Algebra 2 (Linear algebra)

Said AISSAOUI

March 4, 2025

Contents

1 Vector spaces **2**

1.1 Vector spaces over field $\mathbb{F}$ 2

1.2 Vector subspaces 6

1.3 Linear Dependence, Spanning Sets and Bases 9

1.4 Finite dimensional vector spaces 16

1.4.1 Dimension of a vector spaces 16

1.4.2 Sums and direct sums 20

Chapter 1

Vector spaces

A vector space over some field $\mathbb{F}$ is an algebraic structure consisting of a non empty set V on which are defined two binary operations referred to addition, and a scalar multiplication in which elements of the vector space are multiplied by elements of the given field $\mathbb{F}$. These two operations are required to satisfy certain axioms .

1.1 Vector spaces over field $\mathbb{F}$

Let $\mathbb{F}$ be a field and V be a non empty set. Assume that there is a binary operation on V called "addition" which assigns to each pair of elements u and v of V a unique sum $u \oplus v \in V$. Assume that there is a second operation , called "scalar multiplication" which assigns to any $k \in \mathbb{F}$ and any $v \in V$ a unique scalar multiple $k \otimes v \in V$.

Definition 1.1.1 *Let V be a non empty set equipped by two binary operations denoted addition ($\oplus$) and scalar multiplication ($\otimes$). We say that $(V, \oplus, \otimes)$ is a **vector space over a field** $\mathbb{F}$ if and only if*

❶ $(V, \oplus)$ *is an abelian group.*

❷ *The scalar multiplication satisfies these conditions* $\forall \alpha, \beta \in \mathbb{F}, \ \forall u, v \in V$

(a) $\alpha \otimes (u \oplus v) = \alpha \otimes u \oplus \alpha \otimes v$

(b) $(\alpha + \beta) \otimes u = \alpha \otimes u \oplus \beta \otimes u$

(c) $(\alpha\beta) \otimes u = \alpha \otimes (\beta \otimes u)$

(d) $1 \otimes u = u$

In other words,

Definition 1.1.2 *Let V be a non empty set together with two binary operations addition ($\oplus$) and scalar multiplication ($\otimes$). $(V, \oplus, \otimes)$ is a vector space over $\mathbb{F}$ if the following axioms are satisfied.*

Axioms for vector addition

(A1) *V is closed under $\oplus$: if u and v are in V, then $u \oplus v$ is in V.*

(A2) *$\oplus$ is commutative : $u \oplus v = v \oplus u$ for all u and v in V.*

(A3) *$\oplus$ is associative : $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ for all u, v and w in V .*

(A4) *Existence of the identity element : An element 0_V in V exists such that $v \oplus 0_V = v = 0_V \oplus v$ for every v in V.*

(A5) *Existence of the symmetric elements : for each v in V, an element $-v$ in V exists such that $-v \oplus v = 0_V$ and $v \oplus (-v) = 0_V$.*

Axioms for scalar multiplication

(S1) *V closed under scalar multiplication $\otimes$:*
if v is in V, then $a \otimes v$ is in V for all a in $\mathbb{F}$.

(S2) *Distributivity property of multiplication over addition :*
$a \otimes (v \oplus w) = a \otimes v \oplus a \otimes w$ for all v and w in V and all a in $\mathbb{F}$.

(S3) *Distributivity property of scalar multiplication :*
$(a + b) \otimes v = a \otimes v \oplus b \otimes v$ for all v in V and all a and b in $\mathbb{F}$.

(S4) *Associativity of scalar multiplication :*
$a \otimes (b \otimes v) = (ab) \otimes v$ for all v in V and all a and b in $\mathbb{F}$.

(S5) *$1 \otimes v = v$ for all v in V .(Where 1 is the unity element of the filed $\mathbb{F}$).*

Then V is called a vector space over $\mathbb{F}$.

Remark 1.1.1

1. *The elements of the underlying field $\mathbb{F}$ are called* ***scalars*** *and the elements of the vector space are called* ***vectors****.*

2. *Note also that we often restrict our attention to the case when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.*

3. *A vector space over a field $\mathbb{F}$ is sometimes called an* ***$\mathbb{F}$- vector space*** *or simply* ***$\mathbb{F}$- space*** *. A vector space over the real field is called a* ***real vector space*** *and a vector space over the complex field is called a* ***complex vector space****.*

Example 1.1.1 *Every field $\mathbb{F}$ is a vector space over $\mathbb{F}$.*
$\mathbb{R}$ is a $\mathbb{R}$ - vector space. $\mathbb{C}$ is a $\mathbb{C}$ - vector space.

Example 1.1.2 *Let $\mathbb{F}$ be a field , let $n \in \mathbb{N}^*$. Then the set $\mathbb{F}^n$ of n -tuples of elements of $\mathbb{F}$ is a vector space over $\mathbb{F}$.*

$$\mathbb{F}^n = \underbrace{\mathbb{F} \times \mathbb{F} \times \mathbb{F} \times \times \cdots \mathbb{F}.}_{n \text{ times}} = \{(x_1, x_2, \ldots, x_n) \ : \ x_i \in \mathbb{F} \text{ for } i = 1, 2, \cdots, n\}$$

where

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$
$$\lambda.(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

for all elements $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ of $\mathbb{F}^n$ and for all elements λ of $\mathbb{F}$.

Example 1.1.3 *Let X a non empty set and V a $\mathbb{F}$ - vector space. we denote $\mathscr{F}(X, V) = \{f : X \longrightarrow V, f \text{ function}\}$* we define two binary operations over $\mathscr{F}(X, V)$

$$\oplus \quad \mathscr{F}(X, V) \times \mathscr{F}(X, V) \longrightarrow \mathscr{F}(X, V) \qquad \otimes \quad \mathbb{F} \times \mathscr{F}(X, V) \longrightarrow \mathscr{F}(X, V)$$
$$(f, g) \mapsto f \oplus g \qquad (\lambda, f) \mapsto \lambda \otimes f$$
$$(f \oplus g)(x) = f(x) + g(x) \qquad (\lambda \otimes f)(x) = \lambda f(x)$$

We show that $(\mathscr{F}(X, V), \oplus, \otimes)$ is a $\mathbb{F}$- vector space.

Example 1.1.4 *Let $\mathbf{P}_2$ be the set of all polynomials of degree at most 2 with coefficients from a field $\mathbb{F}$, i.e., expressions of the form*

$$p(x) = ax2 + bx + c, \text{ where } a, b, c \in \mathbb{F}.$$

Define addition and scalar multiplication of polynomials in the usual way, i.e.,

$$(ax^2 + bx + c) + (a' x^2 + b' x + c') = (a + a')x^2 + (b + b')x + (c + c')$$
$$\alpha.(ax^2 + bx + c) = \alpha a x^2 + \alpha b x + \alpha c.$$

Then $\mathbf{P}_2$ is a vector space .

Proposition 1.1.1 *Let $(V, \oplus, \otimes)$ be a vector space over a field $\mathbb{F}$. for all elements c of $\mathbb{F}$ and elements v of V .The following properties are satisfied*

1. $c \otimes 0_V = 0_V$
2. $0_{\mathbb{F}} \otimes v = 0_V$
3. $(-1) \otimes v = -v$
4. $(-c) \otimes v = -(c \otimes v) = c \otimes (-v)$
5. $(\alpha - \beta) \otimes v = \alpha \otimes v - \beta \otimes v$.
6. *If* $c \otimes v = 0_V$ *then* $c = 0_{\mathbb{F}}$ *or* $v = 0_V$.

Proof 1.1.0.1

1. *The zero element* 0_V *of* V *satisfies* $0_V \oplus 0_V = 0_V$. *Therefore* $c \otimes 0_V \oplus c \otimes 0_V = c \otimes (0_V \oplus 0_V) = c \otimes 0_V$. *So* $c \otimes 0_V = 0_V$ *(just add the additive inverse of* $c \otimes 0_V$*)*

2. *The zero element* $0_{\mathbb{F}}$ *of the field* $\mathbb{F}$ *satisfies* $0_{\mathbb{F}} + 0_{\mathbb{F}} = 0_{\mathbb{F}}$. *Therefore* $0_{\mathbb{F}} \otimes v \oplus 0_{\mathbb{F}} \otimes v = (0_{\mathbb{F}} + 0_{\mathbb{F}}) \otimes v = 0_{\mathbb{F}} \otimes v$ *so* $0_{\mathbb{F}} \otimes v = 0_V$ *(just add the additive inverse of* $0_{\mathbb{F}} \otimes v$*)*

3. $$\begin{aligned} v \oplus ((-1) \otimes v) &= (1 \otimes v) \oplus ((-1) \otimes v) \\ &= (1 + (-1)) \otimes v \\ &= 0_{\mathbb{F}} \otimes v = 0_V. \end{aligned}$$

   So the inverse of v *is* $-v = (-1) \otimes v$

4. *We have* $(-c) \otimes v = ((-1) \otimes c) \otimes v = (-1) \otimes (c \otimes v) = -(c \otimes v)$

5. $$\begin{aligned} (\alpha - \beta) \otimes v &= (\alpha + (-\beta)) \otimes v \\ &= \alpha \otimes v \oplus (-\beta) \otimes v \\ &= \alpha \otimes v - \beta \otimes v \end{aligned}$$

6. *We assume that* $c \otimes v = 0_V$. *If* $c = 0_{\mathbb{F}}$ *then we have* $c \otimes v = 0_V$. *If* $c \neq 0_{\mathbb{F}}$, *since* $\mathbb{F}$ *is a field then* c^{-1} *exists. so* $v = 1 \otimes v = (c.c^{-1})) \otimes v = c^{-1} \otimes (c \otimes v) = c^{-1} \otimes 0_V = 0_V$

Remark 1.1.2 *Be careful, the additive identity of the field is not a vector .*
more generally, nothing in the field is a vector. We regard elements of the field and elements of the vector space as separate.

1.2 Vector subspaces

Definition 1.2.1 *If V is a vector space over a field $\mathbb{F}$, then a subspace H of V is a subset $H \subset V$ such that H is a vector space over $\mathbb{F}$ with the same addition and scalar multiplication as V .*

Proposition 1.2.1 *If V is a vector space over a field $\mathbb{F}$, and H is a subset of V , then H is a vector subspace of V if and only if*

1. $H \neq \emptyset$
2. *For any $u, v \in H$, then $u + v \in H$.*
   (H is closed under addition).
3. *For any $\alpha \in \mathbb{F}$ and $u \in H$, then $\alpha u \in H$.*
   (H is closed under scalar multiplication)

Proposition 1.2.2 *Let V be a vector space over $\mathbb{F}$. H be a subset of V.*

$$H \text{ is a subspace of } V \iff \begin{cases} H \neq \emptyset. \\ \forall \alpha \in \mathbb{F},\ \forall u, v \in H, \alpha u + v \in H. \end{cases}$$

Proof 1.2.0.1

1. *let's show the direct implication* ($\Longrightarrow$)
   We assume that H is a vector subspace then

   (a) *Since H is a vector subspace, then the identity element belongs to H. thus $H \neq \emptyset$*

   (b) *Since H is a vector subspace then $\forall \alpha \in \mathbb{F}, \forall u, v \in H, \alpha u + v \in H$*

2. *Show the inverse implication* ($\Longleftarrow$),
   we assume that $H \neq \emptyset$ and $\forall \alpha \in \mathbb{F}, \forall u, v \in H, \alpha u + v \in H$ and show that H is closed under addition and scalar multiplication.
   Just take $\alpha = 1$, we obtain $u + v \in H, \ \forall u, v \in H$ so H is closed under addition.
   to show that H is closed under scalar multiplication , just take $v = 0$, we obtain $\alpha u \in H, \ \forall \alpha \in \mathbb{F}, u \in H$.

Proposition 1.2.3 (necessary condition) *Let V be a vector space over a field $\mathbb{F}$ and H be a subset of V.*

$$H \text{ is a vector subspace of } V \implies 0_V \in H$$
$$(\ 0_V \text{ is the identity element of } V)$$

Proof 1.2.0.2 *Since H is a vector subspace of V then H is closed under scalar multiplication, that means $\forall \alpha \in \mathbb{F}, \forall v \in H, \alpha . v \in H$.*
If we put $\alpha = 0_{\mathbb{F}}$, then we have $0_{\mathbb{F}}\ .\ v = 0_V \in V$ (see the proposition above).

Remark 1.2.1 *The converse of this implication is false as this following example shows*

Example 1.2.1 *In the $\mathbb{R}$ - vector space $\mathbb{R}^2$, the subset*

$$F = \left\{(x, y) \in \mathbb{R}^2 \ : \ xy = 0\right\}$$

is not a vector space, although it contains the identity element $(0,0)$. We have F is not closed under addition , $(1,0), (0,1) \in F$ but $(1,0) + (0,1) = (1,1) \notin F$.

Example 1.2.2

1. *The set $\{0\} \subset V$ is always a subspace of V .*

2. *The set $V \subset V$ is always a subspace of V.*

3. $\mathbb{R}^2$ *is an* $\mathbb{R}$ *- vector space.*
   (a) $F_1 = \{(x,y) \in \mathbb{R}^2 \ : \ y = 0\}$ *is an* $\mathbb{R}$ *- vector subspace of* $\mathbb{R}^2$
   (b) $F_2 = \{(x,y) \in \mathbb{R}^2 \ : \ x + y = 2\}$ *is not an* $\mathbb{R}$ *- vector subspace of* $\mathbb{R}^2$
   (c) $F_3 = \{(x,y) \in \mathbb{R}^2 \ : \ x + y = 0\}$ *is an* $\mathbb{R}$ *- vector subspace of* $\mathbb{R}^2$

Proposition 1.2.4 *Let* V *be a* $\mathbb{F}$ *- vector space. We consider a set of vector subspaces* $(H_i)_{i\in I}$ *then* $\bigcap_{i\in I} H_i$ *is a vector subspace.*

Proof 1.2.0.3

1. *From the above proposition the* 0_V *vector is in all subspaces* $H_i, \forall i \in I,$ *then it is in* $\bigcap_{i\in I} H_i$ *which means that* $\bigcap_{i\in I} H_i \neq \emptyset$.
2. $\forall \alpha \in \mathbb{F}, \forall u, v \in \bigcap_{i\in I} H_i$
   $\alpha u + v \in^? \bigcap_{i\in I} H_i$

$$\begin{aligned} u, v \in \bigcap_{i\in I} H_i &\Longrightarrow u, v \in H_i, \forall i \in I. \\ &\Longrightarrow \alpha u + v \in H_i, \forall i \in I. \quad (\textit{because } H_i \textit{ is a vector subspace of } V) \\ &\Longrightarrow \alpha u + v \in \bigcap_{i\in I} H_i. \end{aligned}$$

   Therefore $\bigcap_{i\in I} H_i$ *is a vector subspace of* V.

Remark 1.2.2 *The union of subspaces is not a subspaces, in general.*

Example 1.2.3 $\mathbb{R}^2$ *is a vector space over* $\mathbb{R}$*. Let*

$$F = \{(x,y) \in \mathbb{R}^2 \ : \ x = 0\}.$$
$$G = \{(x,y) \in \mathbb{R}^2 \ : \ y = 0\}.$$

are two vector spaces of $\mathbb{R}^2$.

$$F \cup G = \{(x,y) \in \mathbb{R}^2 \ : \ x = 0 \ or \ y = 0\}.$$

is not a subspace , because it is not closed under the addition. we have

$$(1,0) \in F \cup G \ and \ (0,1) \in F \cup G \ but \ (1,0). + (0,1) = (1,1) \notin F \cup G.$$

Proposition 1.2.5 *The union of two subspaces is a subspace if and only if one of the subspaces is contained in the other.In other words let F_1, F_2 two subset of vector space E. we have this equivalence*

$$F_1 \cup F_2 \text{ is a vector space of } E \iff F_1 \subset F_2 \text{ or } F_2 \subset F_1$$

Proof 1.2.0.4

1. *Show* $(\Longleftarrow)$ *This is the easy direction.*
   If $F_1 \subset F_2$ *or* $F_2 \subset F_1$ *then* $F_1 \cup F_2 = F_1$ *or* $F_1 \cup F_2 = F_2$ *is a subspace of* E.

2. *Show* $(\Longrightarrow)$) *This is the harder direction.*
   We suppose that $F_1 \cup F_2$ *is a subspace , prove that* $F_1 \subset F_2$ *or* $F_2 \subset F_1$. *By using a contradiction reasoning, we assume that* $F_1 \not\subseteq F_2$ *and* $F_2 \not\subseteq F_1$
   which means there exists x, y *such that* $x \in F_1$ *and* $x \notin F_2$ *and* $y \in F_2$ *and* $y \notin F_1$.
   We have

$$\begin{aligned} x, y \in F_1 \cup F_2 &\Longrightarrow x + y \in F_1 \cup F_2 \\ &\Longrightarrow x + y \in F_1 \text{ or } x + y \in F_2 \\ &\Longrightarrow (-x) + (x + y) \in F_1 \text{ or } (x + y) + (-y) \in F_2 \\ &(\text{because } F_1 \text{ and } F_2 \text{ are subspaces then the inverse of } x \in F_1 \text{ exist} \\ &\text{and the inverse of } y \in F_2 \text{ exist also.} \\ &\Longrightarrow y \in F_1 \text{ or } x \in F_2 \end{aligned}$$

Contradiction , because we assumed that $y \notin F_1$ *and* $x \notin F_2$. *Therefore, the union of two subspaces is a subspace if and only if one of the subspaces is contained in the other.*

1.3 Linear Dependence, Spanning Sets and Bases

Definition 1.3.1 (Linear Combinations) *Let* V *be an arbitrary vector space over field* $\mathbb{F}$, *and let* $v_1, v_2, \cdots, v_n$ *be elements of* V. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be scalars (elements of* $\mathbb{F}$*). An expression of type*

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n$$

is called a ***linear combination*** *of* $v_1, v_2, \ldots, v_n$

Spanning Set of vectors

Definition 1.3.2 *The collection of all linear combinations of elements* $v_1, v_2, \ldots, v_n$ *is denoted* $span\{v_1, v_2, \ldots, v_n\}$. *or* $\langle v_1, v_2, \ldots, v_n\rangle$.

Proposition 1.3.1 *Let* $W = span\{v_1, v_2, \ldots, v_n\}$ *be the set of all linear combinations of* $v_1, v_2, \ldots, v_n$ *then* W *is a subspace of* V.

Proof 1.3.0.1

1. *Show that* W *is closed under addition.*
   Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$ *be scalars. Then*

$$\begin{aligned}&\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + \cdots + \beta_n v_n \\ &\quad = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + (\alpha_3 + \beta_3) v_3 + \cdots + (\alpha_n + \beta_n) v_n\end{aligned}$$

   Thus the sum of two elements of W *is again an element of* W.

2. *Show that* W *is closed under scalar multiplication.*
   if λ *is a scalar, then*

$$\lambda(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n) = \lambda\alpha_1 v_1 + \lambda\alpha_2 v_2 + \lambda\alpha_3 v_3 + \cdots + \lambda\alpha_n v_n$$

   is a linear combination of $v_1, v_2, \ldots, v_n$, *and hence is an element of* W.

3. *We have*

$$0 = 0.v_1 + 0.v_2 + 0.v_3 + \cdots + 0.v_n$$

   is an element of W

This proves that W *is a subspace of* V.

Definition 1.3.3 *We call* $Span\{v_1, \ldots, v_n\}$ *the subspace* ***spanned*** *(or* ***generated****) by* $\{v_1, \ldots, v_n\}$.
Given any subspace H *of* V, *a* ***spanning*** *(or* ***generating****) set for* H *is a set* $\{v_1, \ldots, v_n\}$ *in* H *such that* $H = Span\{v_1, \ldots, v_n\}$.

Definition 1.3.4 (Span of a set of vectors) *Let* V *be a vector space over some field* $\mathbb{F}$, *and let* S *be a set of vectors (i.e. a subset of* V*). The span of* S *is the set of all linear combinations of elements of* S. *In symbols, we have*

$$\boxed{span\ S = \left\{a_1 u_1 + \ldots + a_k u_k \ : \ u_1, \ldots, u_k \in S \text{ and } a_1, \ldots, a_k \in \mathbb{F}\right\}}.$$

Remark 1.3.1 *Be careful, even when the set S is infinite, each individual element $v \in spanS$ is a linear combination of only finitely many elements $u_1, \ldots, , u_k$ of S. The definition does not talk about infinite linear combinations $a_1 u_1 + a_2 u_2 + a_3 u_3 + \ldots$.*

Linear dependence

Definition 1.3.5 *Let V be a vector space over the field $\mathbb{F}$, and let $v_1, v_2, \ldots, v_n$ be elements of V. We shall say that $v_1, v_2, \ldots, v_n$ are **linearly dependent over** $\mathbb{F}$ if there exist elements $a_1, a_2, \ldots, a_n$ in $\mathbb{F}$ not all equal to 0 such that*

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n = 0$$

*If there do not exist such numbers, then we say that $v_1, v_2, \ldots, v_n$ are **linearly independent**. In other words, vectors $v_1, v_2, \ldots, v_n$ are linearly independent if and only if the following condition is satisfied:*

$$\forall a_1, a_2, \ldots, a_n \in \mathbb{F}: \; a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n = 0 \implies a_1 = a_2 = \cdots = a_n = 0.$$

Example 1.3.1 *Show that the vectors $(1,1)$ and $(-3,2)$ are linearly independent. Let α, β be two numbers such that*

$$\alpha(1,1) + \beta(-3,2) = (0,0).$$

Writing this equation in terms of components, we find

$$a - 3\beta = 0, \;\; \alpha + 2\beta = 0.$$

This is a system of two equations which we solve for α and β. Subtracting the second from the first, we get $-5\beta = 0$, whence $\beta = 0$. Substituting in either equation, we find $\alpha = 0$. Hence α, β are both 0, and our vectors are linearly independent.

Example 1.3.2

1. *Let $V = \mathbb{K}^n$ and consider the vectors*

$$\begin{aligned} e_1 &= (1,0,0,\ldots,0) \\ e_2 &= (0,1,0,\ldots,0) \\ &\vdots \\ e_n &= (0,0,0,\ldots,1) \end{aligned}$$

Then $e_1, \ldots, e_n$ *are linearly independent. Indeed, let* $\alpha_1, \cdots, \alpha_n$ *be numbers such that*

$$\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$$

since $\alpha_1 e_1 + \cdots + \alpha_n e_n = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, *it follow that all* $\alpha_i = 0$.

2. *Let* V *be the vector space of all functions. Let* $f_1, \cdots, f_n$ *be* n *functions.*
To say that they are linearly dependent is to say that there exists n numbers $\alpha_1, \ldots, \alpha_n$ *an not all equal to* 0 *such that*

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \ + \alpha_n f_n(x) = 0 \text{ for all value of } x.$$

The two functions e^x, e^{2x} *are linearly independent.*

$$\alpha e^x + \beta e^{2x} = 0 \Longrightarrow \alpha = \beta = 0.$$

Theorem 1.3.1 *Let* V *be a vector space. Let* $v_1, \ldots, v_n$ *be linearly independent elements of* V. *Let* $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_n$ *be scalars. Suppose that we have*

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n.$$

Then $\alpha_i = \beta_i, \ \forall i = 1, \ldots, n$.

Proof 1.3.0.2 *Subtracting the right-hand side from the left-hand side, we get*

$$\alpha_1 v_1 + \cdots + \alpha_n v_n - \beta_1 v_1 - \cdots + \beta_n v_n = 0 \Longleftrightarrow (\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n = 0$$

Since $v_1, \ldots, v_n$ *are linearly independent , then we deduce that*

$$\alpha_i - \beta_i = 0, \ \forall i = 1, \ldots, n.$$

Thereby proving our assertion.

Definition of Basis of vector space

If elements $v_1, \ldots, v_n$ of V generate V and in addition are linearly independent, then $\{v_1, \ldots, v_n\}$ is called a **basis** of V. We shall also say that the elements $v_1, \cdots, v_n$ constitute or form a **basis** of V.

Definition 1.3.6 *A collection of vectors in V which is both linearly independent and spans V is called a **basis** of V.*

Example 1.3.3

1. *The set $B = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ is a basis of the vector space $\mathbb{R}^3$. Ideed, First we prove that B spans $\mathbb{R}^3$*
   Given any $(x, y, z) \in \mathbb{R}^3$ we have

   $$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

   So, for any $(x, y, z) \in \mathbb{R}^3$, $(x, y, z) \in span(B)$. So,

   $$\mathbb{R}^3 = Span(B).$$

   Secondly, B is linearly independent, because

   $$\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = (0,0,0) \Longrightarrow \alpha = \beta = \gamma = 0.$$

   So, B is a basis of $\mathbb{R}^3$.

2. *Similarly, a basis of the vector space $\mathbb{R}^n$ is given by the set*

   $$B = \{e_1, e_2, \ldots, e_n\}$$

   where,

   $$\begin{cases} e_1 = & (1,0,0,\ldots,0) \\ e_2 = & (0,1,0,\ldots,0) \\ e_3 = & (0,0,1,\ldots,0) \\ \vdots & \vdots \\ e_n = & (0,0,0,\ldots,1) \end{cases}$$

   *This one is called the **standard basis** of $\mathbb{R}^n$.*

3. *Let P_3 be a vector space of all polynomials of degree less of equal to 3. Then*

   $$B = \{1,\ x,\ x^2,\ x^3\}$$

is a basis of P_3.
Indedd, clearly $span(B) = P_3$. *Also B is linearly independent, because*

$$\alpha_0 1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = 0_{P_3} \Longrightarrow \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

where, 0_{P_3}, *the identity element, is equals to :* $0_{P_3} = 0 + 0.x + 0.x^2 + 0.x^3$. *Also, as in* $\mathbb{R}^n$, *a basis of* P_n, *the vector space of all polynomials of degree less of equal to* n, *is given by the set*

$$B = \{1, x, x^2, \dots, x^n\}$$

we called B *a* ***standard basis*** *of* P_n

Coordinates of a vector

Let V be a vector space, and let $\{v_1, \dots, v_n\}$ be a basis of V. The elements v of V can be represented by n-tuples relative to this basis,

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

The n-tuple $(\alpha_1, \dots, \alpha_n)$ is uniquely determined by v.(according to theorem 1.3.1). We call $(\alpha_1, \dots, \alpha_n)$ the **coordinates** of v with respect to basis, and we call α_i the i-th coordinate.

Example 1.3.4 *Find the coordinates of* $(1,0)$ *with respect to the two vectors* $(1,1)$ *and* $(-1,2)$, *which form a basis. We must find numbers* α, β *such that*

$$\alpha(1,1) + \beta(-1,2) = (1,0).$$

Writing this equation in terms of coordinates, we find

$$\alpha - \beta = 1, \ \alpha + 2\beta = 0.$$

Solving for α *and* β, *we find* $\beta = \frac{-1}{3}$ *and* $\alpha = \frac{2}{3}$. *Hence the coordinates of* $(1,0)$ *with respect to* $(1,1)$ *and* $(-1,2)$ *are* $(-\frac{2}{3}, -\frac{1}{3})$.

Example 1.3.5 *Show that the vectors* $(1,1)$ *and* $(-1,2)$ *form a basis of* $\mathbb{R}^2$. *We have to show that they are* ***linearly independent*** *and that they generate* $\mathbb{R}^2$.

1. *To prove linear independence, suppose that* a, b *are scalars such that*

$$\alpha(1,1)+\beta(-1,2)=(0,0).$$

Then

$$\alpha-\beta=0 \text{ and } \alpha+2\beta=0.$$

Subtracting the first equation from the second , we obtain $3\beta=0$, *so that* $\beta=0$. *then from the first equation,* $\alpha=0$, *thus proving that our vectors are* ***linearly independent***.

2. *Next, let* (x, y) *be an arbitrary element of* $\mathbb{R}^2$. *We have to show that there exist numbers* a, b *such that*

$$(x, y)=\alpha(1,1)+\beta(-1,2).$$

In other words, we must solve the system of equations

$$\begin{cases} -\beta=x \\ \alpha+2\beta=y \end{cases}$$

Again subtract the first equation from the second. We find $3b=y-x$,*whence.*

$$b=\frac{y-x}{3}$$

and finally

$$\alpha=\beta+x=\frac{y-x}{3}+x$$

According to our definitions, (α, β) *are the coordinates of* (x, y) *with respect to the basis* $\{(1,1),(-1,2)\}$.

Exercise 1.3.1 *Let* v, w *be elements of a vector space and assume that* $v \neq 0$. *If* v, w *are linearly dependent, show that there is a scalar* λ *such that* $w=\lambda v$.

Exercise 1.3.2 *Let* (x, y) *and* $(x^{'}, y^{'})$ *be two vectors in the vector space* $\mathbb{R}^2$. *If* $xy^{'}-yx^{'}=0$, *show that they are linearly dependent. If* $xy^{'}-yx^{'}\neq 0$, *show that they are linearly independent.*

Exercise 1.3.3 *Show that the following vectors are linearly independent (over* $\mathbb{C}$ *or* $\mathbb{R}$*).*

1. $(1,1,1)$ *and* $(0,1,-2)$.
2. $(-1,1,0)$ *and* $(0,1,2)$.
3. $(\pi,0)$ *and* $(0,1)$.
4. $(1,1,0),(1,1,1),$ *and* $(0,1,-1)$.
5. $(1,0)$ *and* $(1,1)$.
6. $(2,-1)$ *and* $(1,0)$.
7. $(1,2)$ *and* $(1,3)$.
8. $(0,1,1),(0,2,1),$ *and* $(1,5,3)$.

Exercise 1.3.4 *Express the given vector* X *as a linear combination of the given vectors* A, B, *and find the coordinates of* X *with respect to* A, B.

1. $X=(1,0), A=(1,1), B=(0,1)$.
2. $X=(2,1), A=(1,-1), B=(1,1)$.
3. $X=(1,1), A=(2,1), B=(-1,0)$.
4. $X=(4,3), A=(2,1), B=(-1,0)$.

1.4 Finite dimensional vector spaces

Definition 1.4.1 *A vector V is called **finite dimensional** if it is spanned by a finite set of vectors. Otherwise , V is called infinite dimensional.*

1.4.1 Dimension of a vector spaces

The main result of this section is that any two bases of a vector space have the same number of elements. To prove this, we first have an intermediate result.

Theorem 1.4.1 *Let V be a vector space over the field $\mathbb{F}$. Let $\{v_1, \ldots, v_m\}$ be a basis of V over $\mathbb{F}$. Let $w_1, \ldots, w_n$ be elements of V, and assume that $n > m$. Then $w_1, \ldots, w_n$ are linearly dependent.*

Proof 1.4.1.1 *Assume that* $w_1, \ldots, w_n$ *are linearly independent. Since* $\{v_1, \ldots, v_m\}$ *is a basis, there exist elements* $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ *such that*

$$w_1 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_m v_m$$

Since $w_1, \ldots, w_n$ *are linearly independent, then* $w_1 \neq 0$. *So some scalar* $\alpha_i \neq 0$. *After re- numbering* $v_1, \ldots, v_m$ *if necessary, we may assume without loss of generality that say* $\alpha_1 \neq 0$. *then*

$$\alpha_1 v_1 = w_1 - \alpha_2 v_2 - \alpha_3 v_3 - \cdots - \alpha_n v_n.$$
$$v_1 = \alpha_1^{-1} w_1 - \alpha_1^{-1}\alpha_2 v_2 - \alpha_1^{-1}\alpha_3 v_3 - \cdots - \alpha_1^{-1}\alpha_n v_n$$

The subspace of V *generated by* $w_1, \ldots, v_m$ *contain* v_1, *and hence must be all of* V *since* $v_1, \ldots, v_m$, *generate* V. *The idea is now to continue our procedure stepwise, and to replace successively* $v_2, v_3 \ldots$ *by* $w_1, w_2, \ldots$ *until all the elements* $v_1, \ldots, v_m$ *are exhausted and* $w_1, \ldots, w_m$ *generate* V.
Let us now assume by induction that there is an integer r with $1 < r < m$ *such that, after a suitable renumbering of* $v_1, \ldots, v_m$, *the elements* $w_1, \ldots, w_r, v_{r+1}, \ldots, v_m$ *generate* V. *There exist elements* $\beta_1, \beta_2, \ldots, \beta_r, \gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_m$ *such that*

$$w_{r+1} = \beta_1 w_1 + \cdots + \beta_r w_r + \gamma_{r+1} v_{r+1} + \cdots + \gamma_m v_m.$$

We cannot have $\gamma_j = 0$ *for* $j = r+1, \ldots, m$, *for otherwise, we get a relation of linear dependence between* $w_1, \ldots, w_{r+1}$, *contradicting our assumption. After renumbering* $v_{r+1}, \ldots, v_m$ *if necessary, we may assume without loss of generality that say* $\gamma_{r+1} \neq 0$. *We then obtain*

$$\gamma_{r+1} v_{r+1} = w_{r+1} - \beta_1 w_1 - \cdots - \beta_r w_r - \gamma_{r+2} v_{r+2} - \cdots - \gamma_m v_m.$$

Dividing by γ_{r+1} *we conclude that* v_{r+1} *is in the subspace generated by*
$w_1, \ldots, w_{r+1}, v_{r+2}, \ldots, v_m$

By our induction assumption, it follows that $w_1, \ldots, w_{r+1}, v_{r+2}, \ldots, v_m$ *generate V. Thus by induction, we have proved that* $w_1, \ldots, w_m$ *generate* V. *If* $n > m$, *then there exist elements* $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ *such that*

$$w_n = \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m.$$

therefore , proving that $w_1, \ldots, w_n$ *are linearly dependent. This proves our theorem.*

Theorem 1.4.2 *Let V be a vector space and suppose that one basis has n elements, and another basis has m elements. Then $m = n$.*

Proof 1.4.1.2 *We apply Theorem* (1.4.1) *to the two bases. Theorem 1.4.1 implies that both alternatives $n > m$ and $m > n$ are impossible, and hence $m = n$.*

Definition 1.4.2 *Let V be a vector space having a basis consisting of n elements. We shall say that n is the* ***dimension*** *of V.*

Remark 1.4.1

1. *If $V = \{0\}$, then V does not have a basis, and we shall say that V has dimension* 0.

2. *The dimension of a vector space V over $\mathbb{F}$ will be denoted by* $\dim_{\mathbb{F}} V$, *or simply* $\dim V$.

3. *A vector space which has a basis consisting of a finite number of elements, or the zero vector space, is called* ***finite dimensional****. Other vector spaces are called* ***infinite dimensional****.*

4. *Whenever we speak of the dimension of a vector space in the sequel, it is assumed that this vector space is* ***finite dimensional****.*

Example 1.4.1 *Let $\mathbb{F}$ be a field. Then $\mathbb{F}$ is a vector space over itself, and it is of dimension 1. In fact, the element 1 of $\mathbb{F}$ forms a basis of $\mathbb{F}$ over $\mathbb{F}$, because any element $x \in \mathbb{F}$ has a unique expression as $x = x.1$.*

Example 1.4.2 *The vector space $\mathbb{R}^n$ has dimension n over $\mathbb{R}$, the vector space $\mathbb{C}^n$ has dimension n over $\mathbb{C}$. More generally for any field $\mathbb{F}$, the vector space $\mathbb{F}^n$ has dimension n over $\mathbb{F}$. Indeed, the n vectors*

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, \ldots, 0),\ \ldots,\ e_n = (0, 0, \ldots, 1)$$

form a basis of $\mathbb{F}^n$ over $\mathbb{F}$.

Definition 1.4.3 (Maximal subset of linearly independent) *Let $\{v_1,, \ldots, v_n\}$ be a set of elements of a vector space V. Let r be a positive integer less than n. We shall say that $\{v_1,, \ldots, v_r\}$ is a* ***maximal*** *subset of* ***linearly independent*** *elements if $v_1,, \ldots, v_r$ are linearly independent, and if in addition, given any v_i with $i > r$, the elements $v_1,, \ldots, v_r, v_i$ are* ***linearly dependent****.*

Theorem 1.4.3 *Let* $\{v_1,, \dots, v_n\}$ *be a set of generators of a vector space* V*. Let* $\{v_1, \dots, v_r\}$ *be a* ***maximal*** *subset of linearly independent elements. Then* $\{v_1, \dots, v_r\}$ *is a basis of* V*.*

Proof 1.4.1.3 *We must prove that* $v_1, \dots, v_r$ *generate* V*. We shall first prove that each* v_i *(for* $i > r$*) is a linear combination of* $v_1, \dots, v_r$*. By hypothesis, given* v_i *there exist scalars* $\alpha_1, \dots, \alpha_r, \beta$ *not all* 0*.*

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r + \beta v_i = 0$$

Furthermore, $\beta \neq 0$*, because otherwise, we would have a relation of linear dependence for* $v_1, \dots, v_r$ *Hence we can solve for* v_i

$$v_i = \frac{\alpha_1}{-\beta} v_1 + \frac{\alpha_2}{-\beta} v_2 + \dots + \frac{\alpha_r}{-\beta} v_r$$

thereby showing that V_i *is a linear combination of* $v_1, \dots, v_r$ *.*
Next, let v *be any element of* V*. There exist numbers* $c_1, c_2, \dots, c_n$ *such that*

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

In this relation, we can replace each v_i $(i > r)$ *by a linear combination of* $v_1, \dots, v_r$*, then we collect terms, we find that we have expressed* v *as a linear combination of* $v_1, \dots, v_r$ *This proves that* $v_1, \dots, v_r$ *generate* V *, and hence form a basis of* V*.*

Theorem 1.4.4 *Let* V *be a vector space of dimension* n*, and let* $v_1, \dots, v_n$ *be linearly independent elements of* V*. Then* $v_1, \dots, v_n$ *constitute a basis of* V*.*

Proof 1.4.1.4 *According to the theorem 1.4.1,* $v_1, \dots, v_n$ *is a maximal set of linearly independent elements of* V*. Hence it is a basis by Theorem 1.4.3.*

Proposition 1.4.1 *Let* V *be a* $\mathbb{F}$*-vector space of finite dimension* n*. Let* $B \subset V$ *be a subset of* V *. If* $|B| = n$ *(The cardinality of* B *is equal to* $\dim V$*), then*

$$B \text{ is a basis} \iff B \text{ is linearly independent} \iff B \text{ generates } V.$$

Corollary 1.4.1 *Let* V *be a vector space and let* W *be a subspace. If* $\dim W = \dim V$ *then* $V = W$*.*

Proof 1.4.1.5 *A basis for* W *must also be a basis for* V *by Theorem 1.4.4.*

Corollary 1.4.2 *Let V be a vector space of dimension n. Let r be a positive integer with $r < n$, and let $v_1, \dots, v_r$ be linearly independent elements of V. Then there exist elements $v_{r+1}, \dots, v_n$ such that $\{v_1, v_2, \dots, v_n\}$ is a basis of V.*

Theorem 1.4.5 *Let V be a finite dimensional vector space* $\dim V = n$. *Let W be a subspace which does not consist of 0 alone. Then W has a basis, and* $\dim W \leqslant \dim V$.

1.4.2 Sums and direct sums

Definition 1.4.4 *Let V be a vector space over the field $\mathbb{K}$. Let U, W be subspaces of V. We define the **sum** of U and W to be the subset of V consisting of all sums $u + w$ with $u \in U$ and $w \in W$. We denote this sum by $U + W$*

Proposition 1.4.2 *Let V be a vector space over the field $\mathbb{K}$. Let U, W be subspaces of V. then the subset $U + V$ is a **subspace** of V.*

Proof 1.4.2.1 *Indeed, if $u_1, u_2 \in U$ and $w_1, w_2 \in W$ then $(u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \in U + W$. So, $U + W$ is closed under addition.*
If $c \in \mathbb{K}$, then

$$c(u_1 + w_1) = cu_1 + cw_1 \in U + W.$$

So $U + W$ is closed under scalar multiplication.
We have $0_V + 0_V = 0_V \in U + V$, so $U + V \neq \emptyset$. This prove that $U + W$ is a subspace of V.

Theorem 1.4.6 (Grassmann Formula) *Let V be a vector space and U and W two vector subspaces of V then*

$$\boxed{\dim(U + W) = \dim U + \dim W - \dim(U \cap W)}$$

Proof 1.4.2.2 *Let $B_{U\cap W} = \{v_1, \dots, v_m\}$ be a base of $U \cap W$. If we extend the basis to $B_U = \{v_1, \dots, , v_m, u_{m+1}, \dots, u_r\}$ and $B_W = \{v_1, \dots, v_m, w_{m+1}, \dots, w_s\}$ then*

$$S = \{v_1, v_2, \dots, v_m, u_{m+1}, \dots, u_r, \ w_{m+1}, \dots, w_s\}$$

is a generating set of $U + W$. . Now I have to prove that S is linearly independent:

$$0 = \sum_{ð=1}^{m} \alpha_i v_i + \sum_{j=m+1}^{r} \beta_j u_j + \sum_{k=m+1}^{s} \lambda_k w_k \Longrightarrow v = \sum_{i=1}^{m} \alpha_i v_i + \sum_{j=m+1}^{r} \beta_j u_j = -\sum_{k=m+1}^{s} \lambda_k w_k$$

is a vector of $U \cap W$. *and then*

$$\sum_{j=m+1}^{r} \beta_j u_j = v - \sum_{i=1}^{m} \alpha_i v_i \in U \cap W.$$

So , $\sum_{j=m+1}^{r} \beta_j u_j = 0$ *because the vectors* $u_{m+1}, \ldots, u_r$ *are not in* $V \cap W$.*We deduce that* $\beta_j = 0$, *since* B_U *is independent.*
Therefore

$$0 = \sum_{i=1}^{m} \alpha_i v_i + \sum_{k=m+1}^{s} \lambda_k w_k \tag{1.4.1}$$

and, since B_W *is independent we have that* $\alpha_i = \lambda_k = 0.$ *, and*

$$\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}.$$

Definition 1.4.5 *We shall say that* V *is a* ***direct sum*** *of* U *and* W *if for every element* v *of* V *there exist* ***unique elements*** $u \in U$ *and* $w \in W$ *such that* $v = u + w$. *when* V *is the direct sum of subspaces* U, W *we write*

$$V = U \oplus W$$

Theorem 1.4.7 *Let* V *be a vector space over the field* $\mathbb{K}$, *and let* U, W *be subspaces. If* $U + W = V$, *and if* $U \cap W = \{0\}$, *then* V *is the* ***direct sum*** *of* U *and* W.

Proof 1.4.2.3 *Given* $v \in V$, *by the first assumption, there exist elements* $u \in U$ *and* $w \in W$ *such that* $v = u + w$. *Thus* V *is the sum of* U *and* W. *To prove it is the direct sum, we must show that these elements* u, w *are uniquely determined. Suppose there exist elements* $u' \in U$ *and* $w' \in W$ *such that* $v = u' + w'$. *Thus*

$$u + w = u' + w'.$$

then $u - u' = w' - w$. *But* $u - u' \in U$ *and* $w' - w \in W$ *. By the second assumption, we conclude that* $u - u' = 0$ *and* $w' - w = 0$ *, whence* $u = u'$ *and* $w = w'$*, so proving our theorem.*

Complementary subspaces

Theorem 1.4.8 *Let V be a finite dimensional vector space over the field $\mathbb{F}$. Let W be a subspace. Then there exists a subspace U such that V is the direct sum of W and U.*

Proof 1.4.2.4 *We select a basis of W, and extend it to a basis of V, using Corollary 1.4.2. The assertion of our theorem is then clear. In the notation of that theorem, if $\{v_1, \dots, v_r\}$ is a basis of W, then we let U be the space generated by $\{v_{r+1}, \dots, v_n\}$*

Example 1.4.3 *Let $U = \{(x,0) : x \in \mathbb{R}\}$, $W = \{(0,y) : y\mathbb{R}\}$ be two subspaces of $\mathbb{R}^2$ then*

$$U + W = \{(x,y) : x, y \in \mathbb{R}\}$$

Example 1.4.4 *Let $U = \{(a,0,0) : a \in \mathbb{R}\}$, $W = \{(0,b,0) : a\mathbb{R}\}$ be two subspaces of $\mathbb{R}^3$.Then*

$$U + W = \{(a,b,0) : a, b \in \mathbb{R}\}$$

Example 1.4.5 *Let $U = \{(x,y,0) : x, y \in \mathbb{R}\}$, $W = \{(0,0,z) : z \in \mathbb{R}\}$ two subspaces of $\mathbb{R}^3$ then*

$$U + W = \{(x,y,z) : x, y, z \in \mathbb{R}\} = \mathbb{R}^3$$

One unique way to write

$$(x,y,z) = (x,y,0) + (0,0,z).$$

Any vector in $\mathbb{R}^3$ can be written as a unique way, so U and W are in direct sum of $\mathbb{R}^3$. we write $U \oplus W = \mathbb{R}^3$.

Example 1.4.6 *Let $U = \{(a,b,0) : a, b \in \mathbb{R}\}$, $W = \{(0,c,d) : c, d \in \mathbb{R}\}$ two subspaces of $\mathbb{R}^3$ then*

$$U + W = \{(a, b + c, d) : a, b, c, d \in \mathbb{R}\} = \mathbb{R}^3$$

we can see that there is many way to write an element of $\mathbb{R}^3$ as sum of element of V and element of W.

$$\begin{aligned}(1,2,3) &= (1,2,0) + (0,0,3) \ \text{or} \\ &= (1,0,0) + (0,2,3) \ \text{or} \\ &= (1,1,0) + (0,1,3) \ \text{or} \\ &\ \vdots\end{aligned}$$

so U and W are not in direct sum of $\mathbb{R}^3$

Definition 1.4.6 *Let V be a vector space over $\mathbb{F}$, U and W two subspaces of V. U and W are called* ***complementary subspaces*** *in V if $U+W$ is direct sum and equal to V. Thus*

$$U \text{ and } W \text{ are } \textbf{complements} \text{ in} V \iff V = U \oplus W.$$
$$\iff U \cap W = 0_V \text{ and } U + W = V.$$

Remark 1.4.2

1. *We call U a **complement** of W in V. Note that this complement is not unique in general.*
2. *We note that given the subspace W, there exist usually many subspaces U such that V is the direct sum of W and U.*

Theorem 1.4.9 *If V is a finite dimensional vector space over $\mathbb{F}$, and is the direct sum of subspaces U, W then*

$$\dim V = \dim U + \dim W.$$

Proof 1.4.2.5 *We can apply the grassmann formula, since $U \cap W = \{0_V\}$, then*

$$\begin{aligned} \dim(U+V) &= \dim U + \dim W - \dim(U \cap W) \\ &= \dim U + \dim W - 0 \\ &= \dim U + \dim W. \end{aligned}$$

.

Rank of a set of vectors.

Definition 1.4.7 (Rank) *Let V be a vector space over $\mathbb{F}$ and $S = \{v_1, v_2, \ldots, v_m\}$ be a set of vectors of V. **The rank** of S is the dimension of the subspace spanned by S or, equivalently the maximum number of independent vectors of S.*

Example 1.4.7 *Let $S = \{v_1 = (1,0),\ v_2 = (-1,0), v_3 = (4,0)\}$ be a set of vectors of $\mathbb{R}^2$. we can easily see that the rank of S is 1. ($rank(S) = 1$) there is only one vector which linearly independent.*

Example 1.4.8 *Let* $S = \{w_1 = (1,0,0),\ w_2 = (1,0,1), w_3 = (0,0,1)\}$ *be a set of vectors of* $\mathbb{R}^3$. *we can easily see that the rank of S is 3. (* $rank(S) = 3$*) because S is linearly independent.*

$$\forall \alpha, \beta, \gamma \in \mathbb{R},\ \alpha w_1 + \beta w_2 + \gamma w_3 = (0,0,0) \Longrightarrow \alpha = \beta = \gamma = 0.$$

let to reader the check .