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Lecture Note

Transfer Function

For a linear time-invariant (LTI) system, the transfer function is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input, assuming zero initial conditions.
For an LTI system with input $x(t)$ and output $y(t)$, the transfer function $H(s)$ is defined as:

$$H(s) = \frac{Y(s)}{X(s)}$$

Where:

- $X(s) = \mathcal{L}\{x(t)\}$ is the Laplace transform of the input.
- $Y(s) = \mathcal{L}\{y(t)\}$ is the Laplace transform of the output.
- $s = j\omega$ is the complex frequency variable.

Derivation from Differential Equations:

Consider an nth-order LTI system described by:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

Taking Laplace transform with zero initial conditions:

$$\mathcal{L}\left\{\frac{d^k y(t)}{dt^k}\right\} = s^k Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

Where:

- $N(s)$ = Numerator polynomial of degree m
- $D(s)$ = Denominator polynomial of degree n
- **Zeros** z_i: Values of s where $N(s) = 0 \rightarrow H(s) = 0$
- **Poles** p_i: Values of s where $D(s) = 0 \rightarrow H(s) \rightarrow \infty$
- **K:** is the gain constant.

First-Order Systems:

Standard Form:

$$\boxed{\boldsymbol{H(s) = \frac{K}{\tau s + 1}}}$$

Where:

- $\boldsymbol{K}$ = DC Gain (steady-state value)
- $\boldsymbol{\tau}$ = Time Constant (seconds)
- Pole: $\boldsymbol{s = -\frac{1}{\tau}}$

Second-Order Systems:
Standard Form:

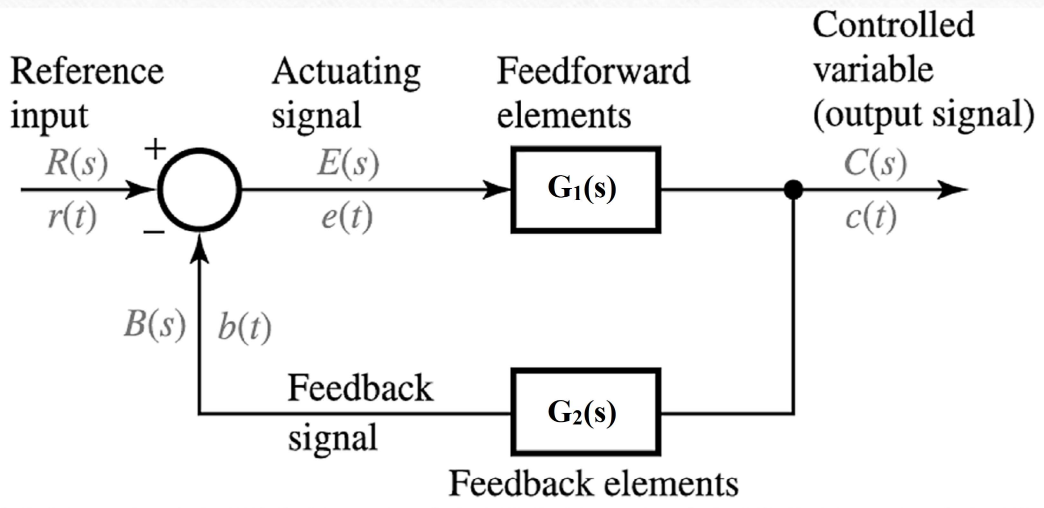
$$H(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Where:

- K: steady-state gain
- ω_n: natural frequency (rad/s)
- ξ: damping ratio (dimensionless)

Block Diagram

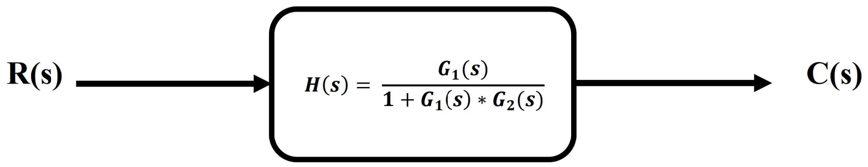
Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.



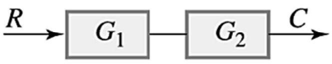
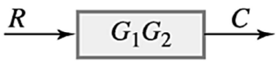
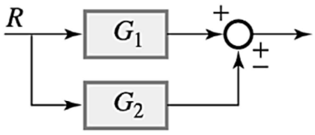
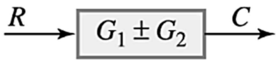
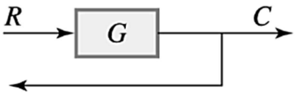
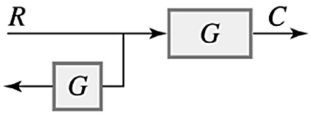
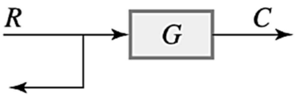
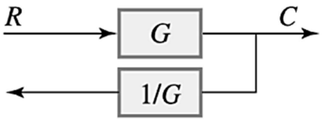
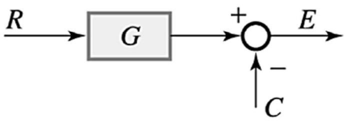
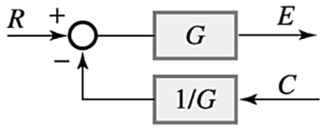
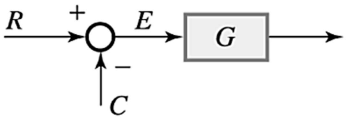
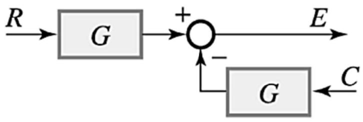
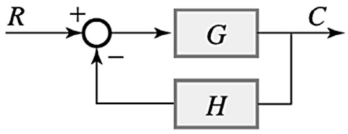
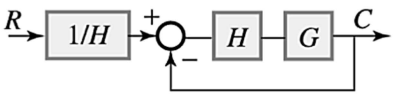
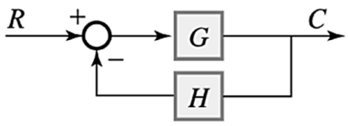
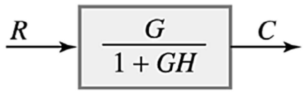
Where:

- **R(s):** Reference input.
- **C(s):** Output signal.
- **B(s):** Feedback signal B(s) = C(s)*G2(s).
- **E(s):** Error E(s) = R(s) – B(s).
- **G_1(s):** Forward path transfer function.
- **G_2(s):** Feedback path transfer function.
- **H(s):** Closed loop transfer function $\boldsymbol{H(s) = \frac{G_1(s)}{1+G_1(s)*G_2(s)}}$
- **L(s):** Open loop transfer function $\boldsymbol{L(s) = G_1(s) * G_2(s)}$
- Characteristic equation $\boldsymbol{1 + G_1(s) * G_2(s) = 0}$

Reduced block diagram:



Block diagram reduction technique table:

Original Block Diagram	Manipulation	Modified Block Diagram
	Cascaded elements	
	Addition or subtraction (eliminating auxiliary forward path)	
	Shifting of pickoff point ahead of block	
	Shifting of pickoff point behind block	
	Shifting summing point ahead of block	
	Shifting summing point behind block	
	Removing H from feedback path	
	Eliminating feedback path	

Time-domain analysis of linear systems

First-Order Systems:

Standard Form:

$$\boxed{\boldsymbol{H(s) = \frac{K}{\tau s + 1}}}$$

<u>STEP RESPONSE</u>: ($X(s) = \frac{1}{s}$)

$$Y(s) = H(s) * X(s) = \frac{\boldsymbol{K}}{\boldsymbol{\tau s + 1}} \cdot \frac{\boldsymbol{1}}{\boldsymbol{s}} = \frac{\boldsymbol{K}}{\boldsymbol{s}} - \frac{\boldsymbol{K}}{\boldsymbol{s + 1/\tau}}$$

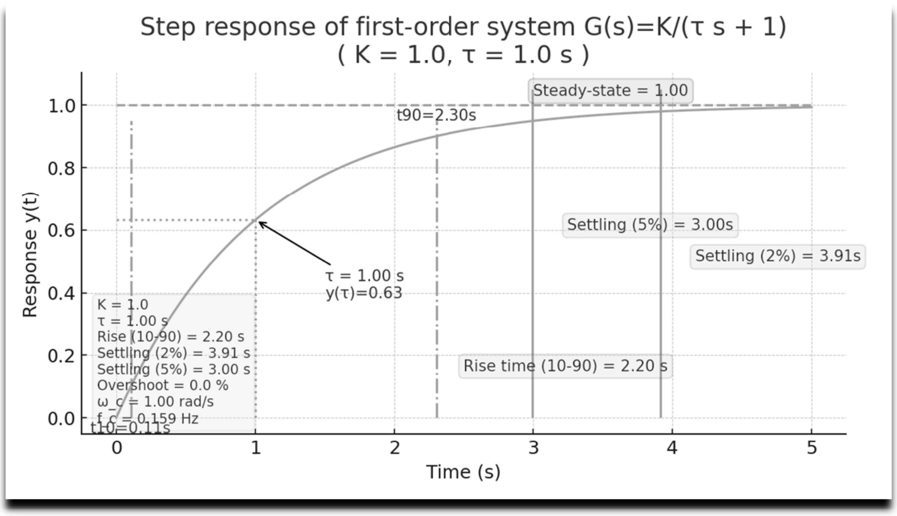
Time Domain Solution:

$$\boxed{\boldsymbol{y(t) = K\left(1 - e^{-t/\tau}\right) \quad \text{for } t \geq 0}}$$

Key Time Points:

Time	Response Value	% of Final Value
$\boldsymbol{t = 0}$	$\boldsymbol{y(0) = 0}$	0%
$\boldsymbol{t = \tau}$	$\boldsymbol{y(\tau) = K(1 - e^{-1}) \approx 0.632K}$	63.2%
$\boldsymbol{t = 2\tau}$	$\boldsymbol{y(2\tau) = K(1 - e^{-2}) \approx 0.865K}$	86.5%
$\boldsymbol{t = 3\tau}$	$\boldsymbol{y(3\tau) = K(1 - e^{-3}) \approx 0.950K}$	95.0%
$\boldsymbol{t = 4\tau}$	$\boldsymbol{y(4\tau) = K(1 - e^{-4}) \approx 0.982K}$	98.2%
$\boldsymbol{t = 5\tau}$	$\boldsymbol{y(5\tau) = K(1 - e^{-5}) \approx 0.993K}$	99.3%

Quantity	**Formula**
Standard TF	$\boldsymbol{H(s) = \dfrac{K}{\tau s + 1}}$
Pole	$\boldsymbol{s = -\dfrac{1}{\tau}}$
Step Response	$\boldsymbol{y(t) = K(1 - e^{-t/\tau})}$
63.2% Time	$\boldsymbol{t = \tau}$
Rise Time (10-90%)	$\boldsymbol{t_r = 2.2\tau}$
Settling Time (2%)	$\boldsymbol{t_s = 4\tau}$
DC Gain	$\boldsymbol{H(0) = K}$
Impulse Response	$\boldsymbol{h(t) = \dfrac{K}{\tau} e^{-t/\tau}}$



Second-Order Systems:

Standard Form:

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Poles of the System:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Cases:

- $\zeta = 0 \rightarrow$ Undamped
- $0 < \zeta < 1 \rightarrow$ Underdamped (oscillatory)
- $\zeta = 1 \rightarrow$ Critically damped
- $\zeta > 1 \rightarrow$ Overdamped

Underdamped System ($0 < \zeta < 1$):

- **Damped frequency**:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

- **Rise time**:

$$T_r \approx \frac{\pi - \theta}{\omega_d}, \quad \theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

- **Peak time**:

$$T_p = \frac{\pi}{\omega_d}$$

- **Maximum overshoot**:

$$M_p = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100\%$$

- **Settling time** (2% criterion):

$$T_s \approx \frac{4}{\zeta\omega_n}$$

Frequency-domain analysis of linear systems

The gain margin (GM) and phase margin (PM):

In frequency-domain analysis, gain margin (GM) and phase margin (PM) are defined from **the open-loop transfer function**

$$\boldsymbol{L(s) = G_1(s) * G_2(s)}$$

Gain margin:

Phase crossover frequency ω_{pc}

When: $\omega = \omega_{pc}$ $\quad \arg\left(L(j\omega_{pc})\right) = -180°$

Formula (linear scale)

$$\boxed{\mathrm{GM} = \frac{1}{|L(j\omega_{pc})|}}$$

Formula (in dB)

$$\boxed{\mathrm{GM_{dB}} = -20 * \log_{10}(|L(j\omega_{pc})|)}$$

Phase margin:

Gain crossover frequency ω_{gc}

When: $\omega = \omega_{gc}$: $|L(j\omega_{gc})| = 1$ (0 dB)

Formula

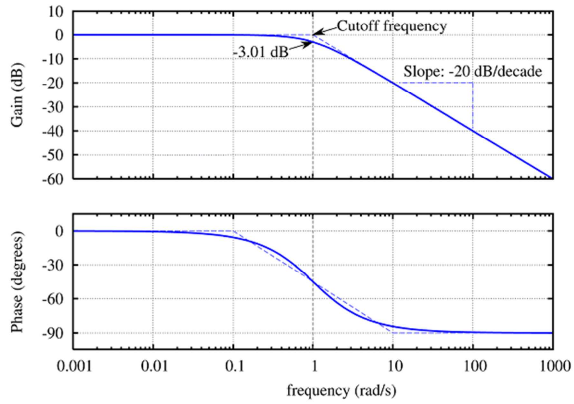
$$\boxed{\mathrm{PM} = 180° + \arg(L(j\omega_{gc}))}$$

Stability interpretation

Margin	Condition
PM > 0°	Stable
PM ≈ 30-60°	Well-damped
GM > 1 (or > 0 dB)	Stable
GM ≤ 1 (≤ 0 dB)	Unstable

Bode Diagram: for the open-loop transfer function

Bode plots are logarithmic frequency response plots of a system's transfer function $G(s)$ evaluated at $s = j\omega$.



- **Magnitude:** $\boldsymbol{MG_{dB}} = 20 * \log_{10}|G(j\omega)|$
- **Phase:** $\boldsymbol{\varphi}^{\circ} = \arg(G(j\omega))$

Constant Gain (K)

- **Magnitude:** $20\log_{10}|K|$ dB → **Horizontal line**
- **Phase:** 0° if K>0, ±180° if K<0

Integrator/Differentiator

Factor	Magnitude Slope	Phase
$\frac{1}{j\omega}$ (Integrator)	**-20 dB/dec**	**-90°**
$j\omega$ (Differentiator)	**+20 dB/dec**	**+90°**

1st Order transfer function:

$$H(s) = \frac{K}{1 + \tau.s}$$

$$H(\omega) = \frac{K}{1 + j(\tau.\omega)}$$

Magnitude (dB):

$$\boldsymbol{MG_{dB}} = |G(\omega)|_{dB} = 20 * \log_{10}(K) - 20\log_{10}\sqrt{1 + (\omega\tau)^2}$$

$$\boldsymbol{MG_{dB}} = |G(\omega)|_{dB} = 20 * \log_{10}(K) - 10 * \log_{10}(1 + (\omega\tau)^2)$$

Phase (º):

$$\varphi = \arg\big(H(\omega)\big) = \arg(K) - \arg\,(1 + j(\tau.\omega))$$

$$\varphi = \arg\big(H(\omega)\big) = -\text{arctg}(\tau.\omega)$$

Corner frequency $\boldsymbol{\omega_c : \frac{1}{\tau}}$

2nd Order transfer function:

$$H(s) = \frac{K}{(s - p_1)(s - p_2)}$$

$$H(s) = \mathrm{K}\left(\frac{1}{(s - p_1)} \times \frac{1}{(s - p_2)}\right)$$

$$H(s) = \mathrm{K}\left(\frac{-1/p_1}{1 + \left(\frac{-1}{p_1}\right).\mathrm{s}} \times \frac{-1/p_2}{1 + \left(\frac{-1}{p_2}\right).\mathrm{s}}\right)$$

$$H(s) = H_1(s) \times H_2(s)$$

$$H(\omega) = H_1(\omega) \times H_2(\omega)$$

Magnitude:

$$|G(j\omega)| = |G_1(j\omega)| \cdot |G_2(j\omega)|$$

$$|H(\omega)| = |H_1(\omega)| \times |H_2(\omega)|$$

Magnitude in Decibels (dB):

$$|H(\omega)|_{dB} = 20\log_{10}|H(\omega)| = 20\log_{10}|H_1(\omega)| + 20\log_{10}|H_2(\omega)|$$

$$\boxed{|H(\omega)|_{dB} = |H_1(\omega)|_{dB} + |H_2(\omega)|_{dB}}$$

Phase:

$$\arg\,(H(\omega)) = \arg\big(H_1(\omega) \times H_2(\omega)\big)$$

$$\arg\big(H(\omega)\big) = \arg\big(H_1(\omega)\big) + \arg\big(H_2(\omega)\big)$$

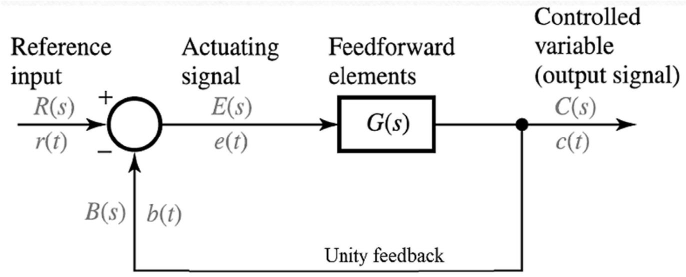
Special case:

For $G(s) = K \cdot G_1(s)$: and K is real number

- **Magnitude:** $|G(j\omega)|_{dB} = 20\log_{10}K + |G_1(j\omega)|_{dB}$
- **Phase:** $\arg(G(j\omega)) = \arg(G_1(j\omega))$

System Performance

Steady-State Error e_{ss}



- $E(s)$: Error Signal (s-domain) $E(s) = R(s) - C(s) = \frac{R(s)}{1+G(s)}$
- $L(s)$: is open-loop transfer function $L(s) = \ G(s) * 1$
- $H(s)$: is closed-loop transfer function $H(s) = \frac{G(s)}{1+G(s)}$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \ s.E(s)$$

Using error constants:

- **Position constant**:

$$K_p = \lim_{s\to 0} G(s) \quad \Rightarrow \quad e_{ss} = \frac{1}{1+K_p}$$

- **Velocity constant**:

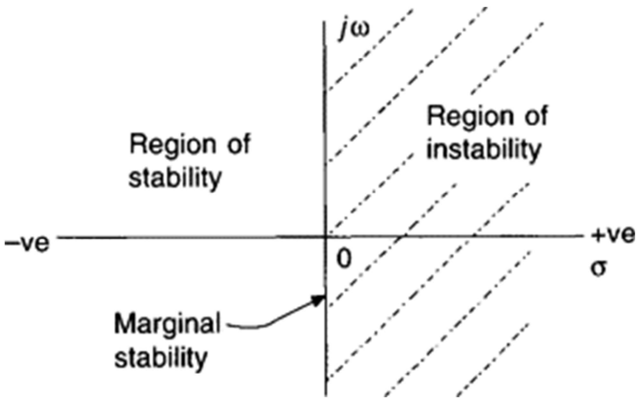
$$K_v = \lim_{s\to 0} s.G(s) \quad \Rightarrow \quad e_{ss} = \frac{1}{K_v}$$

- **Acceleration constant**:

$$K_a = \lim_{s\to 0} s^2.G(s) \quad \Rightarrow \quad e_{ss} = \frac{1}{K_a}$$

System stability:

A. **The basic rule:** A continuous-time LTI system is asymptotically stable if and only if all poles have negative real parts (i.e., Re(p) < 0).
B. **Visual Tool:** The s-plane. Stability region is the left-half plane (excluding the imaginary axis).



C. Routh-Hurwitz Criterion: The Routh–Hurwitz criterion is an algebraic method used to determine the stability of **closed-loop LTI system** $\boldsymbol{H(s)} = \frac{G_1(s)}{1+G_1(s)*G_2(s)}$.

The Characteristic Equation: $$\boldsymbol{D(s) = 1 + G_1(s) * G_2(s) = 0}$$

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Step 1: Necessary Condition: all coefficients $a_0, a_1, \ldots, a_n$ Must be > 0.

- If any coefficient is missing (zero) or negative, the system is **definitely unstable**.
- If they are all positive, you must proceed to Step 2.

Step 2: Construct the Routh Array: The Routh array is structured as follows:

Row	s^n	s^{n-1}	s^{n-2}	s^{n-3}	…
s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	…
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	…
s^{n-2}	b_1	b_2	b_3	b_4	…
s^{n-3}	c_1	c_2	c_3	c_4	…
…	…	…	…	…	…
s^0	…	…	…	…	…

The elements b_1, b_2, c_1, c_2, etc., are calculated systematically:

$$b_1 = \frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} = \frac{-(a_n \cdot a_{n-3} - a_{n-1} \cdot a_{n-2})}{a_{n-1}}$$

$$b_2 = \frac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}} = \frac{-(a_n \cdot a_{n-5} - a_{n-1} \cdot a_{n-4})}{a_{n-1}}$$

$$c_1 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1}$$

$$c_2 = \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1}$$

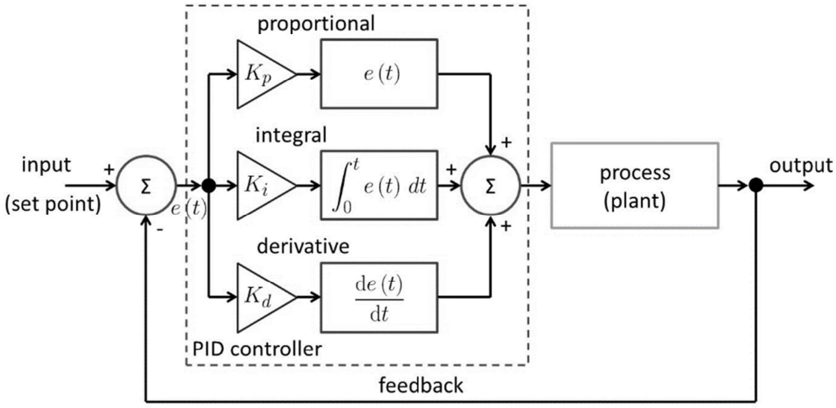
Step 3: The Stability Criterion The system is stable **if and only if** all the elements in the **first column** of the Routh table are **positive**.

D. Nyquist-Bode Stability Criterion

- **Stable:** PM > 0º and GM > 0 dB. ➔ $\omega_{gc} < \omega_{pc}$.
- **Marginally Stable:** PM = 0º or GM = 0 dB.
- **Unstable:** PM < 0º or GM < 0 dB. ➔ $\omega_{gc} > \omega_{pc}$.

System Correction

System correction using a **PID controller** is crucial in control systems used to make a system's output follow a desired reference accurately and stably.



The control law is:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau + K_d \frac{de(t)}{dt}$$

In Laplace domain:

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s$$

Effect of PID on system performance

Parameter	Rise Time	Overshoot	Steady-State Error	Stability
Increase K_p	↓	↑	↓	↓
Increase K_i	↓	↑	→ 0	↓
Increase K_d	~	↓	~	↑

- **Increase** K_p: Faster response, but more overshoot.
- **Increase** K_i: Eliminates offset, but may cause oscillations.
- **Increase** K_d: Damping, reduces overshoot, improves stability.

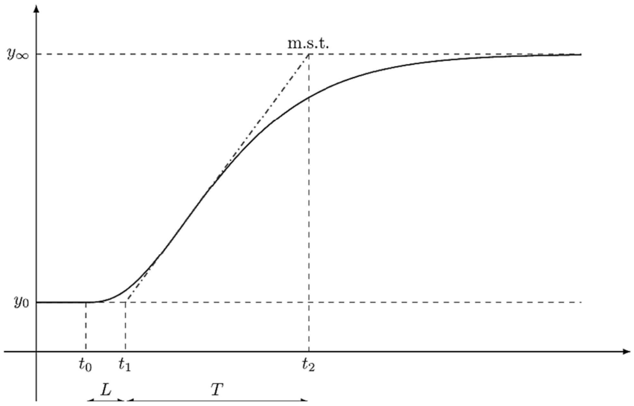
Tuning the PID Controller (Finding Kp, Ki, Kd)

Method	Approach	Use Case
Manual Tuning	Adjust parameters based on step response observation.	Simple systems, initial tuning.
Ziegler-Nichols	Find ultimate gain (K_u) and oscillation period (P_u), use formulas.	When process model is unknown.
Pole placement	Use feedback to "move" the open-loop poles to pre-selected desirable locations in the complex plane.	if and only if the system is completely controllable.

Ziegler-Nichols:

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = K_p + \frac{K_i}{s} + K_d s$$

A. Step Response Method (Process Reaction Curve)



1. Apply a **unit step input** to the open-loop system.
2. Obtain an **S-shaped step response**.
3. Determine:
   - **L** = delay time $L = t_1 - t_0$
   - **T** = time constant $T = t_2 - t_1$
   - **K** = gain constant $K = y_\infty - y_0$
4. Use Z-N tuning formulas.

Tuning Formulas

Controller	K_p	T_i	T_d
P	$\frac{T}{K \cdot L}$	-	-
PI	$\frac{0.9T}{K \cdot L}$	$3.33L$	-
PID	$\frac{1.2T}{K \cdot L}$	$2L$	$0.5L$

B. Ultimate Gain Method (Closed-Loop Method)

1- Set controller to **P-only** mode:
   - $T_i = \infty, T_d = 0$

2- Increase K_p until:
   - Output shows **sustained oscillations**

3- Record:
   - **Ultimate gain** K_u
   - **Ultimate period** P_u

Tuning Formulas

Controller	K_p	K_i	K_d
P	$0.5K_u$	-	-
PI	$0.45K_u$	$1.2K_p/T_u$	-
PID	$0.6K_u$	$2K_p/T_u$	$K_pT_u/8$

Divers:

Partial Fraction Decomposition:

1- *The Basic Idea for Linear Factors*

$$\frac{N(s)}{D(s)} = \frac{N(x)}{(s-a_1)(s-a_2)\dots(s-a_n)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \dots + \frac{A_n}{s-a_n}$$

Then to find A_k:

$$A_k = \lim_{s \to a_k} (s-a_k) \cdot \frac{N(s)}{D(s)}$$

2- *Extension to Repeated Linear Factors*

$$\frac{N(s)}{(s-a)^m} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_m}{(s-a)^m}$$

To find A_m (highest power)

$$A_m = \lim_{s \to a}(s-a)^m \cdot \frac{N(s)}{D(s)}$$

To find A_{m-1}

Differentiate after multiplying by $(s-a)^m$, then take limit:

$$A_{m-1} = \lim_{s \to a} \frac{d}{ds}\left[(s-a)^m \cdot \frac{N(s)}{D(s)}\right]$$