Algebra 2 (Linear algebra)

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Chapter 1

Vector spaces

A vector space over some field $\mathbb{F}$ is an algebraic structure consisting of a non empty set V on which are defined two binary operations referred to addition, and a scalar multiplication in which elements of the vector space are multiplied by elements of the given field $\mathbb{F}$. These two operations are required to satisfy certain axioms .

1.1 Vector spaces over field $\mathbb{F}$

Let $\mathbb{F}$ be a field and V be an non empty set. Assume that there is a binary operation on V called "addition" which assigns to each pair of elements u and v of V a unique sum $u \oplus v \in V$. Assume that there is a second operation , called "scalar multiplication" which assigns to any $k \in \mathbb{F}$ and any $v \in V$ a unique scalar multiple $k \otimes v \in V$.

Definition 1.1.1 *Let V be a non empty set equipped by two binary operations denoted addition ($\oplus$) and scalar multiplication ($\otimes$). We say that $(V, \oplus, \otimes)$ is a* ***vector space over a field*** $\mathbb{F}$ *if and only if*

❶ $(V, \oplus)$ *is an abelian group.*

❷ *The scalar multiplication satisfies theses conditions* $\forall \alpha, \beta \in \mathbb{F}$, $\forall u, v \in V$

(a) $\alpha \otimes (u \oplus v) = \alpha \otimes u \oplus \beta \otimes v$

(b) $(\alpha + \beta) \otimes u = \alpha \otimes u \oplus \beta \otimes u$

(c) $(\alpha\beta) \otimes u = \alpha \otimes (\beta \otimes v)$

(d) $1 \otimes u = u$

In other words,

Definition 1.1.2 *Let V be a non empty set together with two binary operations addition ($\oplus$) and scalar multiplication ($\otimes$). $(E, \oplus, \otimes)$ is a vector space over $\mathbb{F}$ if the following axioms are satisfied.*

Axioms for vector addition

A1 *If u and v are in V, then $u \oplus v$ is in V.*
V is closed under $\oplus$.

A2 *$u \oplus v = v \oplus u$ for all u and v in V.*
$\oplus$ is commutative

A3 *$u \oplus (v \oplus w) = (u \oplus v) \oplus w$ for all u, v and w in V.*
$\oplus$ is associative

A4 *An element 0_V in V exists such that $v \oplus 0_V = v = 0_V \oplus v$ for every v in V.*
there exists an identity element denoted 0_V.

A5 *For each v in V, an element $-v$ in V exists such that $-v \oplus v = 0$ and $v \oplus (-v) = 0$.*
Each element v has an inverse denoted $-v$ under $\oplus$.

Axioms for scalar multiplication

S1 *If v is in V, then $a \otimes v$ is in V for all a in $\mathbb{F}$.*
V closed under scalar multiplication $\otimes$.

S2 *$a \otimes (v \oplus w) = a \otimes v \oplus a \otimes w$ for all v and w in V and all a in $\mathbb{F}$.*
distributivity property

S3 *$(a + b) \otimes v = a \otimes v \oplus b \otimes v$ for all v in V and all a and b in $\mathbb{F}$.*
distributivity property

S4 *$a \otimes (b \otimes v) = (ab) \otimes v$ for all v in V and all a and b in $\mathbb{F}$.*
associativity of scalar multiplication.

S5 *$1 \otimes v = v$ for all v in V.*

Then V is called a vector space over $\mathbb{F}$.

Remark 1.1.1

1. *The elements of the underlying field $\mathbb{F}$ are called* ***scalars*** *and the elements of the vector space are called* ***vectors****.*

2. *Note also that we often restrict our attention to the case when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.*

3. *A vector space over a field $\mathbb{F}$ is sometimes called an* ***$\mathbb{F}$- vector space*** *or simply* ***$\mathbb{F}$-space*** *. A vector space over the real field is called a* ***real vector space*** *and a vector space over the complex field is called a* ***complex vector space****.*

Example 1.1.1 *Every field $\mathbb{F}$ is a vector space over $\mathbb{F}$.*
$\mathbb{R}$ is a $\mathbb{R}$ - vector space. $\mathbb{C}$ is a $\mathbb{C}$ - vector space.

Example 1.1.2 *Let $\mathbb{F}$ be a field , let $n \in \mathbb{N}^*$. Then the set $\mathbb{F}^n$ of n-tuples of elements of $\mathbb{F}$ is a vector space over $\mathbb{F}$.*

$$\mathbb{F}^n = \underbrace{\mathbb{F} \times \mathbb{F} \times \mathbb{F} \times \times \cdots \mathbb{F}}_{n \text{ times}} = \left\{(x_1, x_2, ..., x_n) \,:\, x_i \in \mathbb{F} \text{ for } i = 1, 2, \cdots, n\right\}$$

where

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n),$$
$$c.(x_1, x_2, ..., x_n) = (cx_1, cx_2, ..., cx_n)$$

for all elements $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ of $\mathbb{F}^n$ and for all elements c of $\mathbb{F}$.

Example 1.1.3 *Soit X a non empty set and V a $\mathbb{F}$-space. we denote $\mathscr{F}(X, V) = \{f : X \longrightarrow V, f \text{ function}\}$* we define two binary operations over $\mathscr{F}(X, V)$

$$\oplus \quad \mathscr{F}(X, V) \times \mathscr{F}(X, V) \longrightarrow \mathscr{F}(X, V) \qquad \otimes \quad \mathbb{F} \times \mathscr{F}(X, V) \longrightarrow \mathscr{F}(X, V)$$
$$(f, g) \longmapsto f \oplus g \qquad (\lambda, f) \longmapsto \lambda \otimes f$$
$$(f \oplus g)(x) = f(x) + g(x) \qquad (\lambda \otimes f)(x) = \lambda f(x)$$

We show that $(\mathscr{F}(X, V), \oplus, \otimes)$ is a $\mathbb{F}$-space.

Example 1.1.4 *Let $\mathbf{P}_2$ be the set of all polynomials of degree at most 2 with coefficients from a field $\mathbb{F}$, i.e., expressions of the form*

$$p(x) = ax_2 + bx + c, \text{ where } a, b, c \in \mathbb{F}.$$

Define addition and scalar multiplication of polynomials in the usual way, i.e.,

$$(ax_2 + bx + c) + (a^{â^2} x^2 + b^{â^2} x + c^{'}) = (a + a^{â^2})x^2 + (b + b^{â^2})x + (c + c^{â^2})$$
$$\alpha.(ax^2 + bx + c) = \alpha ax^2 + \alpha bx + \alpha c.$$

Then $\mathbf{P}_2$ *is a vector space .*

Proposition 1 *Let* $(V, \oplus, \otimes)$ *be a vector space over a field* $\mathbb{F}$*. for all elements* c *of* $\mathbb{F}$ *and elements* v *of* V *.The following properties are satisfied*

1. $c \otimes 0_V = 0_V$
2. $0_{\mathbb{F}} \otimes v = 0_V$
3. $(-1) \otimes v = -v$
4. $(-c) \otimes v = -(c \otimes v) = c \otimes (-v)$
5. $(\alpha - \beta) \otimes v = \alpha \otimes v - \beta \otimes v.$
6. *If* $c \otimes v = 0_V$ *then* $c = 0_{\mathbb{F}}$ *or* $v = 0_V$.

Proof 1.1.0.1

1. *The zero element* 0_V *of* V *satisfies* $0_V \oplus 0_V = 0_V$. *Therefore* $c \otimes 0_V \oplus c \otimes 0_V = c \otimes (0_V \oplus 0_V) = c \otimes 0_V$. *So* $c \otimes 0_V = 0_V$ *(just add the additive inverse of* $c \otimes 0_V$ *)*

2. *The zero element* $0_{\mathbb{F}}$ *of the field* $\mathbb{F}$ *satisfies* $0_{\mathbb{F}} + 0_{\mathbb{F}} = 0_{\mathbb{F}}$. *Therefore* $0_{\mathbb{F}} \otimes v \oplus 0_{\mathbb{F}} \otimes v = (0_{\mathbb{F}} + 0_{\mathbb{F}}) \otimes v = 0_{\mathbb{F}} \otimes v$ *so* $0_{\mathbb{F}} \otimes v = 0_V$ *(just add the additive inverse of* $0_{\mathbb{F}} \otimes v$*)*

3. $$\begin{aligned} v \oplus ((-1) \otimes v) &= (1 \otimes v) \oplus ((-1) \otimes v) \\ &= (1 + (-1)) \otimes v \\ &= 0_{\mathbb{F}} \otimes v = 0_V. \end{aligned}$$

   So the inverse of v *is* $-v = (-1) \otimes v$

4. *We have* $(-c) \otimes v = ((-1) \otimes c) \otimes v = (-1) \otimes (c \otimes v) = -(c \otimes v)$

5.

$$\begin{aligned}(\alpha-\beta)\otimes v &= (\alpha+(-\beta))\otimes v\\ &= \alpha\otimes v\oplus(-\beta)\otimes v\\ &= \alpha\otimes v-\beta\otimes v\end{aligned}$$

6. *We assume that* $c\otimes v = 0_V$. *If* $c = 0_{\mathbb{F}}$ *then we have* $c\otimes v = 0_V$. *If* $c \neq 0_{\mathbb{F}}$, *since* $\mathbb{F}$ *is a field then* c^{-1} *exists. so* $v = 1\otimes v = (\hat{o}\check{r} c\, c^{-1})\hat{o}\check{r}\otimes v = c^{-1}\otimes(c\otimes v) = c^{-1}\otimes 0_V = 0_V$

Remark 1.1.2 *Be careful, the additive identity of the field is not a vector .*
more generally, nothing in the field is a vector. We regard elements of the field and elements of the vector space as separate.

1.2 Vector subspaces

Definition 1.2.1 *If* V *is a vector space over a field* $\mathbb{F}$, *then a subspace* W *of* V *is a subset* $W\subset V$ *such that* W *is a vector space over* $\mathbb{F}$ *with the same addition and scalar multiplication as* V *.*

Proposition 2 *If* V *is a vector space over a field* $\mathbb{F}$, *and* W *is a subset of* V , *then* W *is a vector subspace of* V *if and only if*

1. $W \neq$
2. *For any* $w, w' \in W$, *then* $w+w' \in W$.
   (W *is closed under addition).*
3. *For any* $\alpha\in\mathbb{F}$ *and* $w\in W$, *then* $\alpha\hat{a}w\in W$.
   (W *is closed under scalar multiplication)*

Example 1.2.1

1. *The set* $\{0\}\subset V$ *is always a subspace of* V *.*
2. *The set* $V\subset V$ *is always a subspace of* V.

Proposition 3 *Let* V *be a vector space over a field* $\mathbb{F}$ *and* H *be a subset of* V *then if* H *is a vector subspace of* V *then* H *contains the identity element of* V *(* $0_V\in H$ *).*

Proof 1.2.0.1 *Since H is a vector subspace of V then H is closed under scalar multiplication, that means $\forall \alpha \in \mathbb{F}, \forall v \in H, \alpha.v \in H$.*
If we put $\alpha = 0_{\mathbb{F}}$, then we have $0_{\mathbb{F}}$. v=$0_V \in V$ (see the proposition above).

1.3 Linear Dependence, Spanning Sets and Bases

Definition 1.3.1 (Linear Combinations) *Let V be an arbitrary vector space over field $\mathbb{F}$, and let $v_1, v_2, \cdots, v_n$ be elements of V. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be scalars (elements of $\mathbb{F}$). An expression of type*

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n$$

*is called a **linear combination** of $v_1, v_2, \cdots, v_n$*

Spanning Set of vectors

Definition 1.3.2 *The collection of all linear combination of element $\{v_1, v_2, \ldots, v_n\}$ is denoted $span\{v_1, v_2, \ldots, v_n\}$.*

$$\boxed{span\{v_1, v_2, \ldots, v_n\} = \left\{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n, \text{ for any } \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}\right\}}$$

Proposition 4 *Let $W = span\{v_1, v_2, \ldots, v_n\}$ be the set of all linear combinations of $v_1, v_2, \ldots, v_n$ then W is a subspace of V.*

Proof 1.3.0.1

1. *Show that W is closed under addition.*
   Let, $\beta_1, \beta_2, \cdots, \beta_n$ be scalars. Then

$$\begin{aligned}\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \cdots + \beta_n v_n \\ = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + (\alpha_3 + \beta_3) v_3 + \cdots + (\alpha_n + \beta_n) v_n\end{aligned}$$

   Thus the sum of two elements of W is again an element of W

2. *Show that W is closed under scalar multiplication.*
   if λ is a scalar, then

$$\lambda(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n) = \lambda\alpha_1 v_1 + \lambda\alpha_2 v_2 + \lambda\alpha_3 v_3 + \cdots + \lambda\alpha_n v_n$$

   is a linear combination of $v_1, v_2, \ldots, v_n$, and hence is an element of W.

3. *We have*

$$0 = 0.v_1 + 0.v_2 + 0.v_3 + \cdots + 0.v_n$$

is an element of W

This proves that W *is a subspace of* V

Definition 1.3.3 *We call* $Span\{v_1, \ldots, v_n\}$ *the subspace* ***spanned*** *(or* ***generated****) by* $\{v_1, \ldots, v_n\}$.
Given any subspace H *of* V, *a* ***spanning*** *(or* ***generating****) set for* H *is a set* $\{v_1, \ldots, v_n\}$ *in* H *such that* $H = Span\{v_1, \ldots, v_n\}$.

Definition 1.3.4 (Span of a set of vectors) *Let* V *be a vector space over some field* $\mathbb{F}$, *and let* S *be a set of vectors (i.e., a subset of* V*). The span of* S *is the set of all linear combinations of elements of* S. *In symbols, we have*

$$\boxed{span\ S = \left\{a_1 u_1 + \ldots + a_k u_k\ :\ u_1, \ldots, u_k \in S \text{ and } a_1, \ldots, a_k \in \mathbb{F}\right\}}.$$

Remark 1.3.1 *Be careful, even when the set* S *is infinite, each individual element* $v \in spanS$ *is a linear combination of only finitely many elements* $u_1, \ldots, u_k$ *of* S. *The definition does not talk about infinite linear combinations* $a_1 u_1 + a_2 u_2 + a_3 u_3 + \ldots$ *Indeed, such infinite sums do not typically exist. However, different elements* $v, w \in spanS$ *can be linear combinations of a different (finite) number of vectors of* S. *For example, it is possible that* v *is a linear combination of* 5 *elements of* S, *and* w *is a linear combination of* 50 *elements of* S.

Definition 1.3.5 *Let* V *be a vector space over the field* $\mathbb{F}$, *and let* $v_1, v_2, \ldots, v_n$ *be elements of* V. *We shall say that* $v_1, v_2, \ldots, v_n$ *are* ***linearly dependent over*** $\mathbb{F}$ *if there exist elements* $a_1, a_2, \cdots, a_n$ *in* $\mathbb{F}$ *not all equal to* 0 *such that*

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n = 0$$

If there do not exist such numbers, then we say that $v_1, v_2, \ldots, v_n$ *are* ***linearly independent****. In other words, vectors* $v_1, v_2, \ldots, v_n$ *are linearly independent if and only if the following condition is satisfied:*

$$\forall a_1, a_2, \ldots, a_n \in \mathbb{F},\ \text{if } a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n = 0 \text{ then } a_i = 0, \text{ for all } i = 1, \ldots, n.$$

1.4 Finite-Dimensional Vector Spaces