak{R}v$ (by the transitivity of $\mathfrak{R}$.)
- ✓ Let $x, y \in E$ such that $\dot{x} \cap \dot{y} \neq \emptyset$, we show that $\dot{x} = \dot{y}$.
We have $\dot{x} \cap \dot{y} \neq \emptyset$ then $\exists z \in E$, tel that $z \in \dot{x} \cap \dot{y}$

$$\begin{aligned} z \in \dot{x} \cap \dot{y} &\implies z \in \dot{x} \text{ et } z \in \dot{y} \\ &\implies z\mathfrak{R}x \text{ et } z\mathfrak{R}y, \\ &\implies \dot{z} = \dot{x} \text{ and } \dot{z} = \dot{y}, \text{ (see the first property)} \\ &\implies \dot{x} = \dot{y}. \end{aligned}$$

- ✓ Let us show that the quotient set forms a partition of E

- ❶ We have $\forall \dot{x} \in E/\mathfrak{R}$, $\dot{x} \neq \emptyset$, because $x \in \dot{x}$. (the relation $\mathfrak{R}$ is reflexive, $x\mathfrak{R}x$).
- ❷ We showed previously that all distinct equivalence classes are disjoint.

③ Rest à show that

$$E = \bigcup_{x \in E} \dot{x},$$

we have an obvious inclusion

$$\bigcup_{y \in E} \dot{y} \subset E,$$

let's show the other inclusion.

$$E \subset \bigcup_{y \in E} \dot{y}.$$

Let $x \in E$ then $x \in \dot{x}$ and therefore $x \in \bigcup_{y \in E} \dot{y}$ hence

$$E \subset \bigcup_{y \in E} \dot{y}$$

in conclusion, the set $E/\mathfrak{R}$ is a partition of E .

3.2.3 Canonical composition of a function

Definition 3.2.3 Let E and F be two sets, $f : E \longrightarrow F$ be a function, and let $\mathfrak{R}$ be a relation defined on E by:

$$x \mathfrak{R} y \iff f(x) = f(y).$$

This relation $\mathfrak{R}$ is an equivalence relation, it is called an equivalence relation *associated to* f . Let $f(E) = \{f(x) \mid x \in E\}$ is the image set of E by f , i the **canonical injection** of $F(E)$ into F and π **the canonical surjection** of E in $E/\mathfrak{R}$.

$$\begin{aligned} i : f(E) &\longrightarrow F \\ x &\longmapsto x \end{aligned}$$

$$\begin{aligned} \pi : E &\longrightarrow E/\mathfrak{R} \\ x &\longmapsto \dot{x} \end{aligned}$$

Remark 3.2.1 We easily verify, by construction, that the function i is injective and the function π is surjective.

Theorem 3.2.1 Let E, F be two sets, and $f : E \longrightarrow F$ be a map

❶ The binary relation $\mathfrak{R}$ defined on E by:

$$x\mathfrak{R}y \iff f(x) = f(y).$$

is an equivalence relation on E called *associated* to f .

❷ . Let π be the canonical surjection of E on $E/\mathfrak{R}$ and i the canonical injection of $f(E)$ in F . Then there exists a bijective map unique

$$\begin{aligned} \tilde{f} : E/\mathfrak{R} &\longrightarrow f(E) \\ \dot{x} &\longmapsto \tilde{f}(\dot{x}) = f(x). \end{aligned}$$

such that $f = i \circ \tilde{f} \circ \pi$.

Proof 3.2.3.1 ❶ It is easy to verify that $\mathfrak{R}$ is an equivalence relation on E .

❷ Indeed,

$$\begin{aligned} \dot{x} = \dot{y} &\iff f(x) = f(y) \\ &\iff \tilde{f}(\dot{x}) = \tilde{f}(\dot{y}). \end{aligned}$$

We have also just shown that $\tilde{f}$ is injective.

$\tilde{f}$ is surjective by construction. Thus, $\tilde{f}$ is a bijection of $E/\mathfrak{R}$ into $f(E)$, $\tilde{f}$ is called the bijection canonical associated to f .

If there existed another application $g : E/\mathfrak{R} \longrightarrow F$ such that $f = g \circ \pi$, we

would have for all $x \in E$, $\tilde{f}(\dot{x}) = f(x) = g(\dot{x})$. hence $\tilde{f} = g$, which proves the uniqueness of $\tilde{f}$.

it is clear that for all $x \in E$, we then have:

$$\boxed{\forall x \in E, f(x) = i(f(x)) = i(\tilde{f}(\dot{x})) = (i \circ \tilde{f} \circ \pi)(x).}$$

Hence, $f = i \circ \tilde{f} \circ \pi$: This is the canonical decomposition of f into the composition of an injection, a bijection and a surjection. We have just established the theorem of the canonical decomposition of an application, we have the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \pi \downarrow & & \uparrow i \\ E/\mathfrak{R} & \xrightarrow{\tilde{f}} & f(E) \end{array}$$

Exercise 3.2.2 Let $\mathfrak{R}$ be the relation defined on $\mathbb{R}$ by:

$$\forall x, y \in \mathbb{R}, x \mathfrak{R} y \iff x^2 = y^2$$

- ❶ Show that $\mathfrak{R}$ is an equivalence relation on $\mathbb{R}$:
- ❷ Determine the equivalent class of $a \in \mathbb{R}$;
- ❸ Determine the quotient set $\mathbb{R}/\mathfrak{R}$.
- ❹ The function f defined by:

$$\begin{aligned} f : \mathbb{R}/\mathfrak{R} &\longrightarrow [0, +\infty[\\ \dot{x} &\longmapsto f(\dot{x}) = x^2. \end{aligned}$$

is it well defined? is it bijective?

3.3 Order relations

3.3.1 Definitions and examples

Definition 3.3.1 Let $\mathfrak{R}$ be a binary relation on a set E . We say that $\mathfrak{R}$ is **an order relation** if $\mathfrak{R}$ is reflexive, antisymmetric and transitive.

Example 3.3.1 We define in $\mathbb{N}$ the binary relation $\mathfrak{R}$ by:

$$\forall x, y \in \mathbb{N}, x \mathfrak{R} y \iff x \text{ divide } y.$$

Let us show that $\mathfrak{R}$ is an order relation.

✓ Let us show that $\mathfrak{R}$ is reflexive. We have

$$\mathfrak{R} \text{ is reflexive} \iff [\forall x \in \mathbb{N}, x \mathfrak{R} x]$$

It is easy to see that $\forall x \in \mathbb{N}, x = 1.x$, which is equivalent to saying that $x \mathfrak{R} x$ hence $\mathfrak{R}$ is reflexive.

✓ Let us show that $\mathfrak{R}$ is antisymmetric. We have by definition

$$R \text{ is antisymmetric} \iff [\forall x, y \in \mathbb{N}, x \mathfrak{R} y \text{ and } y \mathfrak{R} x \implies y = x]$$

and therefore

$$\begin{aligned} \forall x, y \in \mathbb{N}, x \mathfrak{R} y \text{ and } y \mathfrak{R} x &\implies x \text{ divides } y \text{ and } y \text{ divides } x \\ &\implies \exists k, k' \in \mathbb{N}, y = k.x, \text{ and } x = k'.y \\ &\implies \exists k, k' \in \mathbb{N}, y = k.x \quad \text{and} \quad x = k'.k.x. \\ &\implies \exists k, k' \in \mathbb{N}, y = k.x \text{ and } k'.k = 1. \\ &\implies \exists k, k' \in \mathbb{N}, y = k.x \text{ and } k' = k = 1. \\ &\implies y = x. \end{aligned}$$

hence, $\mathfrak{R}$ is anti-symmetric.

✓ Let us show that $\mathfrak{R}$ is transitive. We have by definition

$$R \text{ is transitive} \iff [\forall x, y, z \in \mathbb{N}, x \mathfrak{R} y \text{ and } y \mathfrak{R} z \implies x \mathfrak{R} z]$$

and so

$$\begin{aligned} \forall x, y, z \in \mathbb{N}, x \mathfrak{R} y \text{ et } y \mathfrak{R} z, &\implies \exists k, k' \in \mathbb{N}, y = k.x \text{ and } z = k'.y \\ &\implies \exists k, k' \in \mathbb{N}, z = k'.k.x \\ &\implies \exists k'' \in \mathbb{N}, z = k''x \\ &\implies x \mathfrak{R} z. \end{aligned}$$

hence, $\mathfrak{R}$ is transitive.

We deduce that $\mathfrak{R}$ is an order relation on $\mathbb{N}$.

Exercise 3.3.1   Is the following relation an order relation on $\mathbb{Z}$? :

$$\forall x, y \in \mathbb{Z}, x \mathfrak{R} y \iff x \text{ divide } y.$$

Example 3.3.2 The relations $\mathfrak{R}_2, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_8$ are all order relations, on the other hand the relations

$\mathfrak{R}_1$ (is not antisymmetric), $\mathfrak{R}_3$ (is not antisymmetric), $\mathfrak{R}_6$ (is not reflexive), $\mathfrak{R}_7$ (is not reflexive), $\mathfrak{R}_9$ (is not antisymmetric) $\mathfrak{R}_{10}$ (is neither reflexive nor transitive), are not.

Remark 3.3.1 A set with an ordering relation is called **an ordered set**, and we denote it $(E, \mathfrak{R})$.

Remark 3.3.2 A set with an ordering relation is called **an ordered set**, and we denote it $(E, \mathfrak{R})$.

3.3.2 Total or partial order

A order relation compares elements in a set. This means that we can form the notion of upper and lower bounds of sets. We make the following definition:

Definition 3.3.2 Let $(A, \mathfrak{R})$ be a ordered set. An element $a \in A$ is called **largest element of A** if and only if $x \mathfrak{R} a$ for every $x \in A$. Conversely, an element $a \in A$ is called **smallest element of A** if and only if $a \mathfrak{R} x$ for every $x \in A$.

Definition 3.3.3 Let E be a set ordered by the order relation $\mathfrak{R}$.

- ❶ Let x, y be two elements of E , we say that x and y are **comparable** if $x \mathfrak{R} y$ or $y \mathfrak{R} x$.
- ❷ We say that the relation $\mathfrak{R}$ is a **total order**, or else $(E, \mathfrak{R})$ is **totally ordered**, if any two elements x, y of E are **comparable**. In other words:

The relation $\mathfrak{R}$ **is a total order** $\iff \forall x, y \in E, x \mathfrak{R} y \text{ or } y \mathfrak{R} x$.

Otherwise, we say that the relation $\mathfrak{R}$ is a **partial order**, or else $(E, \mathfrak{R})$ is **partially ordered**. In other words:

The relation $\mathfrak{R}$ **is a partial order** $\iff \exists x, y \in E, x \not\mathfrak{R} y, \text{ and } y \not\mathfrak{R} x$.

Example 3.3.3 ❶ $\leq$ and $\geq$ define a total order on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

❷ The division defines a **partial order** on $\mathbb{N}$.

③ $\subset$ and $, \supset$ define a partial order on $\mathcal{P}(E)$ such that $\text{card}(E) \geq 2$.

3.3.3 Remarkable elements of an ordered set

lower and upper bound of a set

Definition 3.3.4 Let $\mathfrak{R}$ be an order relation on a set E and A , a non-empty part of E .

- ① We say that A is **bounded from above** or **majorized** for the relation $\mathfrak{R}$ if:

$$\boxed{\exists M \in E, \forall x \in A, \text{ such as } x \mathfrak{R} M}.$$

We say that M is a **upper bound** of A or else A is **bounded from above** by M .

- ② We say that A is **bounded from below** or **minorized** for the relation $\mathfrak{R}$ if:

$$\exists m \in E, \forall x \in A, \text{ such that } m \mathfrak{R} x.$$

We say that m is a **lower bound** of A or A is **bounded from below** by m :

Example 3.3.4 ① $A = \{1, 3, 7\}$ is reduced by 1 and increased by 21 for the relation defined on $\mathbb{N}$ by:

$$\boxed{\forall x, y \in \mathbb{N}, x \mathfrak{R} y \iff y \text{ multiple of } x.}$$

Indeed :

✓ Let $M \in \mathbb{N}$

M is an upper bound of $A \implies \forall x \in A, x \mathfrak{R} M.$

$\implies M$ multiple of 1, M multiple of 3 and M multiple of 7

$\implies M$, is the common multiplier of 1, 3 and 7.

$\implies M$ is multiple of 21.

Then the set of upper bounds of A is $\{21k, \text{ such that } k \in \mathbb{N}\}.$

✓ Let $m \in \mathbb{N}$

m is a lower bound of $A \implies \forall x \in A, m \mathfrak{R} x$.

$\implies 1$ multiple of m , 3 multiple of m and 7 multiple of m

$\implies m$ is the common divisor of 1, 3 and 7.

$\implies m = 1$.

Then the set of lower bounds of A is $\{1\}$.

② In the ordered set $(\mathcal{P}(E), \subset)$, $\mathcal{P}(E)$ is minorized by $\emptyset$ and majorized by E .

Upper bound and lower bound of a set

Definition 3.3.5 Let $\mathfrak{R}$ be an order relation on a set E and A be a non-empty part of E .

- ① If A is *majorized* for the relation $\mathfrak{R}$ then the *least upper bound* of A , if it exists, is called *least upper bound* or *supremum*, denoted $\sup A$.
- ② If A is *minorized* for the relation $\mathfrak{R}$ then the *greatest lower bound* of A , if it exists, is called *greatest lower bound* or *infimum*, denoted $\inf A$.

Example 3.3.5 ① $A = \{1, 3, 7\}$ is bounded from below by 1 and bounded from above by 21 for the relation defined on $\mathbb{N}$ by:

$$\forall x, y \in \mathbb{N}, x \mathfrak{R} y \iff y \text{ multiple of } x.$$

✓ We have seen that the set of upper bounds of A is $\{21k, \text{ such that } k \in \mathbb{N}\}$, and so $\sup(A) = 21$ is the smallest multiple of elements of A .

✓ For the lower bound, we have already shown that the set of lower bounds of A is $\{1\}$, so $\inf(A) = 1$.

② A subset A of an ordered set E does not necessarily admit an upper (resp. lower) bound. However, if A admits an upper bound (resp. lower), it is unique but it may not belong to A .

For example, if $E = \mathbb{Q}$, ordered by the usual inequality, and if

$$A = \{x \in \mathbb{Q} : 0 < x \text{ et } x^2 < 2\},$$

then the set A is minimized by any negative or zero rational number. We have $\inf(A) = 0$ but A does not admit an upper bound in $\mathbb{Q}$ since $\sqrt{2} \notin \mathbb{Q}$.

Theorem 3.3.1 Let $(E, \mathcal{R})$ be a totally ordered set, and A be a part of E . For an element M of E to be the upper bound of A , it is necessary and sufficient that M satisfies both conditions.

1. For all $x \in A$, we have: $x \mathcal{R} M$.
2. For any element $c \in E$ such that $c \mathcal{R} M$, $\exists x \in A$, such that $c \mathcal{R} x$.

maximum element, minimum element in a set

Definition 3.3.6 Let $\mathcal{R}$ be an order relation on a set E and A be a non-empty part of E .

- ① We call **largest element** of A (or **maximum** of A), any element of A which is the upper bound of A , we denote it $\max(A)$. In other words:

$$M = \max(A) \iff M \text{ is the largest element of } A \iff [(M \in A) \text{ et } (\forall x \in A, x \mathcal{R} M)]$$

If one exists, this element is unique.

- ② We call **smallest element** of A (or **minimum** of A), any element of A which is lower bound of A , we denote it $\min(A)$. In other words

$$m = \min(A) \iff m \text{ is the smallest element of } A \iff [(m \in A) \text{ and } (\forall x \in A, m \mathcal{R} x)]$$

If one exists, this element is unique

Example 3.3.6 ① . With the usual relation $\leq$ defined on $\mathbb{R}$ i.e. A, B two parts of $\mathbb{R}$.

$$A = \{2, 5, -7\}, \quad B =]0, 1[.$$

- ✓ The set A has a minimal element which is -7 , and a maximal element which is 5 , $\min(A) = -7$, $\max(A) = 5$.
- ✓ The set B has neither a minimal nor a maximal element.

② For the relation $\mathfrak{R}$ on $\mathbb{N}$:

$$\forall x, y \in \mathbb{N}, x \mathfrak{R} y \iff y \text{ multiple of } x.$$

Let the subset $A = \{2, 3, 10\}$ of $\mathbb{N}$.

- ✓ A does not have a maximal element, the upper bounds are the multiples of 30 and the smallest of upper bounds of A is 30, which does not belong to A .
- ✓ A does not have a minimal element, the lower bounds of A are the common divisors of 2, 3 and 10 so the only lower bound of A is 1, which does not belong to A .

Maximal element and minimal element of a set

Definition 3.3.7 Let E be a set with an order relation $\mathfrak{R}$ and A a non-empty part of E .

① We say that $a \in A$ is a **maximal element** of A if

$$\forall x \in A, a \mathfrak{R} x \implies x = a$$

That is, there is no element x in A , other than a , such that a is related to x . (or, there is no element in the set A greater than a , with respect to $\mathfrak{R}$).

② We say that $b \in A$ is a **minimal element** of A if $\forall x \in A, x \mathfrak{R} b \implies x = b$.

That is, there exists no element x in A , other than b , such that x is related to b . (or else, there is no element in the set A lower than b , with respect to the relation $\mathfrak{R}$).

③ We say that an element of E is **extremal** if it is or **maximal** or **minimal**.

Exercise 3.3.2 (Exercise $N^\circ 5$ from the series $N^\circ 3$)

Solution 3.3.1 to enter