Course of Algebra 1

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Contents

Chapter 2

Sets and functions

2.1 Sets

2.1.1 Definitions

a set is a collection of objects. The objects in a set are called **elements** or **members**
If an element x is a **member** of A, we write

$$x \in A.$$

and that x is a member of A, or that x **belongs** to A. If x is not in A, we write

$$x \notin A.$$

The elements in the sets are described in either the **Statement form**, **Roster Form** or **Set Builder Form**.

- **Statement Form** In statement form, the well-defined descriptions of a member of a set are written and enclosed in the curly brackets.

  Example 2.1.1 *the set of even numbers less than* 5*. In statement form, it can be written as*

  $$\left\{ \textit{even numbers less than} 5 \right\}.$$

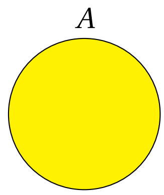
- **Roster Form**, all the elements of a set are listed, separated by commas, within braces;

Example 2.1.2 *The set of natural numbers less than 5. Therefore, the set is* $A = \{0,1,2,3,4\}$.

- **Set Builder Form** The general form is, $A = \{x : \text{property}\}$.

  Example 2.1.3 *Write the following sets in set builder form:* $A = \{2,4,6,8\}$. *So, the set builder form is* $A = \{x : x = 2n, n \in \mathbb{N} \text{ and } 1 \leqslant n \leqslant 4\}$.

Also, **Venn Diagrams** are the simple and best way for visualized representation of sets.



Remark 2.1.1

1. *Repeated elements are listed once* $\{a,b,a,c,b,a\} = \{a,b,c\}$.
2. *There is no order in the set* $\{3,2,1\} = \{1,2,3\} = \{2,1,3\}$.

Example 2.1.4

1. ***Natural Numbers*** : $\mathbb{N} = \{0, 1, 2, \dots\}$
2. ***Integers*** : $\mathbb{Z} = \{\dots, -2, -1, 0\, 1, 2, \dots\}$.
3. ***Rational numbers*** : $\mathbb{Q} = \{\frac{a}{b} : a, \ b \in \mathbb{Z} \text{ and } b \neq 0.\}$.

2.1.2 Types of Sets

✔ An **empty set** , denoted $\emptyset$ or $\{\}$, is a set that does not contain any elements.

✔ Let $E = \{a\}$, consisting of a single element, is called **singleton**.

- ✔ A set which consists of a finite number of elements is called a **finite set**.
  Example: A set of natural numbers up to 10, $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- ✔ A set which is not finite is called an **infinite set**.
  Example: A set of all natural numbers. $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\}$
- ✔ The two sets A and B are said to be **disjoint** if the set does not contain any common element.
  Example: Set $A = \{a, b, c, d\}$ and set $B = \{e, f, g\}$ are disjoint sets, because there is no common element between them.
- ✔ A is a **subset** of B, written $A \subset B$, if and only if every element of A is also an element of B
- ✔ If A is not a subset of B, we write $A \not\subset B$.
- ✔ A is equal to B, written $A = B$, if and only if for any a we have $a \in A$ if and only if $a \in B$.
- ✔ A is a **proper subset** of B, written $A \subsetneq B$, if and only if $A \subset B$ but $A \neq B$.
  Thus A being a proper subset of B means that A is a subset of B and B contains something that A does not contain.
  There is an important way to rephrase the definition of two sets being equal: $A = B$ if and only if $A \subset B$and $B \subset A$. . This is sometimes useful as a proof technique, as you can split a proof of $A = B$ into first checking $A \subset B$ and then checking $B \subset A$

Definition 2.1.1 *The **cardinality** of a set is a measure of how many elements are in the set. If A a finite set. the we denote the cardinality of A by* $|A|$.

Example 2.1.5

- *If* $A = \{a, 2, x, 5\}$, *then* $|A| = 4$
- $|\emptyset| = 0$
- *If* $X = \{a, \{2, x\}, \{1, \emptyset\}\}$, *then* $|X| = 3$
- $|\{\emptyset\}| = 1$

For any set A, we have $\emptyset \subset A$.
How many subsets can a finite set have?
If A a finite set, then A has $2^{|A|}$ subsets, we prove it by induction.

Definition 2.1.2 *If A a set, the set of all subsets of A is called the* **power set** *of A, denoted $\mathscr{P}(A)$.*

Example 2.1.6

❶ *if* $A = \left\{a, b, c\right\}$, *then* $\mathscr{P}(A) = \left\{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\right\}$.

❷ $\mathscr{P}(\emptyset) = \left\{\emptyset\right\}$.

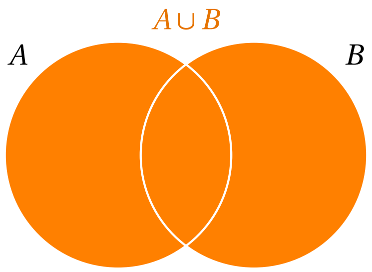
❸ $\mathscr{P}(\{\emptyset\}) = \left\{\emptyset, \{\emptyset\}\right\}$.

2.1.3 Operations on Sets

Union

Let A and B be two set, the union of A and B, denoted as $A \cup B$ (read as A union B) is a set of element that belong to either A or B. The **union** of the sets A and B is the set

$$\boxed{A \cup B = \left\{x : x \in A \text{ or } x \in B\right\}}$$



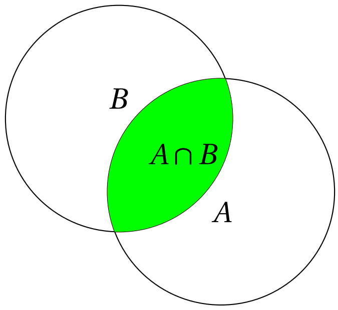
Remark 2.1.2 *We use often these equivalences*

$$\boxed{x \in A \cup B \iff x \in A \text{ or } x \in B}, \qquad \boxed{x \notin A \cup B \iff x \notin A \text{ and } x \notin B.}$$

Intersection

Let A and B be two sets, the **intersection** of A and B, denoted as $A \cap B$ (read as A intersect B) is a set of element that belong to both A and B.

$$\boxed{A \cap B = \left\{x : x \in A \text{ and } x \in B\right\}}$$



We say that A and B are **disjoint** sets if $A \cap B = \emptyset$.

Remark 2.1.3 *In practice, we use these equivalences*

$$\boxed{x \in A \cap B \iff x \in A \text{ and } x \in B}, \qquad \boxed{x \notin A \cap B \iff x \notin A \text{ or } x \notin B.}$$

Proposition 1 *Let A, B, C be subsets of a set E*

❶ $A \cup A = A, \quad A \cap A = A.$

❷ $A \cup \emptyset = A, \ A \cap \emptyset = \emptyset.$

❸ $A \cup B = B \cup A, \quad A \cap B = B \cap A.$ ***(Commutativity)***

❹ $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$ ***(Associativity)***

❺ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$ ***(Distributivity)***

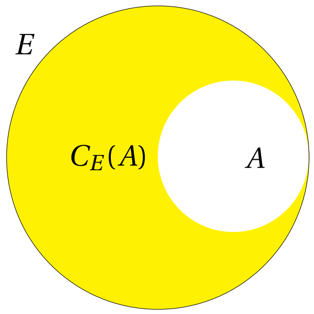
Proof 2.1.3.1 *We prove that* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *Let* $x \in A \cap (B \cup C)$

$$\begin{aligned} x \in A \cap (B \cup C) &\iff [x \in A \text{ and } x \in B \cup C] \\ &\iff [x \in A \text{ and } (x \in B \text{ or } x \in C)] \\ &\iff [(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)] \\ &\iff [(x \in A \cap B) \text{ or } (x \in A \cap C)] \\ &\iff x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

A complement of a set

Let E be a set, if a set A is contained in E, we say that A is a subset or a subset of E. The elements of E that do not belong to set A form a new set called the **complement** of A in E , denoted as A^C or $C_E(A)$,

$$\boxed{\mathrm{C}_E(A) = \left\{x : x \notin A\right\}}.$$



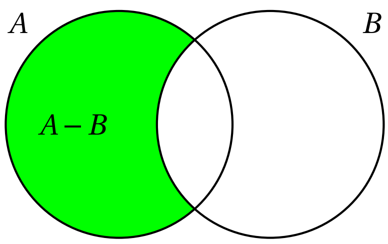
Remark 2.1.4 *In practice, we use these equivalences*

$$\boxed{x \in C_E(A) \iff x \in E \text{ and } x \notin A}, \qquad \boxed{x \notin C_E(A) \iff x \in A}$$

Set difference

Let A , B be two sets of E. The difference of A and B, denoted $A \backslash B$, consists of elements that are in A but not in B. in other words $A \setminus B = A \cap C_E(B)$.

$$\boxed{\mathrm{A} \backslash B = \left\{x : x \in A \text{ and } x \notin B\right\}}$$



The **symetric difference** of A and B, denoted $A\Delta B$, is the set $(A \setminus B) \cup (B \setminus A)$ which is the same

$$\boxed{\mathrm{A}\Delta B = A \setminus B \cup B \setminus A = (A \cup B) \setminus (A \cap B)}.$$

Remark 2.1.5 *In exercises , we usually use these equivalences:*

$$\boxed{x \in A \setminus B \iff x \in A \text{ and } x \notin B}, \qquad \boxed{x \notin A \setminus B \iff x \notin A \text{ or } x \in B}.$$

Proposition 2 *Let A, B be two subsets of E, then*

❶ $A \setminus A = \emptyset$

❷ $A \setminus \emptyset = A$

❸ $A \cup C_E(A) = E$

❹ $A \cap C_E(A) = \emptyset$.

❺ $C_E(C_E(A)) = A$

❻ $C_E(A \cup B) = C_E(A) \cap C_E(B)$.

❼ $C_E(A \cap B) = C_E(A) \cup C_E(B)$.

Proof 2.1.3.2 *We prove that* $C_E(A \cup B) = C_E(A) \cap C_E(B)$. *Let* $x \in C_E(A \cup B)$,

$$\begin{aligned} x \in C_E(A \cup B) &\iff [x \in E \text{ and } x \notin A \cup B] \\ &\iff [x \in E \text{ and } \overline{x \in A \cup B}] \\ &\iff [x \in E \text{ and } \overline{x \in A \text{ or } x \in B}] \\ &\iff [(x \in E \text{ and } \overline{x \in A}) \text{ and } (x \in E \text{ and } \overline{x \in B})] \\ &\iff [(x \in E \text{ and } x \notin A) \text{ and } (x \in E \text{ and } x \notin B)] \\ &\iff [x \in C_E(A) \text{ and } x \in C_E(B)] \\ &\iff x \in C_E(A) \cap C_E(B). \end{aligned}$$

2.1.4 Cartesian product of sets

Definition 2.1.3 *Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the **product**, or **cartesian product**, of A and B.*

$$\boxed{A \times B = \Big\{(a, b) : a \in A \text{ and } b \in B\Big\}}.$$

Remark 2.1.6 *In practice, we use these equivalences.*

$\boxed{(a,b) \in A \times B \iff a \in A \text{ and } b \in B}$, $\boxed{(a,b) \notin A \times B \iff a \notin A \text{ or } b \notin B}$.

Example 2.1.7

1. Let $A = \{1,2\}$ and $B = \{a,b,c\}$ then $A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$
   Also, $A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$.
2. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$.
3. $\mathbb{Z} \times \mathbb{R} = \{(x,y) : x \in \mathbb{Z}, y \in \mathbb{R}\}$.

Proposition 3 *Let A, B, C, D be a subsets of E, then*

❶ $(A \times C) \cup (B \times C) = (A \cup B) \times C$.

❷ $(A \times B) \cup (A \times C) = A \times (B \cup C)$

❸ $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Proof 2.1.4.1 *We prove that* $A \times B) \cup (A \times C) = A \times (B \cup C)$. *Let* $(x,y) \in A \times B) \cup (A \times C)$, *then*

$$\begin{aligned}
(x,y) \in (A \times B) \cup (A \times C) &\iff [(x,y) \in A \times B \text{ or } (x,y) \in A \times C] \\
&\iff [(x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)] \\
&\iff [x \in A \text{ and } (y \in B \text{ or } y \in C)] \\
&\iff [x \in A \text{ and } y \in B \cup C] \\
&\iff (x,y) \in A \times (B \cup C).
\end{aligned}$$

Exercise 2.1.1 *Let* $A = \{a,b,c\}$ *and* $B = \{a,d\}$ *be two subsets of a set* $E = \{a,b,c,d,e\}$.

❶ *Determine* $A \cap B,\ A \cup B,\ C_E(B),\ C_E(A),\ A \setminus B,\ B \setminus A,\ A \Delta B$

❷ *Determine* $A \times B,\ A \times A,\ \mathscr{P}(B),\ B \times \emptyset,\ B \times \{\emptyset\},\ \mathscr{P}(\mathscr{P}(B))$.

Solution 2.1.1 ❶ • $A \cap B = \{x \in E : x \in A \text{ and } x \in B\} = \{a\}$

- $A \cup B = \left\{x \in E : x \in A \; or \; x \in B\right\} = \left\{a, b, c, d\right\}$
- $C_E(B) = \left\{x \in E : x \notin B\right\} = \left\{b, c, e\right\}$
- $C_E(A) = \left\{x \in E : x \notin A\right\} = \left\{d, e\right\}$
- $A \setminus B = \left\{x \in E : x \in A \; and \; x \notin B\right\} = \left\{b, c\right\}$
- $B \setminus A = \left\{x \in E : x \in B \; and \; x \notin A\right\} = \left\{d\right\}$
- $A \Delta B = B \setminus A \cup A \setminus B = \left\{b, c, d\right\}$

❷
- $A \times B = \left\{(x, y) : x \in A \; and \; y \in B\right\} = \left\{(a, a), (a, d), (b, a), (b, d), (c, a), (c, d)\right\}$
- $A \times A = \left\{(x, y) : x \in A \; and \; y \in A\right\} =$ $\left\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\right\}$
- $\mathscr{P}(B) = \left\{\{\}, \{a\}, \{d\}, \{a, d\}\right\}$
- $B \times \emptyset = \emptyset$
- $B \times \left\{\emptyset\right\} = \left\{(a, \emptyset), (d, \emptyset)\right\}$
-

$$\begin{aligned}\mathscr{P}(\mathscr{P}(B) = \Big\{ & \{\}, \{\{\}\}, \{\{a\}\}, \{\{d\}\}, \{\{a, d\}\}, \{\{\}, \{a\}\}, \\ & \{\{\}, \{b\}\}, \{\{\}, \{a, d\}\}, \{\{a\}, \{d\}\}, \\ & \{\{a\}, \{a, d\}\}, \{\{d\}, \{a, d\}\}, \{\{\}, \{a\}, \{d\}\}, \\ & \{\{a\}, \{d\}, \{a, d\}\}, \{\{\}, \{d\}, \{a, d\}\}, \\ & \{\{\}, \{a\}, \{a, d\}\}, \{\{\}, \{a\}, \{d\}, \{a, d\}\}\Big\}\end{aligned}$$

2.1.5 Partitions of set

Definition 2.1.4 *Let A be any nonempty set. A* ***partition*** *of A is a collection* $\left(A_i\right)_{i=1, 2, \ldots, n}$ *of non-empty subsets of A such that:*

❶ $A_i \neq \emptyset$, *where* $i = 1, 2, 3, \ldots, n$.

❷ *The sets of* (A_i) *are mutually disjoint which means* $A_i \cap A_j = \emptyset$ *where* $i \neq j$.

❸ $\bigcup_{i=1}^{n} A_i = A$, *where* $A_1 \cup A_2 \cup \cdots \cup A_n = A$

Example 2.1.8 *Let* $A = \left\{1, 2, 3, n\right\}$, $A_1 = \{1\}$, $A_2 = \{3, n\}, A_3 = \{2\}$
$S = \left\{A_1, A_2, A_3\right\}$ *is a partition on* A, *because it satisfy the three above conditions.*

Exercise 2.1.2 *Consider the following collections of subsets of* $A = \left\{1, 2, 3, 4, 5, 6, 7, 8, 9\right\}$, *which one is a partition of* A?

(I) $\left(\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}\right)$

(II) $\left(\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}\right)$

(III) $\left(\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}\right)$

Solution 2.1.2

(I) is not a partition of A *since* 7 *in* A *does not belong to any of the subsets.*

(II) is not a partition of A *since* $\{1, 3, 5\}$ *and* $\{5, 7, 9\}$ *are not disjoint.*

(III) is a partition of A.

Exercise 2.1.3 *If* A, B, C *are sets, then :*

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Solution 2.1.3 *To prove* (***a***), *we will show that every element in* $A \setminus (B \cup C)$ *is contained in both set* $(A \setminus B)$ *and* $(A \setminus C)$, *and conversely.*
If $x \in A \setminus (B \cup C)$, *then* $x \in A$, *but is not in* $B \cup C$. *Hence* $x \in A$, *but* x *is either in both* B *nor in* C, *therefore,* $x \in (A \setminus B) \cap (A \setminus C)$
Conversely, if $x \in (A \setminus B) \cap (A \setminus C)$, *then* $x \in (A \setminus B)$ *and* $x \in (A \setminus C)$, *hence ,* $x \in A$ *and* $x \notin B$ *and* $x \notin C$. *Therefore,* $x \in A$ *and* $x \notin B \cup C$, *so that* $x \in A \setminus (B \cup C)$.
Since the sets $(A \setminus B) \cap (A \setminus C)$ *and* $A \setminus (B \cup C)$ *contain the same elements, they are equal.*

2.1.6 Generalized unions and Intersections

If we have two sets A and B , we can form the union $A \cup B$ and the intersection $A \cap B$, we want to form the union and intersection of many more than just two sets. We need to generalize the operation of union and intersection so that will apply to more than just two sets. before we will give some definitions

Definition 2.1.5 *A set $\mathscr{F}$, whose elements are sets, is called a* ***family of sets****.*

Definition 2.1.6 *Let I be any set and for each $i \in I$, let A_i be a set. Then we can form the set $\mathscr{F} = \left\{A_i : i \in I\right\}$. The set I is called the* ***index set*** *and $\mathscr{F}$ is called the* ***indexed family of sets****.*

Example 2.1.9 *Assume that for every natural number n we define the set $A_n = \left\{1, 2, 3, \ldots, n\right\}$. Then*

$$\mathscr{F} = \left\{A_n : n \in \mathbb{N}\right\} = \left\{A_1, A_2, A_3, \ldots\right\}$$

is an indexed family of sets, where the set of $\mathbb{N}$ is an index set.

let now extend the definition of union and intersection to more than two sets.
We know that $x \in A \cup B$ means that x is in at least one of two sets A and B, this notion of union can be easily extended to more than two sets. for finitely many sets , say $A_1, A_2, \ldots, A_n$, we shall say that x is in the union $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$. when x is in at least one of the sets $A_1, A_2, \ldots, A_n$, that is $x \in A_i$ for some $1 \leqslant i \leqslant n$. Using $I = \{1, 2, \ldots, n\}$ the index set, we denote the finite union by

$$\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n.$$

So $x \in \bigcup_{i \in I} A_i$ means that $x \in A_i$ for some $i \in I$
We also know that $x \in A \cap B$ means that x is in both of the two sets A and B, this notion of intersection can be easily extended to more than two sets. for finitely many sets , say $A_1, A_2, \ldots, A_n$, we shall say that x is in the intersection $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$. when x is in every one of the sets $A_1, A_2, \ldots, A_n$, that is $x \in A_i$ for every $1 \leqslant i \leqslant n$. Using $I = \{1, 2, \ldots, n\}$ the index set, we denote the finite intersection by

$$\bigcap_{i \in I} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n.$$

So $x \in \bigcap_{i \in I} A_i$ means that $x \in A_i$ for every $i \in I$

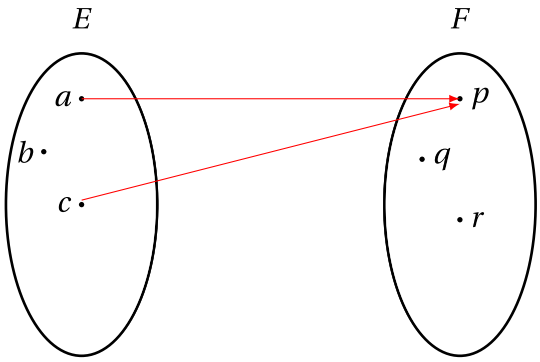
2.2 Functions

2.2.1 Definitions and examples

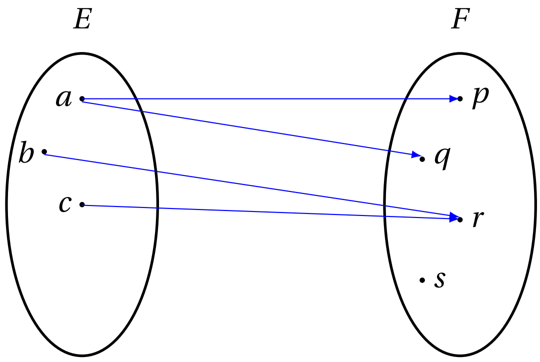
Definition 2.2.1

1. *Let E, F be sets, we say that f is a* ***function*** *from E to F if for every element $x \in E$, there exists a unique element $y \in F$ such that $f(x) = y$, we write $f : E \longrightarrow F$ or $E \xrightarrow{f} F$.*

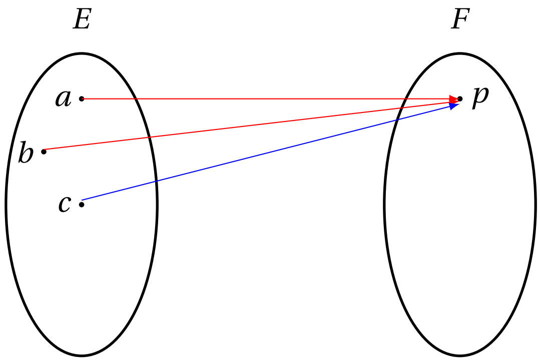
2. *The set E is called the* ***domain*** *and F is called the* ***codomain****. The element x is called the* ***pre-image*** *and y is the* ***image*** *of x under f.*



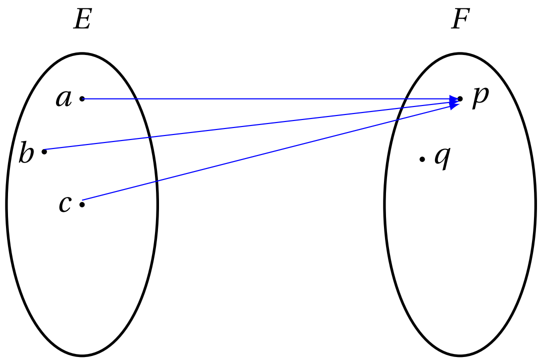
f is not a function



f is not a function



f is a function



f is a function

Example 2.2.1 ✔ *The* ***identity function*** $id_E : E \longrightarrow E$ *on a set* E *is the function* $id_E : x \longmapsto x$ *that maps every element to itself.*

✔ ***Constant function****: The function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by* $\forall x \in \mathbb{R}, f(x) = c$*, where* c *is a real constant , is a constant function.* ***Domain*** *of* $f = \mathbb{R}$*,* ***Range*** *of* $f = \{c\}$*.*

✔ ***Polynomial function****: A real valued function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by* $y = f(x) = a_0 + a_1 x + ... + a_n x^n$*, where* $n \in \mathbb{N}$*, and* $a_0, a_1, a_2, ...a_n \in \mathbb{R}$ *, for each* $x \in \mathbb{R}$ *, is called Polynomial functions.*

✔ *Consider the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $f(x) = |x| =$*. The range of* f *is* $\mathbb{R}^+$*. Hence* f *is also a function from* R *to* $\mathbb{R}^+$*. such function is called* ***absolute value****.*

Definition 2.2.2 *The* ***range****, or* ***image****, of a function* $f : E \longrightarrow F$ *is the set of values*

$$\begin{aligned} ran(f) &= \Big\{ y \in F : y = f(x) \text{ for some } x \in E. \Big\} \\ &= \Big\{ f(x),\ x \in E. \Big\} \end{aligned}$$

A function is onto if its range is all of F; that is , if for every $y \in F$ there exists $x \in E$ such that $y = f(x)$.

Definition 2.2.3 *Let* X, Y, A, B *be sets, and let* $f : X \longrightarrow Y$ *and* $g : A \longrightarrow B$ *be functions. We say that* f *is* ***identically equal*** *to* g*, denoted by* $f \equiv g$*, if the following conditions are met:*

(i) $X = A$

(ii) $Y = B$

(iii) $\forall x \in X, f(x) = g(x)$*.*

Definition 2.2.4 *The* ***graph*** *of a function* $f : E \longrightarrow F$ *is a subset* G_f *of* $E \times F$ *defined by*

$$G_f = \Big\{ (x, y) \in E \times F : x \in E \text{ and } y = f(x). \Big\}$$

Definition 2.2.5 *If* $f : E \longrightarrow F$ *and* $A \subset E$*, then we denote the* ***restriction*** *of* f *to* A *by* $f_{|A} : A \longrightarrow F$*, where* $f_{|A}(x) = f(x)$*, for* $x \in A$*.*

Informally, the restriction of f to A is the same function as f but is only defined on A.

Definition 2.2.6 *A function g is said to be an extension of another function f if whenever x is in the domain of f then x is also in the domain of g and $f(x) = g(x)$. That is, if* $\text{domain } f \subset \text{domain } g$ *and* $g|_{\text{domain } f} = f$.

2.2.2 Direct image and inverse image of a set

Let $f : E \longrightarrow F$ be a function.

Definition 2.2.7

1. *If A is a subset of E, then the **direct image** of A under f is the subset $f(A)$ of F given by*

$$\boxed{f(A)=\left\{f(x) : x\in A\right\}} \quad \text{or} \quad \boxed{f(A)=\left\{y : \exists x \in A \text{ with } f(x) = y\right\}}$$

*If $A = E$, the image of E in F is also called the **range** of f.*

2. *If B is a subset of F, then the **inverse image** of B under f is the subset $f^{-1}(B)$ of E given by*

$$f^{-1}(B) = \left\{x \in E : f(x) \in B\right\}.$$

Remark 2.2.1 *The notation f^{-1} should not be confused with that of an inverse function. Theorem implies that the inverse function exists if and only if the original function is injective (or one - to- one) and surjective (or onto).*

$\boxed{y \in f(A) \iff \exists x \in A : f(x) = y}$, similarly $\boxed{x \in f^{-1}(B) \iff f(x) \in B}$.

Example 2.2.2
Define a function $f : \mathbb{Z} \longrightarrow \mathbb{N}$ by $f(x) = |x| + 1$. To illustrate the concept of image and inverse image, let's consider some subsets of $\mathbb{Z}$ and some subsets of $\mathbb{N}$ here.

1. *If $A_1 = \left\{0,1,2\right\}$, then the image of A_1 is whatever function values are assigned to these 3 numbers. As $f(0) = 1, f(1) = 2, f(2) = 3,$ we have that $f(A_1) = \left\{f(0), f(1), f(2)\right\} = \left\{1,2,3\right\}$.*

❷ *If* $A_2 = \Big\{-2,-1,0,1,2\Big\}$*, then the image of* A_2 *is whatever function values are assigned to these five numbers. Since* $f(-2) = 3, f(-1) = 2, f(0) = 1, f(1) = 2, f(2) = 3$*, we have that* $f(A_2) = \Big\{f(-2), f(-1), f(0), f(1), f(2)\Big\} = \Big\{1,2,3\Big\}$.

❸ *If* $B_1 = \Big\{2\Big\}$*, then the pre-image of* B_1 *is all those elements in* $\mathbb{Z}$ *that map to* 2*; that is, it is all choices of* x *for which* $f(x) = |x| + 1 = 2$*. There are two such elements, namely* ± 1*. Hence* $f^{-1}(B_1) = \Big\{-1,1\Big\}$.

❹ *If* $B_2 = \Big\{0\Big\}$*, then the pre-image of* B_2 *is all those elements in* $\mathbb{Z}$ *that map to* 0*; that is, it is all choices of* x *for which* $f(x) = |x| + 1 = 0$*. There are no such elements. Hence,* $f^{-1}(B_2) = \emptyset$

❺ *If* $B_3 = \Big\{1\Big\}$*, then the pre-image of* B_3 *is all those elements in* $\mathbb{Z}$ *that map to* 1*; which is clearly, just* $x = 0$*. Hence,* $f^{-1}(B_3) = \Big\{0\Big\}$.

Theorem 2.2.1 *Let* $f : E \longrightarrow F$ *be a function . Let* E_1, E_2 *be subsets of* E*, and let* F_1, F_2 *be subsets of* F*. Then*

1. $f(E_1 \cap E_2) \subset f(E_1) \cap f(E_2)$
2. $f(E_1 \cup E_2) = f(E_1) \cup f(E_2)$
3. $f^{-1}(F_1 \cap F_2) = f^{-1}(F_1) \cap f^{-1}(F_2)$
4. $f^{-1}(F_1 \cup F_2) = f^{-1}(F_1) \cup f^{-1}(F_2)$

Proof 2.2.2.1

1. *We prove that* $f(E_1 \cap E_2) \subset f(E_1) \cap f(E_2)$*. Let* $y \in f(E_1 \cap E_2)$*, we will prove that* $y \in f(E_1) \cap f(E_2)$*. Since* $y \in f(E_1 \cap E_2)$*, that means there is an* $x \in E_1 \cap E_2$ *such that* $y = f(x)$ *(this follows from the definition of the direct image of* f*). Because* $x \in E_1 \cap E_2$*, we see that* $x \in E_1$ *and* $x \in E_2$*. Therefore,* $y = f(x) \in f(E_1)$ *and* $y = f(x) \in f(E_2)$*. Thus* $y \in f(E_1) \cap f(E2)$.

2. ($\subset$) *Firstly, We prove that* $f(E_1 \cup E_2) \subset f(E_1) \cup f(E_2)$*. Let* $y \in f(E_1 \cup E_2)$*, we will prove that* $y \in f(E_1) \cup f(E_2)$*. Since* $y \in f(E_1 \cup E_2)$*, that means there is an* $x \in E_1 \cup E_2$ *such that* $y = f(x)$ *(this follows from the definition of the direct image of* f*). Because* $x \in x \in E_1 \cup E_2$*, we see that* $x \in E_1$ *or* $x \in E_2$*. Therefore,* $y = f(x) \in f(E_1)$ *or* $y = f(x) \in f(E_2)$*. Thus* $y \in f(E_1) \cup f(E2)$.

(⊃) *Secondly, we will prove* $f(E_1) \cup f(E_2) \subset f(E_1 \cup E_2)$ *Let* $y \in f(E_1) \cup f(E_2)$*, we will prove that* $y \in f(E_1 \cup E_2)$*. Since* $y \in f(E_1) \cup f(E_2)$*, that means* $y \in f(E_1)$ *or* $y \in f(E_2)$ *, so there is an* $x \in E_1 such that y=f(x) or there is x \in$ E_2 *such that* $y = f(x)$*. (this follows from the definition of the direct image of* f*). Therefore , there is* $x \in A \cup B$ *such that* $y = f(x)$*, thus* $y \in f(E_1 \cup E_2)$*.*

We conclude that $f(E_1 \cap E_2) = f(E_1) \cap f(E_2)$*.*

3. *we prove* $f^{-1}(F_1 \cap F_2) = f^{-1}(F_1) \cap f^{-1}(F_2)$

(⊂) *First we prove* $f^{-1}(F_1 \cap F_2) \subset f^{-1}(F_1) \cap f^{-1}(F_2)$*. Let* $x \in f^{-1}(F_1 \cap F_2)$*. We prove that* $x \in f^{-1}(F_1) \cap f^{-1}(F_2)$

$$\begin{aligned} x \in f^{-1}(F_1 \cap F_2) &\Longrightarrow f(x) \in F_1 \cap F_2 \\ &\Longrightarrow f(x) \in F_1 \text{ and } f(x) \in E_2 \\ &\Longrightarrow x \in f^{-1}(F_1) \text{ and } x \in f^{-1}(F_2). \\ &\Longrightarrow x \in f^{-1}(F_1) \cap f^{-1}(F_2) \end{aligned}$$

Therefore, $f^{-1}(F_1 \cap F_2) \subset f^{-1}(F_1) \cap f^{-1}(F_2)$*.*

(⊃) *Second, we now prove that* $f^{-1}(F_1) \cap f^{-1}(F_2) \subset f^{-1}(F_1 \cap F_2)$*. Let* $x \in f^{-1}(F_1) \cap f^{-1}(F_2), we prove that x \in$ $f^{-1}(F_1 \cap F_2)$ *as follow*

$$\begin{aligned} x \in f^{-1}(F_1) \cap f^{-1}(F_2) &\Longrightarrow x \in f^{-1}(F_1) \text{ and } x \in f^{-1}(F_2) \\ &\Longrightarrow f(x) \in F_1 \text{ and } f(x) \in F_2 \\ &\Longrightarrow f(x) \in (F_1 \cap F_2) \\ &\Longrightarrow x \in f^{-1}(F_1 \cap F_2). \end{aligned}$$

Therefore, $f^{-1}(F_1) \cap f^{-1}(F_2) \subset f^{-1}(F_1 \cap F_2)$*, this complete the proof.*

4. (⊂) *First we prove* $f^{-1}(F_1 \cup F_2) \subset f^{-1}(F_1) \cup f^{-1}(F_2)$*. Let* $x \in f^{-1}(F_1 \cup F_2)$*. We prove that* $x \in f^{-1}(F_1) \cup f^{-1}(F_2)$

$$\begin{aligned} x \in f^{-1}(F_1 \cup F_2) &\Longrightarrow f(x) \in F_1 \cup F_2 \\ &\Longrightarrow f(x) \in F_1 \text{ or } f(x) \in F_2 \\ &\Longrightarrow x \in f^{-1}(F_1) \text{ or } x \in f^{-1}(F_2). \\ &\Longrightarrow x \in f^{-1}(F_1) \cup f^{-1}(F_2) \end{aligned}$$

Therefore, $f^{-1}(F_1 \cup F_2) \subset f^{-1}(F_1) \cup f^{-1}(F_2)$*.*

($\supset$) *Second, we now prove that* $f^{-1}(F_1) \cup f^{-1}(F_2) \subset f^{-1}(F_1 \cup F_2)$. *Let* $x \in f^{-1}(F_1) \cup f^{-1}(F_2)$, *weprovethat* $x \in f^{-1}(F_1 \cup F_2)$ *as follow*

$$\begin{aligned} x \in f^{-1}(F_1) \cup f^{-1}(F_2) &\Longrightarrow x \in f^{-1}(F_1) \text{ or } x \in f^{-1}(F_2) \\ &\Longrightarrow f(x) \in F_1 \text{ or } f(x) \in F_2 \\ &\Longrightarrow f(x) \in (F_1 \cup F_2) \\ &\Longrightarrow x \in f^{-1}(F_1 \cup F_2). \end{aligned}$$

Therefore, $f^{-1}(F_1) \cup f^{-1}(F_2) \subset f^{-1}(F_1 \cup F_2)$, *this complete the proof (4).*

Theorem 2.2.2 *Let* $f : E \longrightarrow F$ *be a function. Let* E_1, E_2 *be two subsets of* E. *If* f *is* ***injective*** *then* $f(E_1 \cap E_2) = f(E_1) \cap f(E_2)$.

Proof 2.2.2.2 *Assume that* f *is injective. We prove that* $f(E_1 \cap E_2) = f(E_1) \cap f(E_2)$. *By previous theorem 2.2.1,* $f(E_1 \cap E_2) \subset f(E_1) \cap f(E_2)$, *we will now show that* $f(E_1) \cap f(E_2) \subset f(E_1 \cap E_2)$. *Let* $y \in f(E_1) \cap f(E_2)$, *we will prove that* $y \in f(E_1 \cap E_2)$. *Since* $y \in f(E_1) \cap f(E_2)$, *then* $y \in f(E_1)$ *and* $y \in f(E_2)$. *because* $y \in f(E_1)$, *then there exist an* $x_1 \in E_1$ *such that* $y = f(x_1)$. *Also , since* $y \in f(E_2)$, *there is an* $x_2 \in E_2$, *such that* $y = f(x_2)$. *Hence ,* $y = f(x_1) = f(x_2)$. *Since* f *is injective, we have* $x_1 = x_2$*; Thus ,* $x_1 \in E_2$ *so,* $x_1 \in E_1 \cap E_2$, *and therefore* $y = f(x) \in f(E_1 \cap E_2)$.*Weconcludethatf(* $E_1 \cap E_2) = f(E_1) \cap f(E_2)$.

2.2.3 One -To - One functions and Onto functions

Definition 2.2.8

❶ *The function* $f : E \longrightarrow F$ *is said to be* ***one- to- one*** *(or* ***injective****) if whether* $x_1 \neq x_2$, *then* $f(x_1) \neq f(x_2)$, *or*

$$\boxed{(f\colon E \longrightarrow F \text{ is one - to- one}) \iff (\forall x_1, x_2 \in E,\ f(x_1) = f(x_2) \Longrightarrow x_1 = x_2)}$$

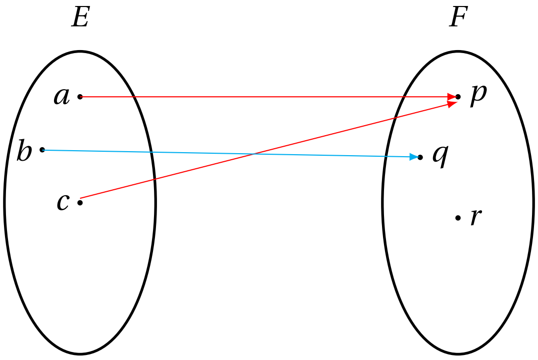
If f *is an one-to-one function, we also say that* f *is an injection .*

❷ *The function* $f : E \longrightarrow F$ *is said to be* ***onto*** *(or* ***surjective****) if for all* $y \in F$, *there exist* $x \in E$ *such that* $f(x) = y$, *or*

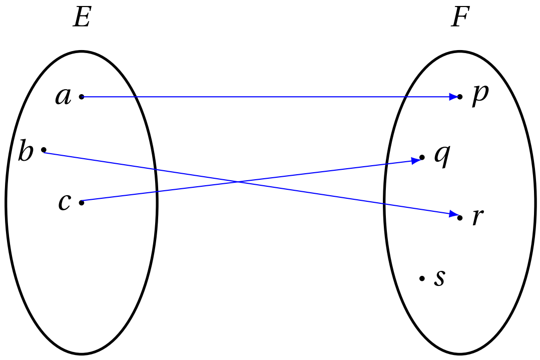
$$\boxed{(f\colon E \longrightarrow F \text{ is onto}) \iff (\forall y \in F, \exists x \in E : f(x) = y)}.$$

Remark 2.2.2 *Note that a function* $f : E \longrightarrow F$ *is*

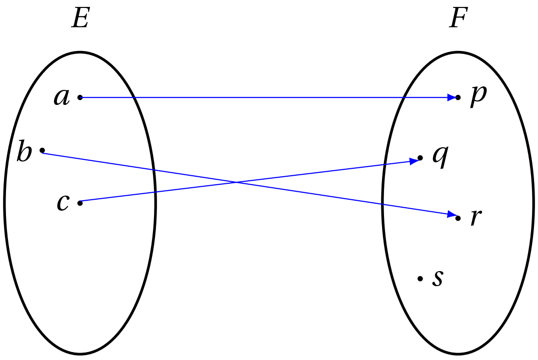
- ***injective*** *if and only if the subset* $f^{-1}\left(\left\{y\right\}\right)$ *contain* ***at most*** *one element for every* $y \in F$.
- ***surjective*** *if and only if the subset* $f^{-1}\left(\left\{y\right\}\right)$ *contain* ***at least*** *one element for every* $y \in F$.
- ***bijective*** *if and only if the subset* $f^{-1}\left(\left\{y\right\}\right)$ *contain* ***precisely*** *one element for every* $y \in F$.



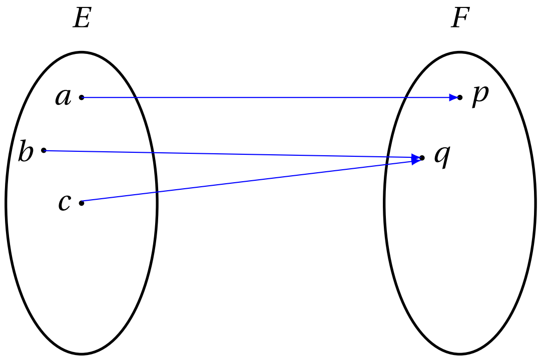
f isn't injective



f is injective



f isn' t surjective



f is surjective

Exercise 2.2.1 *Show that the function* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *defined by* $f(x)=2x+1$ *is injective.*

Solution 2.2.1 *Suppose that* $x_1, x_2 \in \mathbb{R}$, *and that* $f(x_1) = f(x_2)$. *Then by definition of* f, *we have*
$2x_1 + 1 = 2x_2 + 1$, *and thus* $2x_1 = 2x_2$, *and thus* $x_1 = x_2$. *Therefore* f *is injective by definition.*

Exercise 2.2.2 *Show that the function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by* $g(x) = x^2$ *is not injective.*

Solution 2.2.2 *Notice that when* $x = 1$*, we obtain* $g(x) = (1)^2 = 1$*, and when* $x = -1$*, we have* $g(x) = (-1)^2 = 1$*. Therefore, as* $g(1) = g(-1)$*, and* $1 \neq -1$*, we have that* g *is not injective.*

2.2.4 Composition of functions

Definition 2.2.9 *Given two functions* $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$*, one formes the composition function* $g \circ f : X \longrightarrow Z$ *by defining*

$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in X.$$

Theorem 2.2.3 *If* $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *are injectives, then* $g \circ f : X \longrightarrow Z$ *is injective.*

Proof 2.2.4.1 *Let* x_1, x_2 *be two elements of* X*, assume that* $(g \circ f)(x_1) = (g \circ f(x_2)$ *then show that* $x_1 = x_2$*.*

$$\begin{aligned}(g \circ f)(x_1) = (g \circ f)(x_2) &\Longrightarrow g(f(x_1)) = g(f(x_2)) \quad \text{by definition of composition of functions} \\ &\Longrightarrow f(x_1) = f(x_2) \quad (\text{because } g \text{ is injective}) \\ &\Longrightarrow x_1 = x_2 \quad (\text{because } f \text{ is injective}\end{aligned}$$

so, $g \circ f$ is injective.

Theorem 2.2.4 *If* $f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *are surjectives, then* $g \circ f : X \longrightarrow Z$ *is surjective.*

Proof 2.2.4.2 *Let* z *an element of* Z*, we must find a element* x *in* X *such that* $(g \circ f)(x) = z$*.*
since z *an element of* Z *and* g *is surjective, then there exist an element* y *in* Y *such that* $g(y) = z$*. apply the same reasoning for* y *in* Y*.*
since y *an element of* Y *and* f *is surjective, then there exist an element* x *in* X *such that* $f(x) = y$*, therefore there exist* x *in* X *such that* $(g \circ f)(x) = g(f(x)) = g(y) = z$*. so* $g \circ f$ *is surjective.*

2.2.5 Inverse functions

Theorem 2.2.5 *Let $f : X \longrightarrow Y$, be a bijective function, then there is a function $f^{-1} : Y \longrightarrow X$ defined as follow For each $y \in Y$, $f^{-1}(y)$ is defined to be the unique element in X such that $f(x) = y$; That is :*

$$\boxed{\forall y \in Y,\ [f^{-1}(y) = x] \iff [f(x) = y]}. \tag{2.2.1}$$

Definition 2.2.10 (Inverse function) *Given a bijective function $f : X \longrightarrow Y$, the function $f^{-1} : Y \longrightarrow X$ satisfying equation 2.2.1 is called the* ***inverse function*** *of f*

Example 2.2.3

$$\begin{aligned} f :\ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = 2x + 1. \end{aligned}$$

Show that f is bijective then determine its inverse fucntion f^{-1}. We

$$f \ is injective \iff [\forall x_1, x_2 \in \mathbb{R}, f(x_1) = f(x_2) \Longrightarrow x_1 = x_2].$$

$$\begin{aligned} f(x_1) = f(x_2) &\Longrightarrow 2x_1 + 1 = 2x_2 + 1 \\ &\Longrightarrow x_1 = x_2. \end{aligned}$$

therefore f is injective.

$$f \ surjective \iff \forall y \in \mathbb{R}, \exists x \in \mathbb{R} \ such \ that \ \ y = f(x).$$

Let $y \in \mathbb{R}$, find $x \in \mathbb{R}$which satisfy $y = f(x)$, we have to solve this equation on x, $f(x) = y$.

$$\begin{aligned} f(x) = y &\iff 2x + 1 = y \\ &\iff x = \frac{y-1}{2} \in \mathbb{R}. \end{aligned}$$

so

$$\forall y \in \mathbb{R}, \exists x = \frac{y-1}{2} \in \mathbb{R},\ such\ that\ \ y = f(x).$$

then f is surjective ,thus f is bijective. when f is bijective then there exist f^{-1}, the inverse function of f, defined by

$$f(x) = y \Longleftrightarrow f^{-1}(y) = x.$$

$$\begin{aligned} f(x) = y &\Longleftrightarrow 2x + 1 = y \\ &\Longleftrightarrow x = \frac{y-1}{2} = f^{-1}(y). \end{aligned}$$

In conclusion

$$\begin{aligned} f^{-1}: \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f^{-1}(x) = \frac{x-1}{2}. \end{aligned}$$

Example 2.2.4

$$\begin{aligned} f: [0, 1] &\longrightarrow [0, 1] \\ x &\longmapsto f(x) = x^2. \end{aligned}$$

Show that f is bijective and determine its inverse function f^{-1}.

$$f \text{ injective } \Longleftrightarrow [\forall x_1, x_2 \in [0, 1], f(x_1) = f(x_2) \Longrightarrow x_1 = x_2].$$

$$\begin{aligned} f(x_1) = f(x_2) &\Longrightarrow x_1^2 = x_2^2 \\ &\Longrightarrow |x_1| = |x_2| \\ &\Longrightarrow x_1 = x_2 (\text{because } x_1, x_2 \in [0, 1]). \end{aligned}$$

thus f is injective.

$$f \text{ surjective} \Longleftrightarrow \forall y \in [0, 1] \exists x \in [0, 1] \text{ such that } y = f(x).$$

Let $y \in [0, 1]$, find $x \in [0, 1]$ which satisfy $y = f(x)$, we have to solve this equation on x, $f(x) = y$.

$$\begin{aligned} f(x) = y &\Longleftrightarrow x^2 = y \\ &\Longleftrightarrow x = \sqrt{y} \in [0, 1]. \end{aligned}$$

thus

$$\forall y \in [0,\ 1], \exists x = \sqrt{y} \in [0,1],\ \textit{such that}\ \ y = f(x).$$

so f *is surjective , therefore* f *is bijective, if* f *is bijective then there exist the inverse function* f^{-1} *satisfying*

$$f(x) = y \Longleftrightarrow f^{-1}(y) = x.$$

$$\begin{aligned} f(x) = y &\Longleftrightarrow x^2 = y \\ &\Longleftrightarrow x = \sqrt{y} = f^{-1}(y). \end{aligned}$$

In Conclusion

$$\begin{aligned} f^{-1}:\ [0,\ 1] &\longrightarrow [0,\ 1] \\ x &\longmapsto f^{-1}(x) = \sqrt{x}. \end{aligned}$$

Remark 2.2.3 *We can prove the bijectivity of* f *directly by using this definition*

$$\boxed{f\ \ \textit{bijective} \Longleftrightarrow \forall y \in [0,\ 1],\ \exists! x \in [0,\ 1],\ \textit{tel que}\ y = f(x).}$$

Proposition 4 *Soient* E, F, G *be a sets If* $f : E \longrightarrow F$ *and* $g : F \longrightarrow G$ *be two bijective functions then* $g \circ f$ *is also bijective and its inverse function is defined by*

$$\boxed{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}.$$