Notes for Numerical Analysis
Math 5466
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(A Rough Draft)

Contents

Chapter 1

Polynomial Interpolation

1.1 Review

Mean-value theorem: Let $f \in C[a,b]$ and differentiable on (a,b) then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Weighted Mean-value theorem: If $f \in C[a,b]$ and $g(x) > 0$ on $[a,b]$, then there exists $c \in (a,b)$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Rolle's theorem: If $f \in C[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Generalized Rolle's theorem: If $f \in C[a,b]$, $n+1$ times differentiable on (a,b) and admits $n+2$ zeros in $[a,b]$, then there exists $c \in (a,b)$ such that $f^{(n+1)}(c) = 0$.

Intermediate value theorem: If $f \in C[a,b]$ such that $f(a) \neq f(b)$, then for each y between $f(a)$ and $f(b)$ there exists $c \in (a,b)$ such that $f(c) = y$.

1.2 Introduction

From the Webster dictionary the definition of interpolation reads as follows: "Interpolation is the act of introducing something, especially, spurious and foreign, the act of calculating values of functions between values already known"

Our goal is approximate a set of data points or a function by a simpler polynomial function. Given a set of data points $x_i,\ i=0,1,\cdots n,\ x_i \neq x_j,$ if $i \neq j$ we would like to construct a polynomial $p_m(x)$ such that

$$p_m^{(k)}(x_i) = f^{(k)}(x_i),\ i=0,1,\cdots m,\ k=0,1,\cdots n_i,\ \text{with } n = \sum_{i=0}^{m} n_i\ -1.$$

Lagrange Interpolation: $n_i = 0,\ m \geq 1\ ,\ p_n(x_i) = f(x_i),\ i = 0,1,\cdots,n$

Taylor Interpolation: $n_0 > 1,\ m = 0\ ,\ p_n^{(k)}(x_0) = f^{(k)}(x_0),\ k = 0,1,\cdots,n_0.$

Hermite Interpolation: $n_i \geq 1,\ m \geq 1\ ,\ p_n^{(k)}(x_i) = f^{(k)}(x_i),\ \ i = 0,1,\cdots,m,$ $k = 0,1,\cdots n_i.$

Why Interpolation? For instance interpolation is used to approximate integrals

$$\int_a^b f(x)dx \approx \int_a^b p_m(x)dx$$

derivatives

$$f^{(k)}(x) \approx p_m^{(k)}(x)$$

and plays a major role in approximating differential equations.

Taylor interpolation:

We first study Taylor polynomials defined as

$$p_n(x) = f(x_0) + (x-x_0)f^{'}(x_0) + \frac{(x-x_0)^2}{2}f^{(2)}(x_0) + \cdots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0).$$

The interpolation error or remainder formula in Taylor expansions is written as

$$f(x) - p_n(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

Example: $f(x) = sin(x), \; x_0 = 0$

$$p_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

The interpolation error can be written as

$$|sin(x) - p_{2n+1}(x)| = \frac{|x|^{2n+3}}{(2n+3)!} |cos(\xi)| < \frac{|x|^{2n+3}}{(2n+3)!}$$

On the interval $0 < x < 1/2$ the interpolation error is bounded as

$$|sin(x) - p_{2n+1}(x)| < \frac{1}{2^{2n+3}(2n+3)!}.$$

1.3 Lagrange Interpolation

Lagrange form:

Given a set of points $(x_i, f(x_i)), \; i = 0, 1, 2, \cdots n, \; x_j \neq x_i$ we define the Lagrange coefficient polynomials $l_i(x), \; i = 0, 1, \cdots n$. such as

$$l_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

and is defined as

$$l_i(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

The Lagrange form of the interpolation polynomial is

$$p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$$

Example: Let us consider the data set $x = [0,1,2],\ f = [-2,-1,2]$

$$l_0(x) = \frac{(x-1)(x-2)}{(-1)(-2)} = (x^2 - 3x + 2)/2$$

$$l_1(x) = \frac{x(x-2)}{(1)(-1)} = -x^2 + 2x$$

$$l_2(x) = \frac{x(x-1)}{(2)(1)} = (x^2 - x)/2$$

$$p_2(x) = -2l_0(x) - l_1(x) + 2l_2(x) = x^2 - 2$$

Example:

$f(x) = cos(x)^5$ using 8 points $x = [0,1,2,3,4,5,6,7]$ and 14 points $x_i = i * 0.5, i = 0, 2, \cdots 14$

```
A Matlab example

x=[0 1 2 3 4 5 6 7];
y=cos(x).^5;
c = polyfit(x,y,length(x)-1);
xi = 0:0.1:7;
zi = cos(xi).^5;
yi =polyval(c,xi);
subplot(2,1,1)
title('Interpolation')
plot(xi,yi,'-.',xi,zi,x,y,'*');
subplot(2,1,2)
title('Interpolation Error')
plot(xi,zi-yi,x,zeros(1,length(x)),'-*');
```

Newton form and divided differences:

We develop a procedure to compute $a_0,\ a_1, \cdots,\ a_n$ such that the interpolation polynomial has the form

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

where $a_0 = f(x_0)$, $a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0}$ and such that

$$a_k = f[x_0, x_1, \cdots, x_k]$$

is called the k^{th} divided difference. All divided differences are generated by the following recurrence formula

$$f[x_i] = f(x_i)$$

$$f[x_i, x_k] = \frac{f[x_k] - f[x_i]}{x_k - x_i}$$

$$f[x_0, x_1, \cdots, x_{k-1}, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$$

x_i	$f(x_i)$	$1^{st}DD$	$2^{nd}DD$	$3^{rd}DD$	$4^{th}DD$
x_0	$f(x_0)$				
		$f[x_0, x_1]$			
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_2	$f(x_2)$		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
x_3	$f(x_3)$		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$			
x_4	$f(x_4)$				

The forward Newton polynomial can be written as

$$p_4(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +$$

$$f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$+ f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

The backward Newton polynomial can be written as

$$p_4(x) = f(x_4) + f[x_3, x_4](x - x_4) + f[x_2, x_3, x_4](x - x_4)(x - x_3) +$$

$$f[x_1, x_2, x_3, x_4](x - x_4)(x - x_3)(x - x_2)$$

$$+ f[x_0, x_1, x_2, x_3, x_4](x - x_4)(x - x_3)(x - x_2)(x - x_1)$$

Example:

x_i	$f(x_i)$	$1^{st}DD$	$2^{nd}DD$	$3^{rd}DD$	$4^{th}DD$	$5^{th}DD$
-2	16					
		-8				
-1	8		2			
		-4		$-10/3$		
0	4		-8		$10/3$	
		-20		10		-11/6
1	-16		22		$-35/6$	
		24		$-40/3$		
2	8		-18			
		-12				
3	-4					

The forward Newton polynomial

$$p_5(x) = 16 - 8(x+2) + 2(x+2)(x+1) - \frac{10}{3}(x+2)(x+1)x$$

$$+\frac{10}{3}(x+2)(x+1)x(x-1) - \frac{11}{6}(x+2)(x+1)x(x-1)(x-2).$$

The Backward Newton polynomial is given by

$$p_5(x) = -4 - 12(x-3)18(x-3)(x-2) - \frac{40}{3}(x-3)(x-2)(x-1)$$

$$+\frac{35}{6}(x-3)(x-2)(x-1)x - \frac{11}{6}(x-3)(x-2)(x-1)x(x-1).$$

Remarks:

(i) The upper diagonal contains the coefficients for the forward Newton polynomials.

(ii) The lower diagonal contains the coefficients for the backward Newton polynomials.

(iii) $p_k(x)$ interpolates f at $x_0, x_1, \cdots, x_k$ and is obtained as

$$p_k(x) = f(x_0) + \sum_{j=1}^{k} f[x_0, x_1, \cdots, x_j] \prod_{i=0}^{j-1} (x - x_i).$$

(iv) $p_2(x)$ that interpolates f at x_2, x_3 and x_4 is

$$p_2(x) = f(x_2) + f[x_2, x_3](x - x_2) + f[x_2, x_3, x_4](x - x_2)(x - x_3).$$

(v) If we decide to add an additional point, it should be added at the bottom for forward Newton polynomials and at the top for backward Newton polynomials.

(vi) $f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)}(\xi)}{k!}, \qquad \xi \in [a, b].$

Nested mutliplication

An efficient algorithm to evaluate Newton polynomial can be obtained by writing

$$p_n(x) = a_1 + (a_2 + \cdots (a_{n-2} + (a_{n-1} + (a_n + a_{n+1}(x - x_n)) \\ (x - x_{n-1}))(x - x_{n-2}) \cdots)(x - x_1)).$$

Matlab program

```
%input a(i), i=1,2,...,n+1 , x(i),i=1,...,n+1, and x
%
p = a(n+1);
for i=n:-1:1
  p = a(i) + p*(x-x(i));
end;
%p = p_n(x)
```

1.4 Interpolation error and convergence

In this section we study the interpolation error and convergence of interpolation polynomials to the interpolated function.

1.4.1 Interpolation error

Theorem 1.4.1. *Let $f \in C[a,b]$ and $x_0, x_1, x_2, \cdots x_n$, be $n+1$ distinct points in $[a,b]$. Then there exists a unique polynomial p_n of degree at most n such that $p_n(x_i) = f(x_i),\ i = 0, 1, \cdots n$.*

Proof. Existence: we define

$$L_i(x) = \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}$$

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{i = j} \\ 0, & \text{otherwise} \end{cases}$$

and

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

One can verify that

$$p_n(x_j) = f(x_j).$$

Uniqueness:

Assume there are two polynomials $q_n(x)$ and $p_n(x)$ such that

$$q_n(x_j) = p_n(x_j) = f(x_j),\ j = 0, 1, 2, \cdots, n$$

and consider the difference

$$d_n(x) = p_n(x) - q_n(x).$$

$d_n(x_j) = 0,\ i = 0, 1, \cdots, n$ so $d_n(x)$ has $n+1$ roots. By the fundamental theorem of Algebra $d_n(x) = 0$. □

Theorem 1.4.2. *Let $f \in C[a,b]$ $(n+1)$ differentiable on (a,b) and let $x_0,\ x_1, \cdots,\ x_n$, be $(n+1)$ distinct points in $[a,b]$. If $p_n(x)$ is such that $p_n(x_i) = f(x_i)$, $i = 0, 1, \cdots, n$, then for each $x \in [a,b]$ there exists $\xi(x) \in [a,b]$ such that*

$$f(x) - p_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!} W(x),$$

where $W(x) = \prod_{i=0}^{n} (x - x_i)$.

Proof. Let $x \in [a,b]$ and $x \neq x_i,\ i = 0, 1, \cdots, n$ and define the function

$$g(t) = f(t) - p_n(t) - \frac{f(x) - p_n(x)}{W(x)} W(t).$$

We note that g has $(n+2)$ roots, i.e., $g(x_i) = 0,\ i = 0, 1, \cdots n$ and $g(x) = 0$. Using the generalized Rolle's Theorem there exits $\xi(x) \in (a,b)$ such that

$$g^{(n+1)}(\xi(x)) = 0$$

which leads to

$$g^{(n+1)}(\xi(x)) = f^{(n+1)}(\xi(x)) - 0 - \frac{f(x) - p_n(x)}{W(x)} (n+1)! = 0, \qquad (1.1.1)$$

We solve (1.1.1) to find $f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} W(x)$ which completes the proof. □

Corollary 1. *Assume that* $\max_{x \in [a,b]} |f^{(n+1)}(x)| = M_{n+1}$ *then*

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |W(x)|,\ \forall\ x \in [a,b].$$

and

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{x \in [a,b]} |W(x)|.$$

Proof. The proof is straight forward. □

Examples:

$$||f - P_1||_\infty \leq \frac{M_2 h^2}{8}, \qquad [x_0,\ x_0 + h].$$

$$||f - p_2||_\infty \leq \frac{M_3 h^3}{9\sqrt{3}}, \qquad [x_0, x_0 + 2h].$$

$$||f - p_3||_\infty \leq \frac{M_4 h^4}{24}, \qquad [x_0, x_0 + 3h].$$

Example: Let us interpolate $f(x) = e^{\frac{x}{3}}$ on $[0,1]$ at $x_0, x_1, \cdots, x_n$. The $n+1$ derivative is $f^{(n+1)}(x) = \frac{e^{\frac{x}{3}}}{3^{n+1}}$ where

$$M_{n+1} = \max_{x \in [0,1]} |f^{(n+1)}(x)| = \frac{e^{\frac{1}{3}}}{3^{n+1}}.$$

The interpolation error can be bounded as

$$|f(x) - p_n(x)| \leq \frac{e^{1/3}|W(x)|}{3^{n+1}(n+1)!},\ x \in [0,1].$$

For instance, for $n = 4$ and $x = [0, 1/4, 1/2, 3/4, 1]$, $W(x) = x(x - 1/4)(x - 1/2)(x - 3/4)(x - 1)$

The error at $x = 0.3$ can be bounded as

$$|f(0.3) - p_4(0.3)| \leq \frac{e^{1/3}|W(0.3)|}{3^5 5!} \approx 0.45210^{-7}.$$

Example: Let us consider $f(x) = cos(x) + x$ on $[0,2]$ which satisfies $\max_{x \in [0,2]} |f^{(k)}(x)| \leq 1$.

The interpolation error can be bounded as

Case 1: Two interpolation points with $h = 2$,

$$\max_{x \in [0,2]} |f(x) - p_1(x)| \leq \frac{h^2 M_2}{8} \leq 4/8 = 0.5.$$

Case 2: Three interpolation points with $h = 1$,

$$\max_{x\in[0,2]} |f(x) - p_2(x)| \leq \frac{h^2 M_3}{9\sqrt{3}} \leq 1/(9\sqrt{3} \approx 0.0641.$$

Case 3: Four interpolation points with $h = 2/3$,

$$\max_{x\in[0,2]} |f(x) - p_3(x)| \leq \frac{h^4 M_4}{24} \leq (2/3)^4/24 \approx 0.0082.$$

1.4.2 Convergence

We start by reviewing the convergence of functions and defining simple and uniform convergence of sequences of functions.

Let $f_n(x),\ n = 0, 1, \cdots$ be a sequence of continuous functions on $[a, b]$.

Definition 1. *(Simple convergence): $f_n(x)$ converges simply to $f(x)$ if and only if at every $x \in [a, b]$ $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$.*

Definition 2. *(Uniform convergence): f_n converges uniformly to f if and only if $\lim_{n\to\infty} \max_{x\in[a,b]} |f_n(x) - f(x)| = 0$.*

Example: Let us consider the sequence

$$f_n(x) = \frac{1}{1 + nx}, n = 0, 1, \cdots,\ x \in [0, 1].$$

(i) The sequence f_n converges simply to 0 for all $0 < x \leq 1$ while $f_n(0) = 1$. However, f_n does not converge uniformly since $||f_n||_\infty = 1$.

(ii) The sequence f_n converges uniformly to 0 on $[2, 3]$ since $||f_n||_\infty = 1/(1 + 2n)$.

Next, we will study the uniform convergence of interpolation polynomials on a fixed interval $[a, b]$ as the number of interpolation points approaches infinity. Let $h = (b-a)/n$ and $x_i = a + ih,\ i = 0, 1, 2, \cdots, n$, equidistant interpolation points. Let $p_n(x)$ denote the Lagrange interpolation polynomial, i.e., $p_n(x_i) = f(x_i),\ i = 0, \cdots, n$ and let us study the limit

$$\lim_{n\to\infty} ||f - p_n||_\infty = \lim_{n\to\infty} \max_{x\in[a,b]} |f(x) - p_n(x)|.$$

For $x \in [a, b]$, $|x - x_i| \le (b - a)$ which leads to $|W(x)| \le (b - a)^{n+1}$. Thus, the interpolation error is bounded as

$$||f - p_n||_\infty \le \frac{M_{n+1}}{(n+1)!}(b-a)^{n+1}.$$

We have uniform convergence when $\frac{M_{n+1}}{(n+1)!}(b-a)^{n+1} \to 0$ as $n \to \infty$.

Theorem 1.4.3. *Let f be an analytic function on a disk centered at $(a+b)/2$ with a radius $r > 3(b-a)/2$. Then, the interpolation polynomial $p_n(x)$ satisfying $p_n(x_i) = f(x_i),\ i = 0, 1, 2, \cdots n$, converges to f as $n \to \infty$, i.e.,*

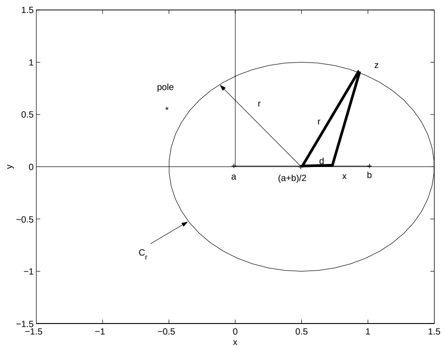
$$\lim_{n\to\infty} ||f - p_n||_\infty = 0.$$

Proof. A function is analytic at $(b+a)/2$ if it admits a power series expansion that converges on a disk of radius r and centered at $(a+b)/2$.

Applying Cauchy's formula

$$f^{(k)}(x) = \frac{k!}{2\pi i}\oint_{C_r} \frac{f(z)}{(z-x)^{k+1}}dz, \qquad x \in [a, b].$$

$$|f^{(k)}(x)| \le \frac{k!}{2\pi}\oint_{C_r} \frac{|f(z)|}{|(z-x)|^{k+1}}dz, \qquad x \in [a, b].$$



Let z be a point on the circle C_r and $x \in [a, b]$. From the triangle with vertices z, $(a+b)/2$ and x the following triangle inequality holds

$$|z - x| + d \geq r.$$

Noting that $d \leq (b-a)/2$ the triangle inequality yields

$$|z - x| \geq r - (b-a)/2.$$

$$|f^{(k)}(x)| \leq \frac{k!}{2\pi} \frac{\max_{z \in C_r} |f(z)|}{|r - (b-a)/2|^{(k+1)}} 2\pi r.$$

Assume $r > \frac{b-a}{2}$ ($[a, b] \subset C_r$) to obtain

$$M_k \leq \frac{r}{r - (b-a)/2} \max_{z \in C_r} |f(z)| \frac{k!}{(r - (b-a)/2)^k}.$$

Using $k = n + 1$ the interpolation error may be bounded as

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!}(b-a)^{(n+1)} \leq$$

$$\max_{z \in C_r} |f(z)| \left(\frac{r(b-a)}{r - \frac{(b-a)}{2}} \right) \left(\frac{b-a}{r - (b-a)/2} \right)^n.$$

Finally, we have uniform convergence if $\frac{b-a}{r-(b-a)/2} < 1$, i.e., $r > \frac{3}{2}(b-a)$ which establishes the theorem. $\square$

Examples of analytic functions are $sin(z)$, e^z, $cos(z^2)$.

Runge phenomenon:

Let $f(x) = \frac{1}{4+x^2}$ is $C^\infty[-10,10]$ but we do not have uniform convergence on $[-10,10]$ when using the $n+1$ equally spaced interpolation points $x_i = -10 + ih,\ i = 0,1,\cdots,n,\ h = 20/n$.

The function $f(z)$ has two poles $z = \pm 2i$, thus, can't be analytic on disk C_r with radius $r > 3(b-a)/2 = 30$ and center at $(a+b)/2$. We cannot apply the previous theorem and we should expect convergence problems for x far away from the origin.

The largest interval $[-a,a]$ satisfying $r > 3(b-a)/2 = 3a$ corresponds to $a < 2/3$. Actually, it may converge on a larger interval because this is a sufficient condition.

Interpolation errors and divided differences:

The Newton form of $p_n(x)$ that interpolates f at $x_i,\ i = 0,1,\cdots,n$ is

$$p_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + \sum_{i=2}^{n} f[x_0,\cdots,x_i]\prod_{j=0}^{i-1}(x-x_j).$$

We proof a theorem relating the interpolation errors and divided differences.

Theorem 1.4.4. *If $f \in C[a,b]$ and $n+1$ times differentiable, then for every $x \in [a,b]$*

$$f(x) - p_n(x) = f[x_0,x_1,\cdots,x_n,x]\prod_{i=0}^{n}(x-x_i)$$

and

$$f[x_0,x_1,x_2,\cdots,x_k] = \frac{f^{(k)}(\xi)}{k!},\ \xi \in [\min_{i=0,\cdots,k} x_i, \max_{i=0,\cdots,n} x_i].$$

Proof. Let us introduce another point x distinct from $x_i, i = 0,1,\cdots n$ and let p_{n+1} interpolate f at $x_0, x_1, \cdots x_n$ and x, thus

$$p_{n+1}(x) = p_n(x) + f[x_0, x_1, \cdots, x_n, x] \prod_{i=0}^{n} (x - x_i).$$

Combining the equation $p_{n+1}(x) = f(x)$ and the interpolation error formula we write

$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x] \prod_{i=0}^{n} (x - x_i) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

This leads to

$$f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)}(\xi)}{k!}, \qquad \xi \in [\min_{i=0,\cdots,k} x_i, \max_{i=0,\cdots,k} x_i],$$

which completes the proof. □

Remark:

$$\lim_{x_i \to x_0,\ i=1,\cdots k} f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)}(x_0)}{k!}$$

1.5 Interpolation at Chebyshev points

In the previous section we have shown that uniform convergence does not occur using uniform interpolation points for some functions.

Now, we study the interpolation error on $[-1, 1]$ where the $(n+1)$ interpolation points, $x_i^*,\ i = 0, 1, \cdots, n$, in $[-1, 1]$ are selected such that

$$||W^*(.)||_\infty = \min_{Q \in \tilde{\mathcal{P}}_{n+1}} ||Q(.)||_\infty$$

where $\tilde{\mathcal{P}}_n$ is the set of the monic polynomials

$$\tilde{\mathcal{P}}_n = [Q \in \mathcal{P}_n | Q = x^n + \sum_{i=1}^{n-1} c_i x^i],$$

and $W^*(x) = \prod_{i}^{n}(x - x_i^*)$.

Question: Are there interpolation points $x_i^*,\ i = 0, 1, 2, \cdots, n$ in $[-1, 1]$ such that

$$||W^*||_\infty = \min_{x_i \in [a,b], i=0,1,\cdots,n} ||W||_\infty$$

If the above statement is true, the interpolation error can be bounded by

$$||E_n||_\infty \leq \frac{M_{n+1}}{(n+1)!}||W^*||_\infty$$

The Answer : The best interpolation points $x_i^*,\ i = 0, 1, 2, \cdots, n$ are the roots of the Chebyshev polynomial $T_{n+1}(x)$ defined as

$$T_k(x) = cos(karcos(x)),\ k = 0, 1, 2, \cdots.$$

In the following theorem we will prove some properties of Chebyshev polynomials.

Theorem 1.5.1. *The Chebyshev polynomials $T_k(x),\ k = 0, 1, 2, \cdots,$ satisfy the following properties:*

(i) $|T_k(x)| \leq 1,\ \textit{for all}\ -1 \leq x \leq 1$

(ii) $T_{k+1}(x) = 2xT_k(x) - T_{k-1}$

(iii) $T_k(x)$ *has* k *roots* $x_j^* = cos(\frac{2j+1}{2k}\pi),\ j = 0, 1, \cdots, k-1, \in [-1, 1]$

(iv) $T_k(x) = 2^{k-1}\prod_{j=0}^{k-1}(x - x_j^*)$

(v) If $\tilde{T}_k(x) = \frac{T_k(x)}{2^{k-1}}$ *then* $\max_{x\in[-1,1]} |\tilde{T}_k(x)| = \frac{1}{2^{k-1}}$.

Proof. We obtain (i) by noting that the range of the cosine function is $[-1, 1]$.

To obtain (ii) we write

$$T_{k+1}(x) = cos(karcos(x) + arcos(x)) = cos(k\theta + \theta)$$

where $\theta = arcos(x)$ and write

$T_{k+1} = cos(k\theta + \theta)$

$T_{k-1} = cos(k\theta - \theta)$

Use the trigonometric identity $cos(a \pm b) = cos(a)cos(b) \mp sin(a)sin(b)$ to obtain

$$cos(k\theta + \theta) = cos(k\theta)cos(\theta) - sin(k\theta)sin(\theta)$$

and

$$cos(k\theta - \theta) = cos(k\theta)cos(\theta) + sin(k\theta)sin(\theta)$$

Adding the previous equations to obtain

$T_{k+1}(x) + T_{k-1}(x) = 2cos(k\theta)cos(\theta) = 2xT_k(x)$

This proves (ii).

To obtain the roots we set $cos(karcos(x)) = 0$ which leads to

$$karcos(x) = \frac{(2j+1)}{2}\pi,\ \ j = 0, \pm 1, \pm 2, \cdots .$$

If we solve for x, we obtain

$$x = cos(\frac{(2j+1)}{2k}\pi), j = 0, \pm 1, \pm 2, \cdots ,$$

leads to the roots

$$x_j^* = cos(\frac{(2j+1)}{2k}\pi), j = 0, 1, \cdots , k-1.$$

Use induction to prove (iv) step 1: $T_1(x) = 2^0x$, $T_2(x) = 2^1x^2 - 1$ (iv) is true for $k = 1, 2$.

Step 2: Assume $T_k = 2^{k-1}x^k + \sum_{i=0}^{k-1} c_i x^i$, for $k = 1, 2, \cdots , n$ and use (ii) we write

$$T_{n+1}(x) = 2xT_n - T_{n-1}(x) = 2^n x^{n+1} + \sum_{i=0}^{n} a_i x^i$$

This establishes (iv), i.e., $T_k = 2^{k-1}\prod_{j=0}^{k-1}(x - x_j^*)$.

Applying (iv) we show that (v) is true. □

Corollary 2. *If $\tilde{T}_n(x)$ is the monic Chebyshev polynomial of degree n, then*

$$\max_{-1\le x\le 1} |\tilde{Q}_n(x)| \ge \max_{-1\le x\le 1} |\tilde{T}_n(x)| = \frac{1}{2^{n-1}},\ \ \forall \tilde{Q}_n \in \tilde{\mathcal{P}}_n.$$

Proof. Assume there is another monic polynomial $\tilde{R}_n(x)$ such that

$$\max_{-1\le x\le 1} |\tilde{R}_n(x)| < \frac{1}{2^{n-1}}$$

We also note that

$$\tilde{T}_n(z_k) = \frac{(-1)^k}{2^{n-1}}, \qquad z_k = cos(k\pi/n),\ k = 0, 1, 2, \cdots, n.$$

The $(n-1)$-degree polynomial $d_{n-1}(x) = \tilde{T}_n(x) - \tilde{R}_n(x)$ satisfies

$d_{n-1}(z_0) > 0$, $d_{n-1}(z_1) < 0$, $d_{n-1}(z_2) > 0$, $d_{n-1}(z_3) < 0$ So $d_{n-1}(x)$ changes sign between each pair z_k and z_{k+1}, $k = 0, 1, 2, \cdots, n$ and thus has n roots. Thus $d_{n-1}(x) = 0$, identically, i.e., $\tilde{T}(x)$ and $\tilde{R}_n(x)$ are identical. This leads to a contradiction with the assumption above. □

Below are the first five Chebyshev polynomials.

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1
\end{aligned}$$

Example of Chebyshev points:

k	x_0^*	x_1^*	x_2^*	x_3^*
1	0			
2	$\sqrt{2}/2$	$-\sqrt{2}/2$		
3	$\sqrt{3}/2$	0	$-\sqrt{3}/2$	
4	$cos(\pi/8)$	$cos(3\pi/8)$	$cos(5\pi/8)$	$cos(7\pi/8)$

Application to interpolation:

Let $p_n(x) \in \mathcal{P}_n$ interpolate $f(x) \in C^{n+1}[-1,1]$ at the roots of $T_{n+1}(x)$, $x_j^*,\ j = 0,1,2,\cdots,n$. Thus, we can write the interpolation error formula as

$$f(x) - p_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}\tilde{T}_{n+1}(x)$$

Using (v) from the previous theorem and assuming $||f^{n+1}||_{\infty,[-1,1]} \leq M_{n+1}$ we obtain

$$\max_{x\in[-1,1]} |f(x) - p_n(x)| \leq \frac{M_{n+1}}{2^n(n+1)!}$$

Remarks:

1. We note that this choice of interpolation points reduces the error significantly.

2. With Chebyshev points, p_n converges uniformly to f when $f \in C^1[-1,1]$ only. The function f does not have to be analytic (see Gautschi).

Example 1:

Consider $f(x) = e^x,\ x \in [-1,1]$

Case 1: with three points; n=2:

$$||E_2||_\infty \leq \frac{M_3}{3!2^2} = e/24 = 0.1136.$$

Case 2: with 6 points; n=5:

$$||E_5||_\infty \leq \frac{M_6}{6!2^5} = e/(720 \times 32) = 0.11710^{-3}.$$

How many Chebyshev points are needed to have $||E_n||_\infty < 10^{-8}$

$$||E_n||_\infty \leq \frac{M_{n+1}}{2^n(n+1)!} = \frac{e}{(n+1)!2^n} = 0.13110^{-9}, \text{ for n=9.}$$

Thus, 10 Chebyshev points are needed.

Chebyshev points on $[a, b]$:

Chebyshev points can be used on an arbitrary interval $[a, b]$ using the linear transformation

$$x = \frac{a+b}{2} + \frac{b-a}{2}t, \qquad -1 \leq t \leq 1. \tag{1.1.2a}$$

We also need the inverse mapping

$$t = 2\frac{x-a}{b-a} - 1, \qquad a \leq x \leq b. \tag{1.1.2b}$$

First, we order the Chebyshev nodes in $[-1, 1]$ as

$$t_k^* = cos(\frac{2k+1}{2n+2}\pi - \pi) = -cos(\frac{2k+1}{2n+2}\pi), \ k = 0, 1, 2, \cdots, n.$$

we define the interpolation nodes on an arbitrary interval $[a, b]$ as

$$x_k^* = \frac{a+b}{2} + \frac{b-a}{2}t_k^*, \ k = 0, 1, 2, \cdots, n$$

Remarks:

1. $x_0^* < x_1^* < \cdots < x_n^*$
2. x_k^* are symmetric with respect to the center $(a+b)/2$
3. x_k^* are independent of the interpolated function f

Theorem 1.5.2. *Let $f \in C^{n+1}[a,b]$ and p_n interpolate f at the Chebyshev nodes $x_k^*, k = 0, 1, 2, \cdots, n$, in [a,b]. Then*

$$\max_{x\in[a,b]} |f(x) - p_n(x)| \leq 2M_{n+1}\frac{(b-a)^{n+1}}{4^{n+1}(n+1)!}.$$

Where $M_{n+1} = \max_{x\in[a,b]} |f^{(n+1)}(x)|$.

Proof. It suffices to rewrite $W(x) = \prod_{i=0}^{n}(x - x_i*)$ using the mapping (1.1.2) to find that

$$(x - x_i^*) = \frac{b-a}{2}(t - t_i^*),$$

and

$$W(x) = \prod_{i=0}^{n}(x - x_i*) = (\frac{b-a}{2})^{n+1}\prod_{i=0}^{n}(t - t_i^*) = (\frac{b-a}{2})^{n+1}\tilde{T}_{n+1}(t).$$

Finally, using $||\tilde{T}_{n+1}||_{\infty,[-1,1]} = \frac{1}{2^n}$ we complete proof. □

Example 2:

Consider $f(x) = 3^x = e^{ln(3)x}$, $x \in [0,1]$ whose derivative is $f^{(n+1)}(x) = ln(3)^{n+1}e^{ln(3)x}$. Noting that $f^{(n+1)}$ is a monotonically increasing function, $M_{n+1} = f^{(n+1)}(1) = 3ln(3)^{n+1}$. Therefore,

$$||E_n||_\infty \leq \frac{2 \times 3\, ln(3)^{n+1}}{4^{n+1}(n+1)!} = \frac{6\, ln(3)^{n+1}}{4^{n+1}(n+1)!}.$$

# of Chebyshev points	Error bound
2	0.226
3	0.0207
4	0.00142
5	0.000078
6	$0.358\ 10^{-5}$
7	$0.140\ 10^{-6}$
8	$0.48\ 10^{-8}$
9	$0.14\ 10^{-9}$

1.6 Hermite interpolation

We restate the general Hermite interpolation by Letting $x_0 < x_1 < x_2 \cdots x_m$ be $m+1$ distinct points such that

$$f^{(k)}(x_i) = p_n^{(k)}(x_i), \qquad k = 0, 1, \cdots n_i - 1, \qquad i = 0, 1, \cdots m, \qquad (1.1.3)$$

where $\sum_{i=0}^{m} n_i = n + 1$ and $n_i \geq 1$. We note that n_i =1, $i = 0, \cdots, m$, leads to Lagrange interpolation.

1.6.1 Lagrange form of Hermite interpolation polynomials

Theorem 1.6.1. *There exists a unique polynomial $p_n(x)$ that satisfies (1.1.3) with $n_i = 2$ and $n = 2m + 1$.*

Proof. Existence:

Next, we study the special case $n_i = 2$, $i = 0, 1, 2, \cdots$, where

$$l_{i,1}(x) = (x - x_i) l_i(x)^2. \qquad (1.1.4)$$

and

$$l_{i,0} = l_i(x)^2 - 2l_i'(x_i)(x - x_i)l_i(x)^2. \qquad (1.1.5)$$

Now, we can verify that

$$l_{i,1}(x_j) = 0, \ j = 0, 1, 2, \cdots, m$$

$$l_{i,1}'(x) = 2(x - x_i)l_i'(x)l_i(x) + l_i(x)^2.$$

Thus, $l_{i,1}'(x_j) = \delta_{ij}$.

One can easily check that $l_{i,0}(x_j) = \delta_{ij}$.

For $l_{i,0}'(x)$ we have

$$l_{i,0}'(x) = (1 - 2l_i'(x_i)(x - x_i))2l_i'(x)l_i(x) - 2l_i'(x_i)l_i(x)^2.$$

Thus, $l_{i,0}'(x_j) = 0, \ j = 0, 1, \cdots, m$.

Existence of Hermite interpolation polynomial is established by writing the Lagrange form of Hermite polynomial as

$$p_n(x) = \sum_{i=0}^{m} f(x_i)l_{i,0}(x) + \sum_{i=0}^{m} f'(x_i)l_{i,1}(x). \qquad (1.1.6)$$

Uniqueness:

Assume there are two polynomials $p_n(x)$ and $q_n(x)$ that satisfy (1.1.3) and consider the difference $d_n(x) = p_n(x) - q_n(x)$ which satisfies

$$d_n^{(s)}(x_j) = 0, \ s = 0, 1, \ j = 0, 1, \cdots, m.$$

Thus, $d_n(x)$ is a polynomial of degree at most n and has $(n+1)$ roots counting the multiplicity of each root. The fundamental theorem of Algebra shows that $d_n(x)$ is identically zero. With this we establish the uniqueness of p_n and finish the proof of the theorem. □

1.6.2 Newton form of Hermite interpolation polynomial

Using the following relation

$$f[x_0, x_0+h, x_0+2h, \cdots, x_0+kh] = \frac{f^{(k)}(\xi)}{k!},\ x_0 < \xi < x_0+kh \qquad (1.1.7)$$

and taking the limit when $h \to 0$ we obtain that

$$f[x_0, x_0, x_0, \cdots, x_0] = \frac{f^k(x_0)}{k!}. \qquad (1.1.8)$$

The divided difference table for the data $(x_k+ih, f(x_0+kh)),\ i = 0,1,2,3,4$ will converge to the table shown below where we recover the Taylor polynomial about x_k and with $n_k = 5$.

Using this observation we initialize the table for x_k and $n_k = 5$ as follows:

(i) every point x_i is repeated n_i times

(ii) we set $z_i = x_k,\ z_{i+1} = x_k,\ \cdots,\ z_{i+4} = x_k$

(iii) we initialize $f[z_i, z_{i+1}, \cdots,\ z_{i+s}] = \frac{f^{(s)}(x_k)}{s!}$ as shown in the following table

z_i	x_k	$f(x_k)$				
z_{i+1}	x_k	$f(x_k)$	$f'(x_k)$			
z_{i+2}	x_k	$f(x_k)$	$f'(x_k)$	$f''(x_k)/2!$		
z_{i+3}	x_k	$f(x_k)$	$f'(x_k)$	$f''(x_k)/2!$	$f'''(x_k)/3!$	
z_{i+4}	x_k	$f(x_k)$	$f'(x_k)$	$f''(x_k)/2!$	$f'''(x_k)/3!$	$f^{(4)}(x_k)/4!$

The general formula for divided differences with repeated arguments for $x_0 \leq x_1 \leq \ldots < x_n$ is given by

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \begin{cases} \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{k-1}]}{x_{i+k} - x_i}, & if\ x_i \neq x_{i+k} \\ \frac{f^{(k)}(\xi)}{k!} & \end{cases}$$

Example: Let $f(x) = x^4 + 1,\ f'(x) = 4x^3$. We will construct a polynomial $p_5(x)$ such that $p(x_i) = f(x_i)$ and $p'(x_i) = f'(x_i)$ with $x_i = -1, 0, 1$. The Hermite divided difference table is given as

	z_i	$f(z_i)$	1 DD	2 DD	3 DD	4 DD	5 DD
z_0	-1	2					
z_1	-1	2	$\boxed{f'(-1) = -4}$				
z_2	0	1	-1	3			
z_3	0	1	$\boxed{f'(0) = 0}$	1	-2		
z_4	1	2	1	1	0	1	
z_5	1	2	$\boxed{f'(1) = 4}$	3	2	1	0

The forward Hermite polynomial is given as

$$p_5(x) = 2 - 4(x+1) + 3(x+1)^2 - 2(x+1)^2x + (x+1)^2x^2 = 1 + x^4 \tag{1.1.9}$$

The Hermite polynomial that interpolates f and f' at $x = -1, 0$ is given as

$$p_3(x) = 2 - 4(x+1) + 3(x+1)^2 - 2(x+1)^2x. \tag{1.1.10}$$

The backward Hermite polynomial is given as

$$p_5(x) = 2 - 4(x-1) + 3(x-1)^2 + 2(x-1)^2x + (x-1)^2x^2 = 1 + x^4 \tag{1.1.11}$$

The Hermite polynomial that interpolates f and f' at $x = 0, 1$ is given by

$$p_3(x) = 2 + 4(x-1) + 3(x-1)^2 + 2(x-1)^2x. \tag{1.1.12}$$

Example:

Consider the data with $m = 1,\ n_0 = 1,\ n_1 = 2$ given in the following table

x_i	0	1
$f(x_i)$	1	2
$f'(x_i)$	0	1
$f''(x_i)$	NA	2

We write the divided differences table as

	z_I	$f(z_i)$	1DD	2DD	3 DD	4DD
z_0	0	1				
z_1	0	1	$\boxed{f'(x_0)=0}$			
z_2	1	2	1	1		
z_3	1	2	$\boxed{f'(x_1)=1}$	0	-1	
z_4	1	2	$\boxed{f'(x_1)=1}$	$\boxed{f''(x_1)/2!=1}$	1	2

The Hermite polynomial is given as

$$H_4(x) = 1 + 0(x-0) + (x-0)^2 - (x-0)^2(x-1) + 2(x-0)^2(x-1)^2$$

$$= 1 + 2x^2 - x^3 + 2x^2(x-1)^2. \tag{1.1.13}$$

1.6.3 Hermite interpolation error

Theorem 1.6.2. *Let $f(x) \in C[a,b]$ be $2m+2$ differentiable on (a,b) and consider $x_0 < x_1 < x_2, \cdots, x_m$ in $[a,b]$ with $n_i = 2,\ i = 0,1,\cdots,m$. If p_{2m+1} is the Hermite polynomial such that $p^{(k)}_{2m+1}(x_i) = f^{(k)}(x_i)$, $i = 0,1,\cdots,\ m$, $k = 0,1$, then there exists $\xi(x) \in [a,b]$ such that*

$$f(x) - p_{2m+1}(x) = \frac{f^{(2m+2)}(\xi(x))}{(2m+2)!}W(x) \tag{1.1.14a}$$

where

$$W(x) = \prod_{i=0}^{m}(x-x_i)^2. \tag{1.1.14b}$$

Proof. We consider the special case $n_i = 2$, i.e., $n = 2m+1$, select an arbitrary point $x \in [a,b],\ x \neq x_i,\ i = 0,\cdots,m$ and define the function

$$g(t) = f(t) - p_{2m+1}(t) - \frac{f(x) - p_{2m+1}(x)}{W(x)}W(t). \tag{1.1.15}$$

We note that g has $(m+2)$ roots, i.e., $g(x_i) = 0,\ i = 0, 1, \cdots m$ and $g(x) = 0$. Applying the generalized Rolle's Theorem we show that

$g'(\xi_i) = 0, i = 0, 1, \cdots, m$ where $\xi_i \in [a, b]$, $\xi_i \neq x_j$, $\xi_i \neq x$.

Using (1.1.3) with $n_i = 2$ we have $g'(x_i) = 0,\ i = 0, 1, \cdots, m$. Thus, $g'(t)$ has $2m + 2$ roots in $[a, b]$.

Applying the generalized Rolle's theorem we show that there exists $\xi \in (a, b)$ such that

$$g^{(2m+2)}(\xi) = 0.$$

Combining this with (1.1.15) yields

$$0 = f^{(2m+2)}(\xi) - \frac{f(x) - p_{2m+1}(x)}{W(x)}(2m + 2)!$$

Solving for $f(x) - p_{2m+1}(x)$ leads to (1.1.14). □

Corollary 3. *If $f(x)$ and $p_{2m+1}(x)$ are as in the previous theorem, then*

$$|f(x) - p_{2m+1}(x)| < \frac{M_{2m+2}}{(2m + 2)!}|W(x)|, \qquad x \in [a, b] \tag{1.1.16}$$

and

$$||f(x) - p_{2m+1}(x)||_{\infty,[a,b]} \leq \frac{M_{2m+2}}{(2m + 2)!}(b - a)^{2m+2}. \tag{1.1.17}$$

Proof. The proof is straight forward. □

At this point we would like to note that we can prove a uniform convergence result under the same conditions as for Lagrange interpolation.

Example: Let $f(x) = sin(x),\ x \in [0, \pi/2]$ and $p_5(x)$ interpolate f and f' at $x_i = 0, 0.2, \frac{\pi}{2}$, with $n_i = 2$ and $2m + 2 = 6$

Using the error bound 1.1.16 with $M_{2m+2} = 1$, $n_i = 2$ and

$$W(x) = [(x-0)(x-0.2)(x-\frac{\pi}{2})]^2,$$

we obtain

$$|E_5(1.1)| \le \frac{|W(1.1)|}{6!} \approx \frac{0.46608^2}{6!} \approx 3.017\ 10^{-4}. \tag{1.1.18}$$

1.7 Spline Interpolation

In this section we will study piecewise polynomial interpolation and write the interpolation errors in terms of the subdivision size and the degree of polynomials.

1.7.1 Piecewise Lagrange interpolation

We construct the piecewise linear interpolation for the data $(x_i, f(x_i)),\ i = 0, 1, \cdots, n$ such that $x_0 < x_1 < \cdots < x_n$ as

$$P_1(x) = \begin{cases} p_{1,0}(x) = f(x_0)\frac{(x-x_1)}{x_0-x_1} + f(x_1)\frac{(x-x_0)}{x_1-x_0},\ x \in [x_0, x_1], \\ p_{1,i}(x) = f(x_i)\frac{(x-x_{i+1})}{x_i-x_{i+1}} + f(x_{i+1})\frac{(x-x_i)}{x_{i+1}-x_i},\ x \in [x_i, x_{i+1}]. \\ i = 0, 1, \cdots, n-1. \end{cases} \tag{1.1.19}$$

The interpolation error on (x_i, x_{i+1}) is bounded as

$$||E_{1,i}(x)|| \le \frac{M_{2,i}}{2}|(x-x_i)(x-x_{i+1})|,\ x \in (x_i, x_{i+1}), \tag{1.1.20}$$

where $M_{2,i} = \max\limits_{x_i \le x \le x_{i+1}} |f^{(2)}(x)|$. This can be written as

$$||E_{1,i}||_\infty \le \frac{M_{2,i}h_i^2}{8},\ h_i = x_{i+1} - x_i. \tag{1.1.21}$$

The global error is

$$||E_1||_\infty \le \frac{M_2 H^2}{8}, \ H = \max_{i=0,\cdots,n-1} h_i. \tag{1.1.22}$$

Theorem 1.7.1. *Let $f \in C^0[a,b]$ be twice differentiable on (a,b). If $P_1(x)$ is the piecewise linear interpolant of f at $x_i = a + i * h$, $i = 0, 1, \cdots, n$, $h = (b-a)/n$, then P_1 converges uniformly to f as $n \to \infty$.*

Proof. We prove this theorem using the error estimate (1.1.22).

For piecewise quadratic interpolation we select $h = (b-a)/n$ and construct $p_{2,i}$ that interpolates f at x_i, $(x_i + x_{i+1})/2$ and x_{i+1}. In this case the interpolation error is bounded as

$$||E_{2,i}||_\infty \le \frac{M_{3,i}(h_i/2)^3}{9\sqrt{3}}, \ h_i = x_{i+1} - x_i. \tag{1.1.23}$$

A global bound is

$$||E_2||_\infty \le \frac{M_3 (H/2)^3}{9\sqrt{3}}, \ H = \max_{i=0,\cdots,2n-1} h_i. \tag{1.1.24}$$

□

Theorem 1.7.2. *Let $f \in C^0[a,b]$ and be m+1 times differentiable on (a,b) and $x_0 < x_1 < \cdots < x_n$ with $h_i = x_{i+1} - x_i$ and $H = \max_i h_i$. If $P_m(x)$ is the piecewise polynomial of degree m on each subinterval $[x_i, x_{i+1}]$ and $P_m(x)$ interpolates f on $[x_i, x_{i+1}]$ at $x_{i,k} = x_i + k * \tilde{h}_i$, $k = 0, 1, \cdots, m$, $\tilde{h}_i = h_i/m$, then P_m converges uniformly to f as $H \to 0$.*

Proof. Again we prove this theorem using the error bound

$$||E_m||_\infty \le \frac{M_{m+1} H^{m+1}}{(m+1)!}, \ H = \max_{i=0,1,\cdots,nm-1} h_i. \tag{1.1.25}$$

□

Similarly, we may construct piecewise Hermite interpolation polynomials following the same line of reasoning as for Lagrange interpolation.

1.7.2 Cubic spline interpolation

We use piecewise polynomials of degree three that are C^2 and interpolate the data such as $S(x_k) = f(x_k) = y_k,\ k = 0, 1, \cdots n.$

Algorithm

(i) Order the points $x_k,\ x = 0, 1, \cdots n$ such that

$$a = x_0 < x_1 < x_2 < \cdots x_{n-1} < x_n = b.$$

(ii) Let $S(x)$ be a piecewise spline defined by n cubic polynomials such that

$$S(x) = S_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3, \quad x_k \le x \le x_{k+1}$$

(iii) find $a_k,\ b_k,\ c_k,\ d_k,\ k = 0, 1, \cdots n - 1$ such that

(1) $S(x_k) = y_k, \quad k = 0, 1, \cdots, n$

(2) $S_k(x_{k+1}) = S_{k+1}(x_{k+1}), \quad k = 0, \cdots, n - 2$

(3) $S_k'(x_{k+1}) = S_{k+1}'(x_{k+1}), \quad k = 0, \cdots, n - 2$

(4) $S_k''(x_{k+1}) = S_{k+1}''(x_{k+1}), \quad k = 0, \cdots, n - 2$

Theorem 1.7.3. *If* $\mathbf{A}$ *is an* $n \times n$ *strictly diagonally dominant matrix, i.e.,* $|a_{kk}| > \sum_{i=1, i\ne k}^{n} |a_{k,i}|,\ k = 1, 2, \cdots n,$ *then* A *is nonsingular.*

Proof. By contradiction, we assume that A is singular, i.e., there exists a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$ and let x_k such that $|x_k| = max|x_i|$. This leads to

$$a_{kk}x_k = -\sum_{i=1, i\ne k}^{n} a_{ki}x_i. \tag{1.1.26}$$

Taking the absolute value and using the triangle inequality we obtain

$$|a_{kk}||x_k| \le \sum_{i=1, i\ne k}^{n} |a_{ki}||x_i|. \tag{1.1.27}$$

Dividing both terms by $|x_k|$ we get

$$|a_{kk}| \leq \sum_{i=1,i\neq k}^{n} |a_{ki}| \frac{|x_i|}{|x_k|} \leq \sum_{i=1,i\neq k}^{n} |a_{ki}| \tag{1.1.28}$$

This leads to a contradiction since **A** is strictly diagonally dominant. □

Theorem 1.7.4. *Let us consider the set of data points $(x_i, f(x_i))$, $i = 0, 1, \cdots, n$, such that $x_0 < x_1 < \cdots, x_n$. If $S''(x_0) = S''(x_n) = 0$, then there exists a unique piecewise cubic polynomial that satisfies the conditions (iii).*

Proof. Existence: we assume $S''(x_k) = m_k$ where $h_k = x_{k+1} - x_k$ and use piecewise linear interpolation of S'' to write

$$S_k''(x) = m_k \frac{x - x_{k+1}}{x_k - x_{k+1}} + m_{k+1} \frac{x - x_k}{x_{k+1} - x_k}$$
$$= -\frac{m_k}{h_k}(x - x_{k+1}) + \frac{m_{k+1}}{h_k}(x - x_k), \;\; x_k \leq x \leq x_{k+1}$$

With this definition of S'', condition (4) is automatically satisfied.

Integrating $S_k''(x)$ we obtain

$$S_k(x) = -\frac{m_k}{6h_k}(x - x_{k+1})^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + p_k(x_{k+1} - x) + q_k(x - x_k)$$

Need to find m_k, q_k and p_k, $k = 0, 1, 2, \cdots, n-1$.

In order to enforce the conditions (1) and (2) we write

$$S_k(x_k) = y_k = \frac{m_k}{6} h_k^2 + p_k h_k$$

$$S_k(x_{k+1}) = y_{k+1} = \frac{m_{k+1}}{6} h_k^2 + q_k h_k$$

Solve for p_k and q_k to solve

$$p_k = \frac{y_k}{h_k} - \frac{m_k h_k}{6} \tag{1.1.29a}$$

$$q_k = \frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6}. \tag{1.1.29b}$$

We note that if $m_k,\ k = 0, 1, \ldots, n$ are known, The previous equations may be used to compute p_k and q_k.

Now, substitute p_k and q_k in the equation for S_k to have

$$S_k(x) = -\frac{m_k}{6h_k}(x - x_{k+1})^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 + (\frac{y_k}{h_k} - \frac{m_k h_k}{6})(x_{k+1} - x) +$$

$$(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6})(x - x_k) \tag{1.1.30}$$

Applying condition (3) to enforce the continuity of $S'(x)$

$$S_k'(x_{k+1}) = S_{k+1}'(x_{k+1}),\ k = 0, 1, \cdots, n-1. \tag{1.1.31}$$

where

$$S_k'(x) = -\frac{m_k}{2h_k}(x - x_{k+1})^2 + \frac{m_{k+1}}{2h_k}(x - x_k)^2 -$$

$$(\frac{y_k}{h_k} - \frac{m_k h_k}{6}) + (\frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6}),\ x_k \le x \le x_{k+1}, \tag{1.1.32}$$

and

$$S_{k+1}'(x) = -\frac{m_{k+1}}{2h_{k+1}}(x - x_{k+2})^2 + \frac{m_{k+2}}{2h_{k+1}}(x - x_{k+1})^2 -$$

$$(\frac{y_{k+1}}{h_{k+1}} - \frac{m_{k+1}h_{k+1}}{6}) + (\frac{y_{k+2}}{h_{k+1}} - \frac{m_{k+2}h_{k+1}}{6}),\ x_{k+1} \le x \le x_{k+2}. \tag{1.1.33}$$

Taking the limit from the left at x_{k+1} leads to

$$S_k'(x_{k+1}) = \frac{m_{k+1}h_k}{3} + \frac{m_k h_k}{6} + d_k. \tag{1.1.34}$$

Taking the limit from the right at x_{k+1} yields

$$S'_{k+1}(x_{k+1}) = -\frac{m_{k+1}h_{k+1}}{3} - \frac{m_{k+2}h_{k+1}}{6} + d_{k+1}$$

where

$$d_k = \frac{y_{k+1} - y_k}{h_k}, \ k = 0, 1, \cdots, n-1$$

Using (1.1.31) we obtain the following system having $n+1$ unknowns and $n-1$ equations.

$$(\mathbf{I}) \begin{cases} \ m_k h_k + 2m_{k+1}(h_k + h_{k+1}) + m_{k+2}h_{k+1} = 6(d_{k+1} - d_k), \\ k = 0, 1, 2, \cdots, n-2. \end{cases} \tag{1.1.35}$$

Now we need to close the system by adding two more equations from $S''(x_0) = 0$ and $S''(x_n) = 0$ which leads to

$$m_0 = 0, \qquad m_n = 0. \tag{1.1.36}$$

This is called the natural spline.

The system (1.1.35) and (1.1.36) lead to

$$(I.NAT) \begin{cases} (2h_0 + 2h_1)m_1 + h_1 m_2 = u_0 \\ m_k h_k + 2m_{k+1}(h_k + h_{k+1}) + m_{k+1}h_{k+1} = u_k, \ 1 \le k \le n-3 \\ h_{n-2}m_{n-2} + 2(h_{n-2} + h_{n-1})m_{n-1} = u_{n-2} \end{cases}$$

In matrix form we write

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 & \cdots & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \ddots & 0 \\ 0 & h_2 & 2(h_2+h_3) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & h_{n-2} \\ 0 & \cdots & 0 & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_{n-1} \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-2} \end{bmatrix}$$

The resulting matrix is strictly symmetric positive definite and diagonally dominant and yields a unique solution. □

Other splines include:

Not-a-Knot Spline:

We add the following two conditions:

$S_0'''(x_1) = S_1'''(x_1)$ which leads to

$$\frac{m_1 - m_0}{h_0} = \frac{m_2 - m_1}{h_1} \tag{1.1.37}$$

and the condition

$S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$ which leads to

$$\frac{m_n - m_{n-1}}{h_{n-1}} = \frac{m_{n-1} - m_{n-2}}{h_{n-2}} \tag{1.1.38}$$

Solve (1.1.37) and (1.1.38) for m_0 and m_n to obtain

$$m_0 = (1 + h_0/h_1)m_1 - (h_0/h_1)m_2 \tag{1.1.39}$$

$$m_n = -h_n/h_{n_1}m_{n-2} + (1 + h_n/h_{n-1})m_{n-1} \tag{1.1.40}$$

Substitute into the system (1.1.35) to obtain

$$(I.NK) \begin{cases} (3h_0 + 2h_1 + h_0^2/h_1)m_1 + (h_1 - h_0^2/h_1)m_2 = u_0 \\ m_k h_k + 2m_{k+1}(h_k + h_{k+1}) + m_{k+1}h_{k+1} = u_k, \;\; k = 1, \cdots, n-3 \\ (h_{n-2} - h_{n-1}^2/h_{n-2})m_{n-2} + (2h_{n-2} + 3h_{n-1} + h_{n-1}^2/h_{n-2})m_{n-1} = u_{n-2} \end{cases}$$

where $u_k = 6(d_{k+1} - d_k)$, $k = 0, 1, \cdots, n-2$. We solve the system for $m_1, m_2, \cdots m_{n-1}$ and use (1.1.39) and (1.1.40) to find m_0 and m_n.

Use (1.1.29) to find p_k and $q_k, k = 0, 1, \cdots n-1$. Finally, we use the formula (1.1.30) that defines $S_k(x)$.

Clamped Spline:

We close the system (I) using the following conditions

$S_0^{'}(x_0) = f^{'}(x_0)$

$S_{n-1}^{'}(x_n) = f^{'}(x_n)$

$$S_0'(x) = -\frac{m_0}{2h_0}(x-x_1)^2 + \frac{m_1}{2h_0}(x-x_0)^2 - (\frac{y_0}{h_0} - \frac{m_0 h_0}{6}) + (\frac{y_1}{h_0} - \frac{m_1 h_0}{6})$$

$$S_0'(x_0) = -\frac{m_0 h_0}{3} - \frac{m_1 h_0}{6} + \frac{y_1 - y_0}{h_0}.$$

The boundary equation $S_0'(x_0) = f'(x_0)$ leads to

$$2m_0 h_0 + m_1 h_0 = 6(d_0 - f'(x_0)) \tag{1.1.41}$$

The boundary condition $S_{n-1}'(x_n) = f'(x_n)$ yields the equation

$$S_{n-1}'(x_n) = \frac{m_n h_{n-1}}{3} + \frac{m_{n-1} h_{n-1}}{6} + d_{n-1} = f'(x_n),$$

which leads

$$2m_n h_{n-1} + m_{n-1} h_{n-1} = 6(f'(x_n) - d_{n-1}) \tag{1.1.42}$$

Now, the system (I) is reduced to

$$(I.CL) \begin{cases} (\frac{3h_0}{2} + 2h_1)m_1 + h_1 m_2 = u_0 - 3(d_0 - f'(x_0)) \\ m_k h_k + 2m_{k+1}(h_k + h_{k+1}) + m_{k+1} h_{k+1} = u_k, \ \ k = 1, 2, \cdots, n-3 \\ h_{n-2} m_{n-2} + (2h_{n-2} + \frac{3h_{n-1}}{2}) m_{n-1} = u_{n-2} - 3(f'(x_n) - d_{n-1}) \end{cases}$$

Matrix formulation for the not-a-knot spline

$$AM = U$$

where

$$A = \begin{bmatrix} 3h_0 + 2h_1 + \frac{h_0^2}{h_1} & h_1 - \frac{h_0^2}{h_1} & 0 & \cdots & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \ddots & 0 \\ 0 & h_2 & 2(h_2+h_3) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & h_{n-2} \\ 0 & \cdots & 0 & h_{n-2} - \frac{h_{n-1}^2}{h_{n-2}} & 2h_{n-2} + 3h_{n-1} + \frac{h_{n-1}^2}{h_{n-2}} \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-2} \end{bmatrix}$$

The system admits a unique solution since the matrix is strictly diagonally dominant.

Example 1: consider the function $f(x) = x/(2+x)$ at $-1, 1, 2, 3$

x_i	$f(x_i)$	$1^{st}DD$	6*(2nd Diff)
-1	-1		
		$2/3$	
1	$1/3$		-3
		$1/6$	
2	$1/2$		$-2/5$
		$1/10$	
3	$3/5$		

$h_0 = 2$, $h_1 = 1$, $h_2 = 1$.

$$\begin{bmatrix} 3h_0 + 2h_1 + h_0^2/h_1 & h_1 - h_0^2/h_1 \\ h_1 - h_2^2/h_1 & 2h_1 + 3h_2 + h_2^2/h_1) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -3 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2/5 \end{bmatrix}$$

The solution is $m_1 = -4/15$, $m_2 = -1/15$

Matrix formulation for the clamped spline

$$\begin{bmatrix} (\frac{3h_0}{2}+2h_1) & h_1 & 0 & \cdots & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \ddots & 0 \\ 0 & h_2 & 2(h_2+h_3) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & h_{n-2} \\ 0 & \cdots & 0 & h_{n-2} & (2h_{n-2}+\frac{3h_{n-1}}{2}) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} u_0 - 3(d_0 - f'(x_0)) \\ u_1 \\ u_2 \\ \vdots \\ u_{n-2} - 3(f'(x_n) - d_{n-1}) \end{bmatrix}$$

The system admits a unique solution since the matrix is symmetric diagonally dominant and positive definite (SPD).

Example 1:

Let us interpolate the function $f(x) = x/(2+x)$ where $f'(x) = 2/(2+x)^2$. Thus $S_0'(x_0) = 2 = f'(-1)$ and $S_{n-1}'(x_3) = 2/25 = f'(3)$.

To compute the $u_i,\ i = 0, 1$ we use the following table

x_i	$f(x_i)$	$1^{st}DD$	6*(2nd Diff)
-1	-1		
		$2/3$	
1	$1/3$		-3
		$1/6$	
2	$1/2$		$-2/5$
		$1/10$	
3	$3/5$		

$h_0 = 2,\ h_1 = 1,\ h_2 = 1.$

$$\begin{bmatrix} 3h_0/2 + 2h_1 & h_1 \\ h_1 & 2h_1 + 3h_2/2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} u_0 - 3(d_0 - f'(x_0)) \\ u_1 - 3(f'(x_3) - d_2) \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 1 & 7/2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -17/50 \end{bmatrix}$$

The solution is $m_1 = 64/275$, $m_2 = -9/55$

$m_0 = 3\frac{f[x_0,x_1]-f'(x_0)}{h_0} - m_1/2 = -582/275$

$m_3 = 3\frac{f'(x_3)-f[x_2,x_3]}{h_2} - m_2/2 = 6/275$

On [-1,1] :

$p_0 = y_0/h_0 - (m_0 h_0)/6 = 113/550$, $q_0 = y_1/h_0 - (m_1 h_0)/6 = 49/550$

$$S_0(x) = \frac{582}{12 \times 275}(x-1)^3 + \frac{64}{12 \times 275}(x+1)^3$$
$$+\frac{113}{550}(1-x) + \frac{49}{550}(x+1)$$

On $[1, 2]$

$p_1 = y_1/h_1 - (m_1 h_1)/6 = 81/275$,

$q_1 = y_2/h_1 - (m_2 h_1)/6 = 29/55$

$$S_1(x) = -\frac{64}{6 \times 275}(x-2)^3 - \frac{9}{6 \times 55}(x-1)^3$$
$$+81(2-x)/275 + 29(x-1)/55$$

On [2,3]

$p_2 = y_2/h_2 - (m_2 h_2)/6 = 29/55$, $q_2 = y_3/h_2 - (m_3 h_2)/6 = 164/275$

$$S_2(x) = \frac{9}{6 \times 55}(x-3)^3 - \frac{1}{275}(x-2)^3$$
$$+29(3-x)/55 + 164(x-2)/275$$

Examples of natural spline approximations

Example 1: Let $f(x) = |x|$ and construct the cubic spline interpolation at the points $x_i = -2 + i,\ i = 0, 1, 2, 3, 4$

x_i	$f(x_i)$	$1^{st}DD$	6*(2nd Diff)
-2	2		
		-1	
-1	1		0
		-1	
0	0		12
		1	
1	1		0
		1	
2	2		

We note that $h_i = h, i = 0, 1, 2, 3$, $m_0 = m_4 = 0$ which leads to the following system

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

The solution is $m_1 = -6/7$, $m_2 = 24/7$, $m_3 = -6/7$.

On $[-2, -1]$

$p_0 = (y_0 - \frac{m_0 h_0^2}{6})/h_0 = (2 - 0)/1 = 2$ $q_0 = (y_1 - \frac{m_1 h_0^2}{6})/h_0 = (1 + 1/7) = 8/7$

$$S_0(x) = -\frac{1}{7}(x+2)^3 + 2(-1-x) + \frac{8}{7}(x+2)$$

On $[-1, 0]$

$p_1 = (y_1 - \frac{m_1 h_1^2}{6})/h_1 = (2 - 0)/1 = 8/7$

$q_1 = (y_2 - \frac{m_2 h_1^2}{6})/h_1 = (1 + 1/7) = -4/7$

$$S_1(x) = \frac{1}{7}x^3 + \frac{4}{7}(x+1)^3 + \frac{8}{7}(0-x) - \frac{4}{7}(x+1)$$

On $[0,1]$

$p_2 = (y_2 - \frac{m_2 h_2^2}{6})/h_2 = (0 - \frac{24}{7\times 6}) = -4/7$

$q_2 = (y_3 - \frac{m_3 h_2^2}{6})/h_2 = (1 + \frac{1}{6\times 7}) = 8/7$

$$S_2(x) = -\frac{4}{7}(x-1)^3 - x^3/7 - \frac{4}{7}(1-x) + \frac{8}{7}x$$

On $[1,2]$

$p_3 = (y_3 - \frac{m_3 h_3^2}{6})/h_3 = 8/7$, $q_3 = (y_4 - \frac{m_4 h_3^2}{6})/h_3 = 2$

$$S_3(x) = -(2-x)^3/7 + 2(x-1) + 8(2-x)/7$$

$p = (2, 8/7, -4/7, 8/7)$, $q = (8/7, -4/7, 8/7, 2)$

Example 2:

x_i	$f(x_i)$	$1^{st}DD$	6*(2nd Diff)
-1	-1		
		$2/3$	
1	$1/3$		-3
		$1/6$	
2	$1/2$		$-2/5$
		$1/10$	
3	$3/5$		

$h_0 = 2$, $h_1 = 1$, $h_2 = 1$.

Natural cubic spline leads to $m_0 = m_3 = 0$. The other coefficients satisfy the system

$$\begin{bmatrix} 2(h_0+h_1) & h_1 \\ h_1 & 2(h_1+h_2) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2/5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2/5 \end{bmatrix}$$

which admits the following solution $m_1 = -\frac{58}{115}$, $m_2 = \frac{3}{115}$

On $[-1, 1]$

$p_0 = (y_0 - \frac{m_0 h_0^2}{6})/h_0 = -1/2$, $q_0 = (y_1 - \frac{m_1 h_0^2}{6})/h_0 = 77/230$

On $[1, 2]$

$p_1 = (y_1 - \frac{m_1 h_1^2}{6})/h_1 = 48/115$, $q_1 = (y_2 - \frac{m_2 h_1^2}{6})/h_1 = 57/115$

On $[2, 3]$

$p_2 = (y_2 - \frac{m_2 h_2^2}{6})/h_2 = 57/115$, $q_2 = (y_3 - \frac{m_3 h_2^2}{6})/h_2 = 3/5$

$p = [-1/2, 48/115, 57/115]$, $q = [77/230, 57/115, 3/5]$

Matlab commands for splines

```
x=0:1:10;
y=sin(x);
xi=0:0.2:10;
yi = sin(xi);
%piecewise linear interpolation
y1 = interp1(x,y,xi)
plot(x,y,'0',xi,yi) %plot the exact function
hold on;
plot(xi,y1);
%
y2 = interp1(x,y,xi,'spline') % spline interpolation
y3 = interp1(x,y,'cubic') % piecewise cubic interpolation
y4 = spline(x,y,xi) %not-a-knot spline

plot(xi,y4);
plot(xi,y2);
plot(xi,y3);
```

1.7.3 Convergence of cubic splines

We will study the uniform convergence of the clamped cubic spline for $f \in C^4[a,b]$.
We first write the matrix formulation for the clamped cubic spline as

$$\mathbf{AM} = \mathbf{B}$$

where $\mathbf{M} = [m_0, m_1, \cdots, m_n]^t$, $\mathbf{B} = [b_0, b_1, \cdots, b_n]^t$.

Let us recall that

$$S_0'(x_0) = \frac{h_0 m_0}{3} + \frac{h_0 m_1}{6} = \frac{(y_1 - y_0)}{h_0} - f'(x_0)$$

and

$$S_{n-1}'(x_n) = \frac{h_{n-1} m_{n-1}}{6} + \frac{h_{n-1} m_n}{3} = f'(x_n) - \frac{(y_n - y_{n-1})}{h_{n-1}}.$$

We combine (1.1.41), (1.1.42) and (1.1.35) to write the $(n+1) \times (n+1)$ matrix $\mathbf{A}$ as

$$a_{i,j} = \begin{cases} 2, \text{ if } i = j \\ 1, \text{ if } (i,j) = (1,2) \ or \ (i,j) = (n, n-1) \\ \frac{h_i}{h_i + h_{i-1}}, \text{ if } j = i+1, \ 1 < i < n \\ \frac{h_{i-1}}{h_i + h_{i-1}}, \text{ if } j = i-1, \ 1 \le i < n-1 \\ 0, \ otherwise, \end{cases} \quad . \tag{1.1.43}$$

the $n+1$ by $n+1$ matrix A can be written as

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ \ddots & \ddots & & & & \vdots \\ \ddots & \frac{h_{i-1}}{h_i + h_{i-1}} & 2 & \frac{h_i}{h_i + h_{i-1}} & \dots & 0 \\ \ddots & \ddots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix} \tag{1.1.44}$$

The right-hand side $\mathbf{B}$ is defined by

$$b_0 = \frac{6}{h_0}\left(\frac{y_1 - y_0}{h_0} - f'(x_0)\right), \tag{1.1.45}$$

$$b_i = \frac{6}{h_i + h_{i-1}}(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}),\ i = 1, 2, \cdots, n-1, \tag{1.1.46}$$

$$b_n = \frac{6}{h_{n-1}}(f'(x_n) - \frac{y_n - y_{n-1}}{h_{n-1}}). \tag{1.1.47}$$

Lemma 1.7.1. *Let* $\mathbf{A}$ *be defined in (1.1.43) such that* $\mathbf{Az} = \mathbf{w}$*. Then,*

$$||\mathbf{z}||_\infty \leq ||\mathbf{w}||_\infty$$

Proof. Let z_k be such that $|z_k| = ||\mathbf{z}||_\infty$. The k^{th} equation from $\mathbf{Az} = \mathbf{w}$ leads to

$$a_{k,k-1}z_{k-1} + 2z_k + a_{k,k+1}z_{k+1} = w_k \tag{1.1.48}$$

Applying the triangle inequality we have

$$||\mathbf{w}||_\infty \geq |w_k| = |a_{k,k-1}z_{k-1} + 2z_k + a_{k,k+1}z_{k+1}| \tag{1.1.49}$$

$$\geq 2|z_k| - a_{k,k-1}|z_{k-1}| - a_{k,k+1}|z_{k+1}| \tag{1.1.50}$$

$$\geq (2 - (a_{k,k-1} + a_{k,k+1}))|z_k|. \tag{1.1.51}$$

Using the fact that $a_{k,k-1} + a_{k,k+1} = 1$ we complete the proof. □

In order to state the following lemma we let $\mathbf{F} = [f''(x_0), f''(x_1), \cdots, f''(x_n)]^t$, $\mathbf{R} = \mathbf{B} - \mathbf{AF} = \mathbf{A}(\mathbf{M} - \mathbf{F})$ and $H = \max_{i=0,\cdots n-1} h_i$.

Lemma 1.7.2. *If* $f \in C^4[a,b]$ *and* $||f^{(4)}||_\infty \leq M_4$*, then*

$$||\mathbf{M} - \mathbf{F}||_\infty \leq ||\mathbf{R}||_\infty \leq \frac{3}{4}M_4 H^2. \tag{1.1.52}$$

Proof.

$$r_0 = b_0 - 2f''(x_0) - f''(x_1) = \frac{6}{h_0}(\frac{y_1 - y_0}{h_0} - f'(x_0)) - 2f''(x_0) - f''(x_1). \tag{1.1.53}$$

Using Taylor expansion we write

$$\frac{y_1 - y_0}{h_0} = \frac{f(x_0 + h_0) - f(x_0)}{h_0} \tag{1.1.54}$$

$$= \frac{1}{h_0}(h_0 f'(x_0) + \frac{h_0^2 f''(x_0)}{2} + \frac{h_0^3 f'''(x_0)}{6} + \frac{h_0^4 f^{(4)}(\tau_1)}{24}) \tag{1.1.55}$$

$$f''(x_0 + h_0) = f''(x_0) + h_0 f'''(x_0) + \frac{h_0^2 f^{(4)}(\tau_2)}{2} \tag{1.1.56}$$

which leads to

$$r_0 = \frac{h_0^2 f^{(4)}(\tau_1)}{4} - \frac{h_0^2 f^{(4)}(\tau_2)}{2} \tag{1.1.57}$$

Thus,

$$|r_0| < 3H^2 M_4/4. \tag{1.1.58}$$

Similarly for

$$r_n = b_n - f''(x_{n-1}) - 2f''(x_n). \tag{1.1.59}$$

$$b_n = \frac{6}{h_{n-1}}(f'(x_n) - \frac{y_n - y_{n-1}}{h_{n-1}}) \tag{1.1.60}$$

Using Taylor series

$$\frac{f(x_n) - f(x_{n-1})}{h_{n-1}} = -\frac{f(x_{n-1}) - f(x_n)}{h_{n-1}} =$$

$$\frac{-1}{h_{n-1}}(-h_{n-1} f'(x_n) + \frac{h_{n-1}^2}{2} f''(x_n) - \frac{h_{n-1}^3}{6} f'''(x_n) + \frac{h_{n-1}^4}{24} f^{(4)}(\tau_1)). \tag{1.1.61}$$

$$f''(x_{n-1}) = f''(x_n - h_{n-1}) = f''(x_n) - \frac{h_{n-1}}{2} f'''(x_n) + \frac{h_{n-1}^2}{2} f^{(4)}(\tau_2) \quad (1.1.62)$$

$$|r_n| < 3H^2 M_4/4. \quad (1.1.63)$$

$$r_j = b_j - \mu_j f''(x_{j-1}) - 2f''(x_j) - \lambda_j f''(x_{j+1}), \quad (1.1.64a)$$

where

$$\mu_j = \frac{h_{j-1}}{h_j + h_{j-1}}, \qquad \lambda_j = \frac{h_j}{h_j + h_{j-1}} \quad (1.1.64b)$$

and

$$b_j = \frac{6}{h_{j-1} + h_j}\left(\frac{y_{j+1} - y_j}{h_j} - \frac{y_j - y_{j-1}}{h_{j-1}}\right). \quad (1.1.64c)$$

Using Taylor expansion we write

$$\frac{y_{j+1} - y_j}{h_j} = [f'(x_j) + \frac{h_j f''(x_j)}{2} + \frac{h_j^2 f'''(x_j)}{6} + \frac{h_j^3 f^{(4)}(\tau_1)}{24}], \quad (1.1.64d)$$

$$\frac{y_j - y_{j-1}}{h_{j-1}} = [+f'(x_j) - \frac{h_{j-1} f''(x_j)}{2} + \frac{h_{j-1}^2 f'''(x_j)}{6} - \frac{h_{j-1}^3 f^{(4)}(\tau_2)}{24}], \quad (1.1.64e)$$

$$f''(x_{j-1}) = f''(x_j) - h_{j-1} f'''(x_j) + h_{j-1}^2 f^{(4)}(\tau_3)/2, \quad (1.1.64f)$$

$$f''(x_{j+1}) = f''(x_j) + h_j f'''(x_j) + h_j^2 f^{(4)}(\tau_4)/2. \quad (1.1.64g)$$

Note that $\tau_i \in (x_{j-1}, x_{j+1})$.

Combining (1.1.64) we obtain

$$r_j = \frac{1}{h_j + h_{j-1}}[\frac{h_j^3 f^{(4)}(\tau_1)}{4} + \frac{h_{j-1}^3 f^{(4)}(\tau_2)}{4} - \frac{h_{j-1}^3 f^{(4)}(\tau_3)}{2} - \frac{h_j^3 f^{(4)}(\tau_4)}{2}]. \quad (1.1.65)$$

This can be bounded as

$$|r_j| \leq \frac{3}{4} M_4 \frac{h_j^3 + h_{j-1}^3}{h_j + h_{j-1}} \tag{1.1.66}$$

Without loss of generality we assume $h_j \geq h_{j-1}$ and write

$$\frac{h_j^3 + h_{j-1}^3}{h_j + h_{j-1}} = h_j^2 \frac{1 + (\frac{h_{j-1}}{h_j})^3}{1 + \frac{h_{j-1}}{h_j}} \leq h_j^2 \leq H^2. \tag{1.1.67}$$

Thus,

$$|r_j| < \frac{3}{4} M_4 H^2. \tag{1.1.68}$$

Finally, using Lemma 1.7.1 we have

$$||\mathbf{M} - \mathbf{F}||_\infty \leq ||\mathbf{R}||_\infty \leq \frac{3}{4} M_4 H^2, \tag{1.1.69}$$

which completes the proof. □

Theorem 1.7.5. *Le $f(x) \in C^4[a,b]$, $a \leq x_0 < x_1 < \cdots < x_n \leq b$, $h_j = x_{j+1} - x_j$ and $H = \max_{i=0,\cdots,n-1} h_j$. Assume there exits $K > 0$ independent of H such that*

$$\frac{H}{h_j} \leq K, \; j = 0, 1, \cdots, n-1. \tag{1.1.70}$$

If $S(x)$ is the clamped cubic spline approximation of f at x_i, $i = 0, 1, \cdots, x_n$, then there exists $C_k > 0$ independent of H such that

$$||f^{(k)} - S^{(k)}||_{\infty,[a,b]} \leq C_k M_4 K H^{4-k}, \; k = 0, 1, 2, 3, \tag{1.1.71}$$

where $M_4 = ||f^{(4)}||_\infty$.

Proof. For $k = 3$ and $x \in [x_{j-1} - x_j]$ by adding and subtracting few auxiliary terms the error can be written

$$e'''(x) = f'''(x) - S'''(x) = f'''(x) - \frac{m_j - m_{j-1}}{h_{j-1}}$$

$$= f'''(x) - \frac{m_j - f''(x_j)}{h_{j-1}} + \frac{m_{j-1} - f''(x_{j-1})}{h_{j-1}}$$

$$-\frac{f''(x_j) - f''(x)}{h_{j-1}} + \frac{f''(x_{j-1}) - f''(x)}{h_{j-1}}. \quad (1.1.72)$$

Using Lemma 1.7.2 we bound the following terms

$$|\frac{m_j - f''(x_j)}{h_{j-1}}| \leq \frac{3M_4H^2}{4h_{j-1}}, \; |\frac{m_{j-1} - f''(x_{j-1})}{h_{j-1}}| \leq \frac{3M_4H^2}{4h_{j-1}}. \quad (1.1.73)$$

We use Taylor series to obtain

$$f''(x_j) - f''(x) = (x_j - x)f'''(x) + \frac{(x_j - x)^2}{2} f^{(4)}(\tau_1)$$

and

$$f''(x_{j-1}) - f''(x) = (x_{j-1} - x)f'''(x) + \frac{(x_{j-1} - x)^2}{2} f^{(4)}(\tau_2)$$

we bound the error

$$|e'''(x)| \leq \frac{3M_4H^2}{4h_{j-1}} + \frac{1}{h_{j-1}}|(x_j - x)f'''(x) + (x_j - x)^2 f^{(4)}(\tau_1) \quad (1.1.74)$$

$$-(x_{j-1} - x)f'''(x) - \frac{(x_{j-1} - x)^2}{2} f^{(4)}(\tau_2) - h_{j-1}f'''(x)| \quad (1.1.75)$$

The f''' terms cancel out to give

$$|e'''(x)| \leq \frac{3M_4H^2}{2h_{j-1}} + \frac{M_4}{2h_j}[(x_j - x)^2 + (x_{j-1} - x)^2].$$

Using

$$||(x_j - x)^2 + (x_{j-1} - x)^2||_{\infty,[x_{j-1},x_j]} = h_{j-1}^2 \leq H^2,$$

we write

$$|e'''(x)| \leq \frac{3M_4H^2}{2h_{j-1}} + \frac{M_4H^2}{2h_{j-1}}. \quad (1.1.76)$$

Since $H/h_j \leq K$ we have

$$|f'''(x) - S'''(x)| \leq 2M_4KH, \ \forall x. \tag{1.1.77}$$

For $k = 2$, we note that for each $x \in (a, b)$, there is x_j such that $|x_j - x| \leq H/2$. Now we rewrite $e''(x)$ as

$$e''(x) = f''(x) - S''(x) = f''(x_j) - S''(x_j) + \int_{x_j}^{x} (f'''(t) - S'''(t))dt \tag{1.1.78}$$

Using $|\int g| < \int |g|dx$ and Lemma 1.7.2 we obtain

$$|e''(x)| \leq \frac{3M_4H^2}{4} + |(x - x_j)| \max_{x \in [a,b]} ||f''' - S'''|| \tag{1.1.79}$$

$$\leq \frac{3M_4H^2}{4} + M_4KH^2 \leq \frac{7KM_4H^2}{4}, \ K > 1. \tag{1.1.80}$$

Thus,

$$||f''(x) - S''(x)||_\infty \leq \frac{7KM_4H^2}{4}. \tag{1.1.81}$$

For $k = 1$, we consider $e(t) = f(t) - S(t)$, since $e(x_j) = 0$, by Rolle's theorem there exist $\xi_j, j = 0, 1, \cdots, n-1$ such that $e'(\xi_i) = 0$ and $e'(x_0) = e'(x_n) = 0$. For every $x \in [a, b]$ there exists ξ_i such that $|x - \xi_i| \leq H$ and $e'(x)$ can be written as

$$f'(x) - S'(x) = \int_{\xi_i}^{x} (f''(t) - S''(t))dt \tag{1.1.82}$$

Thus,

$$|e'(x)| \leq |x - \xi_i| ||e''||_\infty \leq \frac{7}{4} M_4KH^3. \tag{1.1.83}$$

For $k = 0$, for every $x \in [a,b]$ there is x_j such that $|x - x_j| \leq H/2$ we also write

$$f(x) - S(x) = \int_{x_j}^{x} (f'(t) - S'(t))dt \tag{1.1.84}$$

$$|e(x)| \leq |x - x_j| ||e'||_\infty \leq \frac{7}{8} M_4 K H^4. \tag{1.1.85}$$

□

We conclude that $S^{(k)}$ converges uniformly to $f^{(k)}$ for $k = 0, 1, 2, 3$ and $H \to 0$.

Optimal bounds are proved by Birkhoff and De Boor (Burden and Faires) as

$$||f - S||_\infty < \frac{5}{384} M_4 H^4. \tag{1.1.86}$$

We recall the Hermite interpolation error

$$|f - H_3| < \frac{1}{24 \times 16} M_4 H^4 = \frac{M_4 H^4}{384}, \tag{1.1.87}$$

Comparing the cubic spline and Hermite interpolation errors, we see that the ratio between the spline and the Hermite errors is only 5. We also note that Hermite interpolation requires the derivative at all the interpolation points while the clamped spline needs the derivatives at the end points only.

Optimality of Splines: The optimality is in the sense that cubic spline has the smallest curvature. For a curve defined by $y = f(x)$ the curvature is defined as

$$\tau = \frac{|f''(x)||}{[1 + (f'(x))^2]^{3/2}}.$$

Here the curvature is approximated by $|f''(x)|$ and $\int_a^b S''(x)^2 dx$ is minimized. More precisely we state the following theorem.

Theorem 1.7.6. *If $f \in C^2[a,b]$ and $S(x)$ is the natural cubic spline that interpolates f at $n+1$ points x_i, $i = 0, 1, \cdots, n$, then*

$$\int_a^b S''(x)^2 dx \leq \int_a^b f''(x)^2 dx. \tag{1.1.88}$$

Proof. We consider the function $e(x) = f(x) - S(x)$ with $e(x_i) = 0$, $i = 0, 1, \cdots, n$, and write the approximate curvature of $f = S + e$ as

$$\int_a^b f''(x)^2 dx = \int_a^b (S''(x) + e''(x))^2 dx =$$

$$\int_a^b S''(x)^2 dx + \int_a^b e''(x)^2 dx + 2\int_a^b S''(x)e''(x)dx. \tag{1.1.89}$$

We complete the proof by showing that the last term in the right-hand side of (1.1.89) is 0.

Integrating by parts we obtain

$$\int_{x_i}^{x_{i+1}} e''(x)S''(x)dx = S''(x)e'(x)|_{x=x_i}^{x=x_{i+1}} - \int_{x_i}^{x_{i+1}} S'''(x)e'(x)dx.$$

Summing over all intervals, using the fact that $e \in C^2$ and $S''(a) = S''(b) = 0$. Noting that $S'''(x) = C_i$ is a constant on (x_i, x_{i+1}) we obtain

$$\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} e''(x)S''(x)dx = -\sum_{i=0}^{n-1} C_i \int_{x_i}^{x_{i+1}} e'(x)dx$$

$$= -\sum_{i=0}^{n-1} C_i[e(x_{i+1}) - e(x_i)] = 0$$

We used the fact that $e(x_i) = 0$ we establish $\int_a^b S''(x)e''(x)dx = 0$. Combining this with (1.1.89) leads to (1.1.88).

□

The same result holds for the clamped cubic spline with $S'(a) = f'(a)$ and $S'(b) = f'(b)$. We follow the same line of reasoning to prove it.
Thus, among all C^2 functions interpolating f at $x_0, \ldots, x_n$, the natural cubic spline has the smallest curvature. This includes the clamped spline and not-a=knot splines.

1.7.4 B-splines

We describe a system of B-splines (B stands for basis) from which other splines can be obtained.

We first start with B-splines of degree 0, i.e., piecewise constant splines defined as

$$B_i^0(x) = \begin{cases} 1 & x_i \leq x < x_{i+1} \\ 0 & otherwise \end{cases}, i \in \mathbf{Z} \tag{1.1.90}$$

Properties of $B_i^0(x)$:

1. $B_i^0(x) \geq 0$, for all x and i
2. $\sum\limits_{i=-\infty}^{\infty} B_i^0(x) = 1$, for all x
3. The support of $B_i^0(x)$ is $[x_i, x_{i+1})$
4. $B_i^0(x) \in C^{-1}$.

We show property (2) by noting that for arbitrary x there exists m such that $x \in [x_m, x_{m+1})$ then write

$$\sum_{i=-\infty}^{\infty} B_i^0(x) = B_m^0(x) = 1.$$

Use the recurrence formula to generate the next basis functions

$$B_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i} B_i^{k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1}^{k-1}(x), \quad for\ k > 0. \tag{1.1.91}$$

Properties of B_i^1:

1. $B_i^1(x)$ are the classical piecewise linear hat functions equal to 1 at x_{i+1} and zero all other nodes.
2. $B_i^1(x) \in C^0$
3. $\sum\limits_{i=-\infty}^{\infty} B_i^1(x) = 1$ for all x

4. The support of $B_i^1(x)$ is (x_i, x_{i+2})
5. $B_i^1(x) \geq 0$ for all x and i

In general for arbitrary k one can show that:

1. $B_i^k(x)$ are piecewise polynomials of degree k
2. $B_i^k(x) \in C^{k-1}$
3. $\sum\limits_{i=-\infty}^{\infty} B_i^k(x) = 1$ for all x
4. The support of $B_i^k(x)$ is (x_i, x_{i+k+1})
5. $B_i^k(x) \geq 0$ for all x and i
6. $B_i^k(x)$, $-\infty < i < \infty$ are linearly independent, i.e., they form a basis.

See Figure 1.7.4 for plots of the first four b-splines.

Interpolation using B-splines:

(i) For $k = 0$, we construct a piecewise constant spline interpolation by writing

$$f(x) \approx P_0(x) = \sum_{i=-\infty}^{\infty} c_i B_i^0(x) \tag{1.1.92}$$

Using the properties of $B_i^0(x)$ we show that $c_i = f(x_i)$.

(ii) For $k = 1$, we construct a piecewise linear spline interpolation by writing

$$f(x) \approx P_1(x) = \sum_{i=-\infty}^{\infty} c_i B_i^1(x) \tag{1.1.93}$$

Again using $B_i^1(x_{j+1}) = \delta_{ij}$ we show that $c_i = f(x_{i+1})$.

(ii) For $k = 3$, we construct a piecewise cubic spline interpolation by writing

$$f(x) \approx P_3(x) = \sum_{i=-\infty}^{\infty} c_i B_i^3(x) \tag{1.1.94}$$

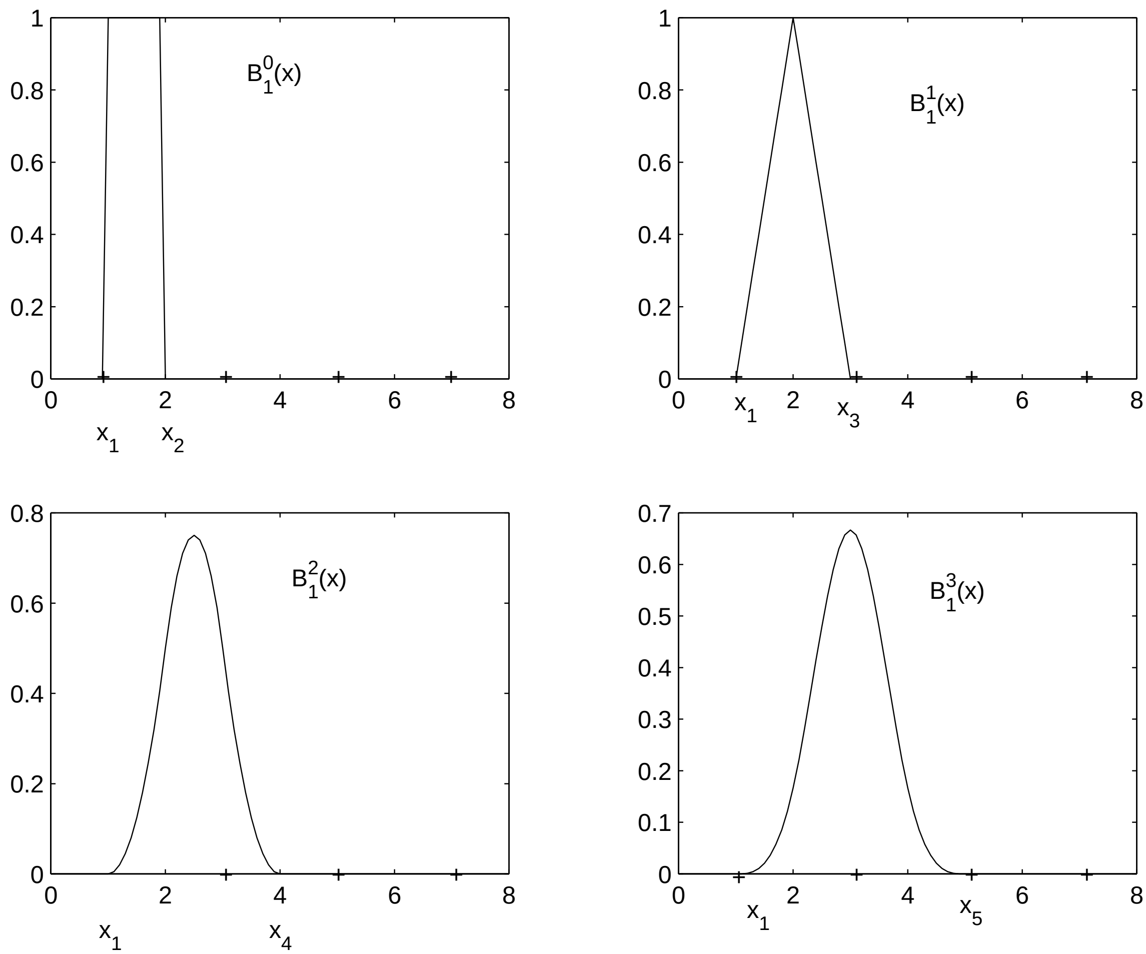


Figure 1.1: B-splines of degree $k = 0, 1, 2, 3$, upper left to lower right.

We recall that $B_i^3 \in C^2$ are piecewise cubic polynomials with support in (x_i, x_{i+3}).

In order to interpolate f at x_i, $i = 0, \cdots, n$, we

1. Write $S(x) = \sum_{i=-3}^{n-1} c_i B_i^3(x)$
   (include basis functions whose support intersect $[x_0, x_n]$).

2. Set $n+1$ equations

$$f(x_i) = c_{i-3}B_{i-3}^3(x_i) + c_{i-2}B_{i-2}^3(x_i) + c_{i-1}B_{i-1}^3(x_i), \quad i = 0, 1, \cdot, n, \tag{1.1.95a}$$

   where $c_{-3}, c_{-2}, c_{-1}, c_0, \cdots, c_n$ are the unknowns.

3. Close the system, for natural Spline, by setting

$$S''(x_0) = 0, \quad S''(x_n) = 0, \tag{1.1.95b}$$

4. Solve the system (1.1.95).

Remarks:

1. If x_i are uniformly distributed we have
   $B_i^2(x_j) = 0$, $j \leq i$ *or* $j \geq i+3$,
   $B_i^2(x_{i+1}) = B_i^2(x_{i+2}) = 1/2$
   $B_i^3(x_j) = 0$, $j \leq i$ *or* $j \geq i+4$,
   $B_i^3(x_{i+1}) = B_i^3(x_{i+3}) = 1/6$, $B_i^3(x_{i+2}) = 2/3$

2. The system (1.1.95) has a unique solution

3. B-splines may be used to construct clamped splines

1.8 Interpolation in multiple dimensions

Read section of 6.10 of textbook (Kincaid and Cheney).

1.9 Least-squares Approximations

Read section 6.8 of Textbook (Kincaid and Cheney).

Bibliography