Course of Algebra 1

Said AISSAOUI

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Contents

Chapter 2

Sets and functions

2.1 Sets

2.1.1 Definitions and examples

a set is a collection of objects. The objects in a set are called **elements** or **members**
If an element x is a **member** of A, we write

$$x \in A.$$

and that x is a member of A, or that x **belongs** to A. If x is not in A, we write

$$x \notin A.$$

The elements in the sets are described in either the **Statement form**, **Roster Form** or **Set Builder Form**.

- **Statement Form** In statement form, the well-defined descriptions of a member of a set are written and enclosed in the curly brackets.

  Example 2.1.1 *the set of even numbers less than* 5*. In statement form, it can be written as*

  $$\left\{ \textit{even numbers less than} 5 \right\}.$$

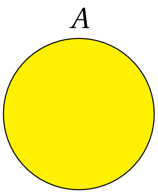
- **Roster Form**, all the elements of a set are listed, separated by commas, within braces;

Example 2.1.2 *The set of natural numbers less than 5. Therefore, the set is* $A = \left\{0, 1, 2, 3, 4\right\}$.

- **Set Builder Form** The general form is, $A = \left\{x : \text{property}\right\}$.

  Example 2.1.3 *Write the following sets in set builder form:* $A = \left\{2, 4, 6, 8\right\}$. *So, the set builder form is* $A = \left\{x : x = 2n, n \in \mathbb{N} \text{ and } 1 \leqslant n \leqslant 4\right\}$.

Also, **Venn Diagrams** are the simple and best way for visualized representation of sets.



Remark 2.1.1

1. *Repeated elements are listed once* $\left\{a, b, a, c, b, a\right\} = \left\{a, b, c\right\}$.
2. *There is no order in the set* $\left\{3, 2, 1\right\} = \left\{1, 2, 3\right\} = \left\{2, 1, 3\right\}$.

Example 2.1.4

1. ***Natural Numbers*** : $\mathbb{N} = \left\{0, 1, 2, \ldots\right\}$
2. ***Integers*** : $\mathbb{Z} = \left\{\ldots, -2, -1, 0\, 1, 2, \ldots\right\}$.
3. ***Rational numbers*** : $\mathbb{Q} = \left\{\frac{a}{b} : a,\ b \in \mathbb{Z} \text{ and } b \neq 0.\right\}$.

2.1.2 Types of Sets

✔ An **empty set** , denoted $\emptyset$ or $\left\{\right\}$, is a set that does not contain any elements.

✔ Let $E = \left\{a\right\}$, consisting of a single element, is called **singleton**.

- ✔ A set which consists of a finite number of elements is called a **finite set**.
  Example: A set of natural numbers up to 10, $A = \{0,1,2,3,4,5,6,7,8,9,10\}$
- ✔ A set which is not finite is called an **infinite set**.
  Example: A set of all natural numbers. $\mathbb{N} = \{0,1,2,3,4,5,6,7,8,9,\dots\}$
- ✔ The two sets A and B are said to be **disjoint** if the set does not contain any common element.
  Example: Set $A = \{a,b,c,d\}$ and set $B = \{e,f,g\}$ are disjoint sets, because there is no common element between them.
- ✔ A is a **subset** of B, written $A \subset B$, if and only if every element of A is also an element of B
- ✔ If A is not a subset of B, we write $A \not\subset B$.
- ✔ A is equal to B, written $A = B$, if and only if for any a we have $a \in A$ if and only if $a \in B$.
- ✔ A is a **proper subset** of B, written $A \subsetneq B$, if and only if $A \subset B$ but $A \neq B$.
  Thus A being a proper subset of B means that A is a subset of B and B contains something that A does not contain.
  There is an important way to rephrase the definition of two sets being equal: $A = B$ if and only if $A \subset B$and $B \subset A$. . This is sometimes useful as a proof technique, as you can split a proof of $A = B$ into first checking $A \subset B$ and then checking $B \subset A$

Definition 2.1.1 *The **cardinality** of a set is a measure of how many elements are in the set. If A a finite set. the we denote the cardinality of A by* $|A|$.

Example 2.1.5

- *If* $A = \{a,2,x,5\}$, *then* $|A| = 4$
- $|\varnothing| = 0$
- *If* $X = \{a,\{2,x\},\{1,\varnothing\}\}$, *then* $|X| = 3$
- $|\{\varnothing\}| = 1$

For any set A, we have $\emptyset \subset A$.
How many subsets can a finite set have?
If A a finite set, then A has $2^{|A|}$ subsets, we prove it by induction.

Definition 2.1.2 *If A a set, the set of all subsets of A is called the **power set** of A, denoted $\mathscr{P}(A)$.*

Example 2.1.6

❶ *if* $A = \left\{a, b, c\right\}$, *then* $\mathscr{P}(A) = \left\{\{\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\right\}$.

❷ $\mathscr{P}(\emptyset) = \left\{\emptyset\right\}$.

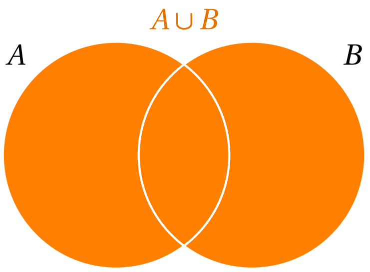
❸ $\mathscr{P}(\{\emptyset\}) = \left\{\emptyset, \{\emptyset\}\right\}$.

2.1.3 Operations on Sets

Union

Let A and B be two set, the union of A and B, denoted as $A \cup B$ (read as A union B) is a set of element that belong to either A or B. The **union** of the sets A and B is the set

$$A \cup B = \left\{x : x \in A \text{ or } x \in B\right\}$$



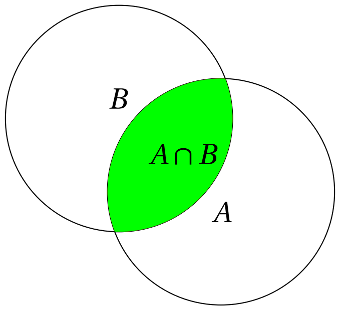
Remark 2.1.2 *We use often these equivalences*

$$\boxed{x \in A \cup B \iff x \in A \text{ or } x \in B}, \qquad \boxed{x \notin A \cup B \iff x \notin A \text{ and } x \notin B.}$$

Intersection

Let A and B be two sets, the **intersection** of A and B, denoted as $A \cap B$ (read as A intersect B) is a set of element that belong to both A and B.

$$\boxed{A \cap B = \left\{x : x \in A \text{ and } x \in B\right\}}$$



We say that A and B are **disjoint** sets if $A \cap B = \emptyset$.

Remark 2.1.3 *In practice, we use these equivalences*

$$\boxed{x \in A \cap B \iff x \in A \text{ and } x \in B}, \qquad \boxed{x \notin A \cap B \iff x \notin A \text{ or } x \notin B.}$$

Proposition 1 *Let A, B, C be subsets of a set E*

1. $A \cup A = A, \quad A \cap A = A.$
2. $A \cup \emptyset = A, \ A \cap \emptyset = \emptyset.$
3. $A \cup B = B \cup A, \quad A \cap B = B \cap A.$ *(**Commutativity**)*
4. $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$ *(**Associativity**)*
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$ *(**Distributivity**)*

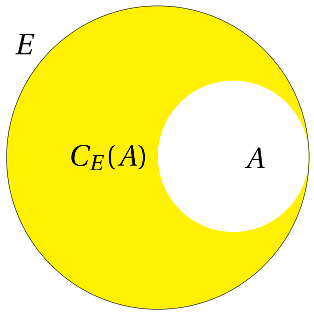
Proof 2.1.3.1 *We prove that* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *Let* $x \in A \cap (B \cup C)$

$$\begin{aligned} x \in A \cap (B \cup C) &\iff [x \in A \text{ and } x \in B \cup C] \\ &\iff [x \in A \text{ and } (x \in B \text{ or } x \in C)] \\ &\iff [(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)] \\ &\iff [(x \in A \cap B) \text{ or } (x \in A \cap C)] \\ &\iff x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

A complement of a set

Let E be a set, if a set A is contained in E, we say that A is a subset or a subset of E. The elements of E that do not belong to set A form a new set called the **complement** of A in E , denoted as A^C or $C_E(A)$,

$$\boxed{\mathrm{C}_E(A) = \left\{x : x \notin A\right\}}.$$



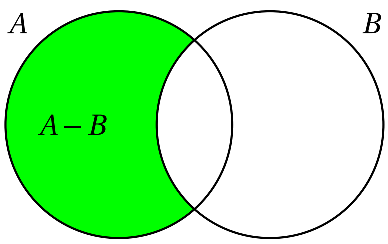
Remark 2.1.4 *In practice, we use these equivalences*

$$\boxed{x \in C_E(A) \iff x \in E \text{ and } x \notin A}, \qquad \boxed{x \notin C_E(A) \iff x \in A}$$

Set difference

Let A , B be two sets of E. The difference of A and B, denoted $A \setminus B$, consists of elements that are in A but not in B. in other words $A \setminus B = A \cap C_E(B)$.

$$\boxed{\mathrm{A} \setminus B = \left\{x : x \in A \text{ and } x \notin B\right\}}$$



The **symetric difference** of A and B, denoted $A\Delta B$, is the set $(A \setminus B) \cup (B \setminus A)$ which is the same

$$\boxed{\mathrm{A}\Delta B = A \setminus B \cup B \setminus A = (A \cup B) \setminus (A \cap B)}.$$

Remark 2.1.5 *In exercises , we usually use these equivalences:*

$$\boxed{x \in A \setminus B \iff x \in A \text{ and } x \notin B}, \qquad \boxed{x \notin A \setminus B \iff x \notin A \text{ or } x \in B}.$$

Proposition 2 *Let A, B be two subsets of E, then*

❶ $A \setminus A = \emptyset$

❷ $A \setminus \emptyset = A$

❸ $A \cup C_E(A) = E$

❹ $A \cap C_E(A) = \emptyset$.

❺ $C_E(C_E(A)) = A$

❻ $C_E(A \cup B) = C_E(A) \cap C_E(B)$.

❼ $C_E(A \cap B) = C_E(A) \cup C_E(B)$.

Proof 2.1.3.2 *We prove that* $C_E(A \cup B) = C_E(A) \cap C_E(B)$. *Let* $x \in C_E(A \cup B)$,

$$\begin{aligned}
x \in C_E(A \cup B) &\iff [x \in E \text{ and } x \notin A \cup B] \\
&\iff [x \in E \text{ and } \overline{x \in A \cup B}] \\
&\iff [x \in E \text{ and } \overline{x \in A \text{ or } x \in B}] \\
&\iff [(x \in E \text{ and } \overline{x \in A}) \text{ and } (x \in E \text{ and } \overline{x \in B})] \\
&\iff [(x \in E \text{ and } x \notin A) \text{ and } (x \in E \text{ and } x \notin B)] \\
&\iff [x \in C_E(A) \text{ and } x \in C_E(B)] \\
&\iff x \in C_E(A) \cap C_E(B).
\end{aligned}$$

2.1.4 Cartesian product of sets

Definition 2.1.3 *Consider two arbitrary sets A and B. The set of all ordered pairs* (a, b) *where* $a \in A$ *and* $b \in B$ *is called the* ***product****, or* ***cartesian product****, of A and B.*

$$\boxed{A \times B = \Big\{(a, b) : a \in A \text{ and } b \in B\Big\}}.$$

Remark 2.1.6 *In pratice, we use these equivalences.*

$\boxed{(a, b) \in A \times B \iff a \in A \text{ and } b \in B}$, $\boxed{(a, b) \notin A \times B \iff a \notin A \text{ or } b \notin B}$.

Example 2.1.7

1. Let $A = \left\{1, 2\right\}$ and $B = \left\{a, b, c\right\}$ then $A \times B = \left\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\right\}$ Also, $A \times A = \left\{(1, 1), (1, 2), (2, 1), (2, 2)\right\}$.
2. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \left\{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\right\}$.
3. $\mathbb{Z} \times \mathbb{R} = \left\{(x, y) : x \in \mathbb{Z}, y \in \mathbb{R}\right\}$.

Proposition 3 *Let A, B, C, D be a subsets of E, then*

❶ $(A \times C) \cup (B \times C) = (A \cup B) \times C$.

❷ $A \times B) \cup (A \times C) = A \times (B \cup C)$

❸ $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Proof 2.1.4.1 *We prove that $A \times B) \cup (A \times C) = A \times (B \cup C)$. Let $(x, y) \in A \times B) \cup (A \times C)$, then*

$$\begin{aligned}
(x, y) \in A \times B) \cup (A \times C) &\iff [(x, y) \in A \times B \text{ or } (x, y) \in A \times C] \\
&\iff [(x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)] \\
&\iff [x \in A \text{ and } (y \in B \text{ or } y \in C)] \\
&\iff [x \in A \text{ and } y \in B \cup C] \\
&\iff (x, y) \in A \times (B \cup C).
\end{aligned}$$

Exercise 2.1.1 *Let $A = \left\{a, b, c\right\}$ and $B = \left\{a, d\right\}$ be two subsets of a set $E = \left\{a, b, c, d, e\right\}$.*

❶ *Determine $A \cap B$, $A \cup B$, $C_E(B)$, $C_E(A)$, $A \setminus B$, $B \setminus A$, $A \Delta B$*

❷ *Determine $A \times B$, $A \times A$, $\mathscr{P}(B)$, $B \times \emptyset$, $B \times \left\{\emptyset\right\}$, $\mathscr{P}(\mathscr{P}(B))$.*

Solution 2.1.1 ❶

- $A \cap B = \left\{x \in E : x \in A \text{ and } x \in B\right\} = \left\{a\right\}$

- $A \cup B = \left\{x \in E : x \in A \; or \; x \in B\right\} = \left\{a, b, c, d\right\}$
- $C_E(B) = \left\{x \in E : x \notin B\right\} = \left\{b, c, e\right\}$
- $C_E(A) = \left\{x \in E : x \notin A\right\} = \left\{d, e\right\}$
- $A \setminus B = \left\{x \in E : x \in A \; and \; x \notin B\right\} = \left\{b, c\right\}$
- $B \setminus A = \left\{x \in E : x \in B \; and \; x \notin A\right\} = \left\{d\right\}$
- $A \Delta B = B \setminus A \cup A \setminus B = \left\{b, c, d\right\}$

❷

- $A \times B = \left\{(x, y) : x \in A \; and \; y \in B\right\} = \left\{(a, a), (a, d), (b, a), (b, d), (c, a), (c, d)\right\}$
- $A \times A = \left\{(x, y) : x \in A \; and \; y \in A\right\} =$
  $\left\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\right\}$
- $\mathscr{P}(B) = \left\{\{\}, \{a\}, \{d\}, \{a, d\}\right\}$
- $B \times \emptyset = \emptyset$
- $B \times \left\{\emptyset\right\} = \left\{(a, \emptyset), (d, \emptyset)\right\}$
-

$$\begin{aligned}\mathscr{P}(\mathscr{P}(B) = \Big\{ & \{\}, \{\{\}\}, \{\{a\}\}, \{\{d\}\}, \{\{a, d\}\}, \{\{\}, \{a\}\}, \\ & \{\{\}, \{b\}\}, \{\{\}, \{a, d\}\}, \{\{a\}, \{d\}\}, \\ & \{\{a\}, \{a, d\}\}, \{\{d\}, \{a, d\}\}, \{\{\}, \{a\}, \{d\}\}, \\ & \{\{a\}, \{d\}, \{a, d\}\}, \{\{\}, \{d\}, \{a, d\}\}, \\ & \{\{\}, \{a\}, \{a, d\}\}, \{\{\}, \{a\}, \{d\}, \{a, d\}\}\Big\}\end{aligned}$$

2.1.5 Partitions of set

Definition 2.1.4 *Let A be any nonempty set. A* ***partition*** *of A is a collection $\left(A_i\right)_{i=1, 2, ..., n}$ of non-empty subsets of A such that:*

❶ *$A_i \neq \emptyset$, where $i = 1, 2, 3, \ldots, n$.*

❷ *The sets of (A_i) are mutually disjoint which means $A_i \cap A_j = \emptyset$ where $i \neq j$.*

❸ *$\bigcup_{i=1}^{n} A_i = A$, where $A_1 \cup A_2 \cup \cdots \cup A_n = A$*

Example 2.1.8 *Let* $A = \left\{1, 2, 3, n\right\}, \quad A_1 = \{1\}, \; A_2 = \{3, n\}, A_3 = \{2\}$
$S = \left\{A_1, A_2, A_3\right\}$ *is a partition on* A, *because it satisfy the three above conditions.*

Exercise 2.1.2 *Consider the following collections of subsets of* $A = \left\{1, 2, 3, 4, 5, 6, 7, 8, 9\right\}$, *which one is a partition of* A?

(I) $\left(\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}\right)$

(II) $\left(\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}\right)$

(III) $\left(\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}\right)$

Solution 2.1.2

(I) *is not a partition of* A *since* 7 *in* A *does not belong to any of the subsets.*

(II) *is not a partition of* A *since* $\{1, 3, 5\}$ *and* $\{5, 7, 9\}$ *are not disjoint.*

(III) *is a partition of* A.

Exercise 2.1.3 *If* A, B, C *are sets, then :*

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Solution 2.1.3 *To prove (**a**), we will show that every element in* $A \setminus (B \cup C)$ *is contained in both set* $(A \setminus B)$ *and* $(A \setminus C)$, *and conversely.*
If $x \in A \setminus (B \cup C)$, *then* $x \in A$, *but is not in* $B \cup C$. *Hence* $x \in A$, *but* x *is either in both* B *nor in* C, *therefore,* $x \in (A \setminus B) \cap (A \setminus C)$
Conversely, if $x \in (A \setminus B) \cap (A \setminus C)$, *then* $x \in (A \setminus B)$ *and* $x \in (A \setminus C)$, *hence ,* $x \in A$ *and* $x \notin B$ *and* $x \notin C$. *Therefore,* $x \in A$ *and* $x \notin B \cup C$, *so that* $x \in A \setminus (B \cup C)$.
Since the sets $(A \setminus B) \cap (A \setminus C)$ *and* $A \setminus (B \cup C)$ *contain the same elements, they are equal.*