Course of Algebra 1

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Contents

Chapter 3

Binary relations

In mathematics, we often seek to compare two elements of a set or the property that two elements of a set are likely to have.

Definition 3.0.1 *Let E and F be two sets, a relation from E in F is an assertion associating each element of E to an element of F, which are true or false. We denote the relation by $\mathfrak{R}$ a relation denotes some kind of relationship between two objects in a set, which may or may not hold*
Formally, a relation $\mathfrak{R}$ over a set X can be seen as a set of ordered pairs (x, y) of members of X .The relation $\mathfrak{R}$ holds between x, and y if (x, y) is a member of $\mathfrak{R}$.

or else

Definition 3.0.2 *Given two sets E and F, a relation from E to F is a correspondence $\mathfrak{R}$ which links elements of E with elements of F.*

or

Definition 3.0.3 *Let E and F be two sets , a relation $\mathfrak{R}$ from E to F is a subset of $E \times F$, we denote $x\mathfrak{R}y$ if $(x, y) \in \mathfrak{R}$.*

Remark 3.0.1

- *The set E is called the domain or the set of departure of $\mathfrak{R}$.*
- *The set F is called the codomain or the set of destination of $\mathfrak{R}$.*

- *For any element x of E and any element y of F satisfying $\mathfrak{R}$, we say that $x \in E$ is related by R with y, and we write $x\mathfrak{R}y$ otherwise $x \not\mathfrak{R} y$.*

- *If $E = F$, the relation $\mathfrak{R}$ is called binary relation on E.*

In the following chapter, we study only the **binary relations on a set**.

Example 3.0.1

1. *In $\mathbb{Z}$, we define the relation $\mathfrak{R}_1$ as follows:*

$$\boxed{\forall x, y \in \mathbb{Z},\ x\,\mathfrak{R}_1\, y \iff y \text{ multiple of } x.}$$

For example , $1\mathfrak{R}_1 x,\ \forall x \in \mathbb{Z}$,
$x\mathfrak{R}_1 0,\ \forall x \in \mathbb{Z}$,
$6\mathfrak{R}_1 12, \quad 4\not\mathfrak{R}_1 10, \quad \dots,$ *etc.*

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2. *In $\mathbb{Z}$, we define the relation $\mathfrak{R}_2$ as follows:*

$$\boxed{\forall x, y \in \mathbb{Z},\ x\,\mathfrak{R}_2\, y \iff x \equiv y[2].}$$

For example : $1\mathfrak{R}_2 1,\ 1\mathfrak{R}_2(-1),\ 2\not\mathfrak{R}_2 3 \dots$ *etc.*

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3. *In $\mathbb{R}$, we define the relation $\mathfrak{R}_3$ as follows:*

$$\boxed{\forall x, y \in \mathbb{R},\ x\,\mathfrak{R}_3\, y \iff x^2 = y^2\,.}$$

For example : $1\mathfrak{R}_3 1,\ 1\mathfrak{R}_3(-1),\ 1\not\mathfrak{R}_3 3,\ 2\mathfrak{R}_3(-2) \dots$ *etc.*

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4. *In $\mathbb{R}$, we define the relation $\mathfrak{R}_4$ as follows:*

$$\boxed{\forall x, y \in \mathbb{R},\ x\,\mathfrak{R}_4\, y \iff x = y\,.}$$

For example : $1\mathfrak{R}_4 1,\ 1\not\mathfrak{R}_4(-1), \dots$ *etc.*

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5. *In* $\mathbb{R}$, *we define the relation* $\mathfrak{R}_5$ *as follows*

$$\boxed{\forall x, y \in \mathbb{R},\ x\,\mathfrak{R}_5\,y \iff x \leq y\ ,.}$$

For example : $1\mathfrak{R}_5 1,\ 1\cancel{\mathfrak{R}_5}(-1),\ \ldots$ *etc.*

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6. *In* $\mathbb{R}$, *we define the relation* $\mathfrak{R}_6$ *as follows:*

$$\boxed{\forall x, y \in \mathbb{R},\ x\,\mathfrak{R}_6\,y \iff x < y\,.}$$

For example : $5\cancel{\mathfrak{R}_6}5,\ 4\mathfrak{R}_6 5,\ \ldots$ *etc*

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7. *Let* $E = \{a, b, c\}$, *in* $P(E)$, *we define the relation* ,$\mathfrak{R}_7$ *as follows:*

$$\boxed{\forall A, B \in P(E),\ A\,\mathfrak{R}_7\,B \iff A \cap B \neq \emptyset\,.}$$

For example : $A\mathfrak{R}_7 B,\ \emptyset\cancel{\mathfrak{R}_7}A,\ A\cancel{\mathfrak{R}_7}H,\ \ldots$ *etc, with* $A = \{a, b\},\ B = \{a\},\ H = \{c\}$.

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8. *Let E be any set, in* $P(E)$, *we define the relation* $\mathfrak{R}_8$ *as follows:*

$$\boxed{\forall A, B \in P(E),\ A\,\mathfrak{R}_8\,B \iff A \subset B\,.}$$

For example : $A\mathfrak{R}_8 E,\ \emptyset\mathfrak{R}_8 A,\ \forall A \in P(E)$.

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9. *In a plane* (P), *we define the relation* $\mathfrak{R}_9$ *on all the lines of the plane* (P) *as follows:*

$$\boxed{\forall (\Delta), (\Delta^{'}) \in P, \quad , (\Delta)\,\mathfrak{R}_9\,(\Delta^{'}) \iff (\Delta) \parallel (\Delta^{'})\,.}$$

For example : *Let three lines:* (Δ_1) *equation* : $y = x$, (Δ_2) *of equation:* $y = x + 1$, (Δ_3) *of equation:* $y = 2x$, *we then have:*
$(\Delta_1)\,\mathfrak{R}_9\,(\Delta_2)$, $(\Delta_1)\cancel{\mathfrak{R}_9}(\Delta_3)$.

$\cancel{\mathfrak{R}}$

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10. *In a plane* (P), *we define the relation* $\mathfrak{R}_{10}$ *on all the lines of the plane* (P) *as follows:*

$$\boxed{\forall (\Delta), (\Delta^{'}) \in P,\ (\Delta)\,\mathfrak{R}_{10}\,(\Delta^{'}) \iff (\Delta) \perp (\Delta^{'})\,.}$$

For example : *Consider three lines:* (Δ_1)*'s equation is:* $y = x$, (Δ_2)*'s equation is* : $y = -x$, (Δ_3) *'s equation is* : $y = 3x$, *we then have:* $(\Delta_1)\mathfrak{R}_{10}(\Delta_2)$, $(\Delta_1)\cancel{\mathfrak{R}_{10}}(\Delta_3)$.

Definition 3.0.4 (Graph of a relation) *Let $\mathfrak{R}$ define a binary relation on a set E, we call the graph of the relation $\mathfrak{R}$, denoted $G_{\mathfrak{R}}$, the subset of $E \times E$ defined by*

$$G_{\mathfrak{R}} = \left\{(x, y) \in E \times E, \text{ such that } x\,\mathfrak{R}\,y\right\}.$$

Example 3.0.2 *Let $E = \{-2, -1, 0, 1, 2, 3\}$ and we define the relation $\mathfrak{R}$, on E by:*

$$\forall x, y \in E, \ x\mathfrak{R}\,y \iff x^2 = y^2.$$

therefore the graph of the relation $\mathfrak{R}$ is defined as follows:

$$G_{\mathfrak{R}} = \{(1,1), (-1,1), (-1,-1), (1,-1), (2,2), (-2,2), (2,-2), (-2,-2), (0,0), (3,3)\}.$$

3.1 Properties of binary relations on a set

Let E be a set, $\mathfrak{R}$ a relation defined on E.

Definition 3.1.1 (Reflexivity) *The relation $\mathfrak{R}$ is said to be reflexive if*

$$\boxed{\forall x \in E, \ x\mathfrak{R}x.}$$

Example 3.1.1 *The relations defined previously $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_8, \mathfrak{R}_9$ are all reflexive, but the relations $\mathfrak{R}_6, \mathfrak{R}_7, \mathfrak{R}_{10}$ are not.*

Remark 3.1.1 *To show that a relation $\mathfrak{R}$ is not reflective, it is enough to find an element $x_0 \in E$ such that $x_0 \not\mathfrak{R}\, x_0$.*
In the previous example the relation $\mathfrak{R}_7$ is not reflexive because $\emptyset \not\mathfrak{R}_7 \emptyset$, $(\emptyset \cap \emptyset = \emptyset)$.

Definition 3.1.2 (Symmetry) *The relation $\mathfrak{R}$. is called symmetric if*

$$\boxed{\forall x, y \in E, \ x\mathfrak{R}y \Longrightarrow y\mathfrak{R}x.}$$

Example 3.1.2 *the relations defined previously $\mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_7, \mathfrak{R}_9, \mathfrak{R}_{10}$ are all symmetric, on the other hand the relations $\mathfrak{R}_1, \mathfrak{R}_5, \mathfrak{R}_6, \mathfrak{R}_8$ are not.*

Remark 3.1.2 *To show that a relation $\mathfrak{R}$ is not symmetric, it is enough to find two elements $x_0, y_0 \in E$ such that $x_0 \mathfrak{R}\, y_0$ and , $y_0 \not\mathfrak{R}\, x_0$*
In the previous example, the relation $\mathfrak{R}_8$ is not symmetric because $\emptyset\, \mathfrak{R}_8\, E$, but $E \not\mathfrak{R}_8 \emptyset$.

Definition 3.1.3 (Antisymmetry) *The relation $\mathfrak{R}$. is said to be antisymmetric if*

$$\boxed{\forall x, y \in E,\ x\,\mathfrak{R}\,y\ ,\ \text{and}\ y\,\mathfrak{R}\,x \Longrightarrow\ x = y.}$$

Example 3.1.3 *The relations defined previously $\mathfrak{R}_2, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_6, \mathfrak{R}_8$ are all antisymmetric, on the other hand the relations $\mathfrak{R}_1, \mathfrak{R}_3, \mathfrak{R}_7, \mathfrak{R}_9, \mathfrak{R}_{10}$ are not.*

Remark 3.1.3 *To show that a relation $\mathfrak{R}$ is not antisymmetric, it is enough to find two elements $x_0, y_0 \in E$ such that $x_0\,\mathfrak{R}\,y_0$ and $y_0\,\mathfrak{R}\,x_0$, but $x_0 \neq y_0$. In the previous examples, the relation $\mathfrak{R}_3$ is not antisymmetric because*
$1\,\mathfrak{R}_3\,(-1)$, *and* $(-1)\,\mathfrak{R}_3\,1$, *but* $1 \neq (-1)$.

Definition 3.1.4 (Transitivity) *The relation $\mathfrak{R}$. is called transitive if*

$$\boxed{\forall x, y, z \in E,\ x\,\mathfrak{R}\,y\ \text{and}\ y\,\mathfrak{R}\,z \Longrightarrow\ x\,\mathfrak{R}\,z.}$$

Example 3.1.4 *The relations defined previously $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_6, \mathfrak{R}_8, \mathfrak{R}_9$ are all transitive, on the other hand the relations $\mathfrak{R}_7, \mathfrak{R}_{10}$ are not.*

Remark 3.1.4 *To show that a relation $\mathfrak{R}$ is not transitive, it is enough to find three elements $x_0, y_0, z_0 \in E$ such that $x_0\,\mathfrak{R}\,y_0$ and $y_0\,\mathfrak{R}\,z_0$, but $x_0 \not\mathfrak{R} z_0$.*
In the previous example, the relation, $\mathfrak{R}_{10}$ is not transitive because if the line (Δ_1) is perpendicular to the line (Δ_2) and the line (Δ_2) is perpendicular to the line (Δ_3) then the line (Δ_1) is not perpendicular to the line (Δ_3), (but $(\Delta_1) \parallel (\Delta_3)$).

3.2 Equivalence relations

3.2.1 Definitions and examples

Definition 3.2.1 *Let $\mathfrak{R}$ be a binary relation on a set E. We say that $\mathfrak{R}$ is **an equivalence relation** if $\mathfrak{R}$ is **reflexive**, **symmetric** and **transitive**.*

Example 3.2.1 *We define in $\mathbb{R}^*$ the binary relation $\mathfrak{R}$ by:*

$$\forall x, y \in \mathbb{R}^*, x\,\mathfrak{R}\,y \iff xy > 0.$$

Let us show that $\mathfrak{R}$ is an equivalence relation.

- *Let us show that* $\mathfrak{R}$ *is reflexive. We have*

$$\boxed{R \text{ is reflexive } \iff [\forall x \in \mathbb{R}^*,\ x\,\mathfrak{R}\,x.]}$$

It is easy to see that $\forall x \in \mathbb{R}^*,\ x^2 > 0,$ *which is equivalent to saying that* $x\,\mathfrak{R}\,x$*. hence* $\mathfrak{R}$ *is reflexive.*

- *Let's show that* $\mathfrak{R}$ *is symmetric. We have by definition*

$$\boxed{R \text{ is symmetric } \iff [\forall x,\ y \in \mathbb{R}^*,\ x\,\mathfrak{R}\,y \implies y\,\mathfrak{R}\,x\,]}$$

we have

$$\begin{aligned} \forall x,\ y \in \mathbb{R}^*\ x\,\mathfrak{R}\,y &\implies x\,y > 0 \\ &\implies y\,x > 0 \\ &\implies y\,\mathfrak{R}\,x\,. \end{aligned}$$

So $\mathfrak{R}$ *is symmetric.*

- *Let us show that* $\mathfrak{R}$ *is transitive. We have by definition*

$$\boxed{R \text{ is transitive } \iff [\forall x,\ y, z\,,\ \in \mathbb{R}^*,\ x\,\mathfrak{R}\,y \text{ and } y\,\mathfrak{R}\,z \implies x\,\mathfrak{R}\,z\,]}$$

we have

$$\begin{aligned} \forall x,\ y,\ z\ \in \mathbb{R}^*,\ x\mathfrak{R}\,y \text{ and } y\,\mathfrak{R}\,z &\implies x\,y > 0,\ \text{and } y\,z > 0 \\ &\implies x\,z\ >\ 0 \end{aligned}$$

$(x$ *has the same sign as* y *and* y *, has the same sign as* z *then necessarily* z *has the same sign that* $x)$

$$\implies x\,\mathfrak{R}\,z\,.$$

hence $\mathfrak{R}$ *is transitive.*

We deduce that $\mathfrak{R}$ *is an equivalence relation on*

Example 3.2.2 *The relations* $\mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5$ *and* $\mathfrak{R}_9$ *are all equivalence relations, on the other hand the relations*
$\mathfrak{R}_1$ *(is not symmetric) „* $\mathfrak{R}_6$ *(is not reflexive) ,* $\mathfrak{R}_7$ *(is not is not reflexive) ,*
$\mathfrak{R}_8$ *(is not reflexive) ,* $\mathfrak{R}_{10}$ *(is neither reflexive nor transitive), are not.*

Exercise 3.2.1 ⚠ ✍ *Is the following relation, an equivalence relation on* $\mathbb{R}$ *? :*

$$\forall x, y \in \mathbb{R}, x \mathfrak{R} y \iff xy \leqslant 0.$$

We can therefore group these elements into "bundles" of elements that are similar, thus defining the notion of equivalence class, to finally construct new sets by " assimilating "similar elements to one and the same element. We then arrive at the notion of set quotient .

3.2.2 Quotient set

Definition 3.2.2 *Let* $\mathfrak{R}$ *be an equivalence relation in a set* E *. For each* x *of* E*, the set of all elements of* E *which are related, by* $\mathfrak{R}$*, with* x *is called equivalence class of* x *denoted by* $\dot{x}$ *or* $\bar{x}$ *or* $cl(x)$ *or* C_x*.*
So, the equivalence class $\dot{x}$ *is the subset of* E *defined by*

$$\dot{x} = \bar{x} = \left\{y \in E \text{ , such that } y \mathfrak{R} x\right\}$$

If $y \in \dot{x}$*,* y *is said to be a representative of the class* $\dot{x}$*.*
The set of equivalence classes is called quotient set of E *by the relation* $\mathfrak{R}$ *denoted by* $E/\mathfrak{R}$*.*

$$E/\mathfrak{R} = \{\dot{x} \mid x \in E\}$$

Example 3.2.3 *In* $\mathbb{Z}$*, we define the relation* $\mathfrak{R}$ *by:*

$$\forall x, y \in \mathbb{Z}, \ x \mathfrak{R} y \iff , x - y = 5k, \ k \in \mathbb{Z}.$$

$\mathfrak{R}$ *is an equivalence relation. the class of* 0.

$$\begin{aligned} \dot{0} = & \ \{x \in \mathbb{Z}, \text{ such that } x\mathfrak{R}, 0\} \\ & = \{x \in \mathbb{Z}, \text{ such that } x - 0 = 5k, k \in \mathbb{Z}\} \\ & = \{5k, \ k \in \mathbb{Z}\}. \end{aligned}$$

In the same way we determine the other equivalence classes, there are exactly five equivalence classes.

$$\dot{0} = \{5k,\ k \in \mathbb{Z}\}, \dot{1} = \{5k+1,\ k \in \mathbb{Z}\}, \dot{2} = \{5k+2,\ k \in \mathbb{Z}\},$$
$$\dot{3} = \{5k+3,\ k \in \mathbb{Z}\}, \dot{4} = \{5k+4,\ k \in \mathbb{Z}\}.$$

For this relation, we note $x \equiv y[5]$, *we read it* x ***congruo to*** y ***modulo*** 5. *The quotient set is denoted* $\mathbb{Z}/5\mathbb{Z}$ *instead of* $\mathbb{Z}/\mathfrak{R}$ *and we then have:*

$$\mathbb{Z}/5\mathbb{Z} = \{\dot{0}, \dot{1}, \dot{2}, ,\dot{3}, \dot{4}\}.$$

Proposition 1 *Let $\mathfrak{R}$ be a relation defined on a set E, we have the following properties:*

❶ *Let $a,\ b \in E$, if , $a \in \dot{b}$ then $\dot{a} = \dot{b}$.*

❷ $\forall a,\ b \in E$, $\dot{a} = \dot{b} \iff a\,\mathfrak{R}\,b$.

❸ *Let $u, v, x \in E$, if $u, v \in \dot{x}$ then $u\,\mathfrak{R}\,v$.*

❹ $\forall x, y \in E$, *we have* $\dot{x} = \dot{y}$ *or* $\dot{x} \cap \dot{y} = \emptyset$.

❺ *The equivalence classes form a partition of the set E.*

$$E = \bigcup_{x\ inE} \dot{x}.$$

Proof 3.2.2.1 ✔ *If $x \in \dot{a}$, then*

$$x\,\mathfrak{R}\,a \text{ and we have } a\mathfrak{R}\,b,$$

we deduce, by transitivity, that

$$x\,\mathfrak{R}\,b,$$

which implies that

$$x \in \dot{b}.$$

same reasoning to show that $\dot{b} \subset \dot{a}$.
Conclusion

$$\dot{a} = \dot{b}.$$

✔ *Let us show the direct implication, we assume that* $\dot{a} = \dot{b}$ *and show that* $a\mathfrak{R}b$. *it is easy to see that* $a \in \dot{a}$ *(because* $\mathfrak{R}$ *is reflexive).*

$$\begin{aligned} a \in \dot{a} &\Longrightarrow a \in \dot{b} \\ &\Longrightarrow a\mathfrak{R}\, b. \end{aligned}$$

Reciprocally if $a\mathfrak{R}\, b$ *then* $\dot{a} = \dot{b}$. *in fact, let* $z \in \dot{a}$

$$\begin{aligned} z \in \dot{a} &\Longrightarrow z\,\mathfrak{R} a, \\ &\Longrightarrow z\ R\ b \ (\text{because } a\ \mathfrak{R}\ b \text{ and } \mathfrak{R} \text{ is transitive.}) \\ &\Longrightarrow z \in \dot{b}. \end{aligned}$$

Hence $\dot{a} \subset \dot{b}$ *Similarly, we show that* $\dot{b} \subset \dot{a}$
Let $z \in \dot{b}$

$$\begin{aligned} z \in \dot{b} &\Longrightarrow z\,\mathfrak{R}\, b, \\ &\Longrightarrow z\ R\ b \ (\text{because } b\ \mathfrak{R}\ a\ (\mathfrak{R} \text{ is symmetric and transitive.}) \\ &\Longrightarrow z \in \dot{a}. \end{aligned}$$

Hence $\dot{b} \subset \dot{a}$.

✔ *We have* $u, v \in \dot{x}$ *then* $u\mathfrak{R}x$ *and* $x\mathfrak{R}v$ *hence* $u\mathfrak{R}v$ *(by the transitivity of* $\mathfrak{R}$.*)*

✔ *Let* $x, y \in E$ *such that* $\dot{x} \cap \dot{y} \neq \emptyset$, *we show that* $\dot{x} = \dot{y}$.
We have $\dot{x} \cap \dot{y} \neq \emptyset$ *then* $\exists z \in E$, *tel that* $z \in \dot{x} \cap \dot{y}$

$$\begin{aligned} z \in \dot{x} \cap \dot{y} &\Longrightarrow z \in \dot{x}\ et\ z \in\ doty \\ &\Longrightarrow z\,\mathfrak{R}\, x\ et\ z\,\mathfrak{R}\ y, \\ &\Longrightarrow \dot{z} = \dot{x} \text{ and } \dot{z} = \dot{y},\ (\text{ see the first property}) \\ &\Longrightarrow \dot{x} = \dot{y}. \end{aligned}$$

✔ *Let us show that the quotient set forms a partition of E*

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❶ *We have* $\forall \dot{x} \in E/\mathfrak{R}$, $\dot{x}\ neq\emptyset$, *because* $x \in \dot{x}$. *(the relation* $\mathfrak{R}$ *is reflexive, ,* $x\mathfrak{R}x$*).*

❷ *We showed previously that all distinct equivalence classes are disjoint.*

❸ *Rest à show that*

$$E = \bigcup_{x \in E} \dot{x},$$

we have an obvious inclusion

$$\bigcup_{y \in E} \dot{y} \subset E,$$

let's show the other inclusion.

$$E \subset \bigcup_{y \in E} \dot{y}.$$

Let $x \in E$ *then* $x \in \dot{x}$ *and therefore* $x \in bigcup_{y \in E} \dot{y}$ *hence*

$$E \subset \bigcup_{y \in E} \dot{y}$$

in conclusion, the set $E/\mathfrak{R}$ *is a partition of* E.

3.2.3 Canonical composition of a function

Definition 3.2.3 *Let* E *and* F *be two sets,* $f : E \longrightarrow F$ *be a function, and let* $\mathfrak{R}$ *be a relation defined on* E *by:*

$$x \,\mathfrak{R}\, y \iff f(x) = f(y).$$

This relation $\mathfrak{R}$ *is an equivalence relation, it is called an equivalence relation associated to* f. *Let* $f(E) = \{f(x) |\, x \in E\}$ *is the image set of* E *by* f, i *the **canonical injection** of* $F(E)$ *into ,*F *and* π ***the canonical surjection** of* E *in ,* $E/\mathfrak{R}$.

$$\begin{aligned} i : f(E) &\longrightarrow F \\ x &\longmapsto x \end{aligned} \qquad\qquad \begin{aligned} \pi : E &\longrightarrow E/\mathfrak{R} \\ x &\longmapsto \dot{x} \end{aligned}$$

Remark 3.2.1 *We easily verify, by construction, that the function* i *is injective and the function* π *is surjective.*

Theorem 3.2.1 *Let* E, F *be two sets, and* $f : E \longrightarrow F$ *be a map*

❶ *The binary relation* $\mathfrak{R}$ *defined on* E *by:*

$$x\mathfrak{R}y \iff f(x) = f(y).$$

is an equivalence relation on E *called associated to* f.

❷ . *Let* π *be the canonical surjection of* E *on* $E/\mathfrak{R}$ *and* i *the canonical injection of* $fE)$ *in* F. *Then there exists a bijective map unique*

$$\begin{aligned} \widetilde{f} : E/\mathfrak{R} &\longrightarrow f(E) \\ \dot{x} &\longmapsto \widetilde{f}(\dot{x}) = f(x). \end{aligned}$$

such that $f = i \circ \widetilde{f} \circ \pi$.

Proof 3.2.3.1 ❶ *It is easy to verify that* $\mathfrak{R}$ *is an equivalence relation on* E.

❷ *Indeed,*

$$\begin{aligned} \dot{x} = \dot{y} &\iff f(x) = f(y) \\ &\iff \widetilde{f}(\dot{x}) = \widetilde{f}(\dot{y})). \end{aligned}$$

We have also just shown that $\widetilde{f}$ *is injective.*
$\widetilde{f}$ *is surjective by construction. Thus,* $\widetilde{f}$ *is a bijection of* $E/\mathfrak{R}$ *into* $f(E)$, $\widetilde{f}$ *is called the bijection canonical associated to* f.
If there existed another application $g : E/\mathfrak{R} \longrightarrow F$ *such that* $f = g \circ \pi$, *we would have for all* $x \in E$, $\widetilde{f}(\dot{x}) = f(x) = g(\dot{x})$. *hence* $\widetilde{f} = g$, *which proves the uniqueness of* $\widetilde{f}$.
it is clear that for all $x \in E$, *we then have:*

$$\boxed{\forall x \in E, f(x) = i(f(x)) = i(\widetilde{f}(\dot{x})) = (i \circ \widetilde{f} \circ \pi)(x).}$$

Hence, $f = i \circ \widetilde{f} \circ \pi$: *This is the canonical decomposition of* f *into the composition of* ***an injection***, ***a bijection*** *and* ***a surjection***. *We have just established the theorem of the canonical decomposition of an application, we have the following diagram:*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & F \\ E/\mathfrak{R} \arrow[r, "\widetilde{f}"] & f(E) \arrow[u, "i"'] \end{tikzcd}$$

Exercise 3.2.2 *Let $\mathfrak{R}$ be the relation defined on $\mathbb{R}$ by:*

$$\forall; x, y \in \mathbb{R}, \ x \mathfrak{R} y \iff x^2 = y^2$$

❶ *Show that $\mathfrak{R}$ is an equivalence relation on $\mathbb{R}$:*

❷ *Determine the equivalent class of $a \in \mathbb{R}$;*

❸ *Determine the quotient set $\mathbb{R}/\mathfrak{R}$.*

❹ *The function f defined by:*

$$\begin{aligned} f : \mathbb{R}/\mathfrak{R} &\longrightarrow [0, +\infty[\\ \dot{x} &\longmapsto f(\dot{x}) = x^2. \end{aligned}$$

is it well defined? is it bijective?

3.3 Order relations

3.3.1 Definitions and examples

Definition 3.3.1 *Let $\mathfrak{R}$ be a binary relation on a set E. We say that $\mathfrak{R}$ is* ***an order relation*** *if $\mathfrak{R}$ is* ***reflexive***, ***antisymmetric*** *and* ***transitive***.

Example 3.3.1 *We define in $\mathbb{N}$ the binary relation $\mathfrak{R}$ by:*

$$\forall x, y \in \mathbb{N}, x \mathfrak{R} y \iff x \text{ divide } y.$$

Let us show that $\mathfrak{R}$ is an order relation.

✔ *Let us show that $\mathfrak{R}$ is reflexive. We have*

$$\mathfrak{R} \text{ is reflexive} \iff [\forall x \in \mathbb{N}, \ x \mathfrak{R} x]$$

It is easy to see that $\forall x \in \mathbb{N}, \ x = 1.x,$ which is equivalent to saying that $x \mathfrak{R} x$ hence $\mathfrak{R}$ is reflexive.

✔ *Let us show that* $\mathfrak{R}$ *is antisymmetric. We have by definition*

$$\boxed{R \text{ is antisymmetric} \iff [\forall x,\ y \in \mathbb{N},\ x\,\mathfrak{R}\,y \text{ and } y\,\mathfrak{R}\,x \Longrightarrow y = x]}$$

and therefore

$$\begin{aligned}
\forall x,\ y \in \mathbb{N},\ x\,\mathfrak{R}\,y \text{ and } y\,R\,x &\Longrightarrow x \text{ divides } y \text{ and } y \text{ divides } x\\
&\Longrightarrow \exists k,\ k' \in \mathbb{N},\ y = k.x\,,\text{ and } x = k'.y\\
&\Longrightarrow \exists k,\ k' \in \mathbb{N},\ y = k.x \quad \text{and} \quad x = k'.k.x\,.\\
&\Longrightarrow \exists k,\ k' \in \mathbb{N},\ y = k.x \text{ and } k'.k = 1\,.\\
&\Longrightarrow \exists k,\ k' \in \mathbb{N},\ y = k.x \text{ and } k' = k = 1\,.\\
&\Longrightarrow\ y = x\,.
\end{aligned}$$

hence, $\mathfrak{R}$ *is anti-symmetric.*

✔ *Let us show that* $\mathfrak{R}$ *is transitive. We have by definition*

$$\boxed{R \text{ is transitive} \iff [\forall x,\ y,\ z \in \mathbb{N},\ x\,\mathfrak{R}\,y \text{ and } ,y\,\mathfrak{R}\,z \Longrightarrow x\,\mathfrak{R}\,z]}$$

and so

$$\begin{aligned}
\forall x,\ y,\ z \in \mathbb{N},\ x\,\mathfrak{R}\,y \text{ et } y\,\mathfrak{R}\,z, &\Longrightarrow \exists k,\ k' \in \mathbb{N},\ y = k.x \text{ and } z = k'.y\\
&\Longrightarrow \exists k,\ k' \in \mathbb{N},\ z = k'.k.x\\
&\Longrightarrow \exists k'' \in \mathbb{N},\ z = k''x\\
&\Longrightarrow x\,\mathfrak{R}\,z\,.
\end{aligned}$$

hence, $\mathfrak{R}$ *is transitive.*

We deduce that $\mathfrak{R}$ *is an order relation on* $\mathbb{N}$.

Exercise 3.3.1 ⚠ ✍ *Is the following relation an order relation on* $\mathbb{Z}$ *? :*

$$\forall x,\ y \in \mathbb{Z},\ x\,\mathfrak{R}\,y \iff x \text{ divide } y.$$

Example 3.3.2 *The relations* $\mathfrak{R}_2, \mathfrak{R}_4, \mathfrak{R}_5, \mathfrak{R}_8$ *are all order relations, on the other hand the relations*
$\mathfrak{R}_1$ *(is not antisymmetric)* , $\mathfrak{R}_3$ *(is not antisymmetric)* , $\mathfrak{R}_6$ *(is not reflexive)* ,
$\mathfrak{R}_7$ *(is not reflexive)* , $\mathfrak{R}_9$, *(is not antisymmetric)* $\mathfrak{R}_{10}$ *(is neither reflexive nor transitive), are not.*

Remark 3.3.1 *A set with an ordering relation is called an ordered set, and we denote it* $_e$ $(E, \mathfrak{R})$.

Remark 3.3.2 *A set with an ordering relation is called an ordered set, and we denote it* $(E, \mathfrak{R})$.

3.3.2 Total or partial order

A order relation compares elements in a set. This means that we can form the notion of upper and lower bounds of sets. We make the following definition:

Definition 3.3.2 *Let* $(A, \mathfrak{R})$ *be a ordered set. An element* $a \in A$ *is called* ***largest element of*** A *if and only if* $x\mathfrak{R}a$ *for every* $x \in A$. *Conversely, an element* $a \in$ $A is called$ ***smallest element of*** $A if and only if a R x for every x$.

Definition 3.3.3 *Let E be a set ordered by the order relation* $\mathfrak{R}$.

1. *Let* x, y *be two elements of* E, *we say that* x *and* y *are* ***comparable*** *if* $x\mathfrak{R}y$ *or* $y\mathfrak{R}x$.

2. *We say that the relation* $\mathfrak{R}$ *is a* ***total order****, or else* $(E, , \mathfrak{R})$ *is totally ordered, if any two elements* x, y *of* E *are comparable. In other words:*

$$\boxed{\text{The relation } R \text{ \textbf{is a total order}} \iff \forall\, x, y \in E,\ x\mathfrak{R}y \ \ textor \ \ y\mathfrak{R}x.}$$

Otherwise, we say that the relation $\mathfrak{R}$ *is a* ***partial order****, or else* , $(E, \mathfrak{R})$ *is partially ordered. In other words:*

$$\text{The relation } \mathfrak{R} \text{ \textbf{is a partial order}} \iff \exists\, x, y \in E,\ \ x \not\mathfrak{R} y\,,\ \text{and } y \not\mathfrak{R} x.$$

Example 3.3.3 1. $\leqslant$ *and* $\geqslant$ *define a total order on* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

2. *The division defines a* ***partial order*** *on* $\mathbb{N}$.

❸ $\subset$ *and*, ,$\supset$ *define a partial order on* $\mathscr{P}(E)$ *such that* $card(E) \geqslant 2$.

3.3.3 Remarkable elements of an ordered set

lower and upper bound of a set

Definition 3.3.4 *Let* $\mathfrak{R}$ *be an order relation on a set* E *and* A *, a non-empty part of* E.

❶ *We say that* A *is* ***bounded from above*** *or* ***majorized*** *for the relation* $\mathfrak{R}$ *if:*

$$\boxed{\exists M \in E, \forall x \in A, \quad \text{such as } \ x\,\mathfrak{R}\,M}.$$

We say that M *is a* ***upper bound*** *of* A *or else* A *is* ***bounded from above*** *by* M.

❷ *We say that A is* ***bounded from below*** *or* ***minorized*** *for the relation* $\mathfrak{R}$ *if:*

$$\exists m \in E, \forall x \in A, \quad \text{such that } \ m\,\mathfrak{R}\,x.$$

We say that m is ***a lower bound*** *of A or A is* ***bounded from below*** *by m:*

Example 3.3.4 ❶ $A = \{1, 3, 7\}$ *is reduced by* 1 *and increased by* 21 *for the relation defined on* $\mathbb{N}$ *by:*

$$\boxed{\forall x, y \in \mathbb{N},\ x\,\mathfrak{R}\,y \iff y \text{ multiple of } x.}$$

Indeed :

✔ *Let* $M \in \mathbb{N}$

$$\begin{aligned} M \text{ is an upper bound of } A &\Longrightarrow \forall x \in A, x\,\mathfrak{R}\,M. \\ &\Longrightarrow M \text{ multiple of } 1,\ M \text{ multiple of } 3 \text{ and } M \text{ multiple of } 7 \\ &\Longrightarrow M\ ,\ \text{is the common multiplier of } 1,3 \text{ and } 7. \\ &\Longrightarrow M \text{ is multiple of } 21. \end{aligned}$$

Then the set of upper bounds of A *is* $\{21k,\ \text{such that } k \in \mathbb{N}\}$.

✔ *Let* $m \in \mathbb{N}$

$$\begin{aligned} m \text{ is a lower bound of } A &\Longrightarrow \forall x \in A, m\,\Re\, x. \\ &\Longrightarrow 1 \text{ multiple of } m,\ 3 \text{ multiple of } m \text{ and } 7 \text{ multiple of } m \\ &\Longrightarrow m \text{ is the common divisor of } 1, 3 \text{ and } 7. \\ &\Longrightarrow m = 1. \end{aligned}$$

Then the set of lower bounds of A *is* $\{1\}$.

❷ *In the ordered set* $(\mathscr{P}(E), \subset)$, $\mathscr{P}(E)$ *is minorized by* $\emptyset$ *and majorized by* E.

Upper bound and lower bound of a set

Definition 3.3.5 *Let* $\Re$ *be an order relation on a set* E *and* A *be a non-empty part of* E.

❶ *If* A *is majorized for the relation* $\Re$ *then the least upper bound of* A, *if it exists, is called* ***least upper bound*** *or* ***supremum***, *denoted* $\sup A$.

❷ *If* A *is minorized for the relation* $\Re$ *then the greatest lower bound of* A, *if it exists, is called* ***greatest lower bound*** *or* ***infimum***, *denoted* $\inf A$.

Example 3.3.5 ❶ $A = \{1, 3, 7\}$ *is bounded from below by* 1 *and bounded from above by* 21 *for the relation defined on* $\mathbb{N}$ *by:*

$$\boxed{\forall x, y \in \mathbb{N},\ x\,\Re\, y \iff y \text{ multiple of } x.}$$

✔ *We have seen that the set of upper bounds of* A *is* $\{21k,$ *such that* $k \in \mathbb{N}\}$, *and so* $sup(A) = 21$ *is the smallest multiple of elements of* A.

✔ *For the lower bound, we have already shown that the set of lower bounds of* A *is* $\{1\}$, *so* $Inf(A) = 1$.

❷ *A subset* A *of an ordered set* E *does not necessarily admit an upper (resp. lower) bound. However, if* A *admits an upper bound (resp. lower), it is unique but it may not belong to* A.
For example, if $E = \mathbb{Q}$, *ordered by the usual inequality, and if*

$$A = \left\{x \in \mathbb{Q} : 0 < x \ et\ x^2 < 2\right\},$$

then the set A is minimized by any negative or zero rational number. We have $inf(A) = 0$ but A does not admit an upper bound in $\mathbb{Q}$ since $\sqrt{2} \notin \mathbb{Q}$.

Theorem 3.3.1 *Let $(E, \mathfrak{R})$ be a totally ordered set, and A be a part of E. For an element M of E to be the upper bound of A, it is necessary and sufficient that M satisfies both conditions.*

1. *For all $x \in A$, we have: $x \mathfrak{R} M$.*

2. *For any element $c \in E$ such that $c \mathfrak{R} M$, $\exists x \in A$, such that $c \mathfrak{R} x$.*

maximum element, minimum element in a set

Definition 3.3.6 *Let $\mathfrak{R}$ be an order relation on a set E and A be a non-empty part of E.*

❶ *We call **largest element** of A (or **maximum** of A), any element of A which is the upper bound of A, we denote it $max(A)$. In other words:*

$$\boxed{M = max(A) \iff M \text{ is the largest element of} A \iff [(M \in A) \text{ et } (\forall x \in A, x \mathfrak{R} M)]}$$

If one exists, this element is unique.

❷ *We call **smallest element** of A (or **minimum** of A), any element of A which is lower bound of A, we denote it $min(A)$. In other words*

$$\boxed{m = min(A) \iff m \text{ is the smallest element of} A \iff [(m \in A) \text{ and } (\forall x \in A, m \mathfrak{R} x)]}$$

If one exists, this element is unique

Example 3.3.6 ❶ . *With the usual relation $\leqslant$ defined on $\mathbb{R}$ i.e. A, B two parts of $\mathbb{R}$.*

$$A = \{2, 5, -7\}, \quad B =]0, 1[.$$

- ✔ *The set A has a minimal element which is -7, and a maximal element which is 5, $min(A) = -7$, $max(A) = 5$.*
- ✔ *The set B has neither a minimal nor a maximal element.*

❷ *For the relation* $\mathfrak{R}$ *on* $\mathbb{N}$:

$$\boxed{\forall x, y \in \mathbb{N},\ x\,\mathfrak{R}\,y \iff y \text{ multiple of } x.}$$

Let the subset $A = \{2, 3, 10\}$ *of* $\mathbb{N}$.

- ✔ A *does not have a maximal element, the upper bounds are the multiples of* 30 *and the smallest of upper bounds of* A *is* 30, *which does not belong to* A.
- ✔ A *does not have a minimal element, the lower bounds of* A *are the common divisors of* $2, 3$ *and* 10 *so the only lower bound of* A *is* 1, *which does not belong to* A.

Maximal element and minimal element of a set

Definition 3.3.7 *Let* E *be a set with an order relation* $\mathfrak{R}$ *and* A *a non-empty part of* E.

❶ *We say that* $a \in A$ *is a maximal element of* A *if*

$$\forall x \in A,\ a\mathfrak{R}x \Longrightarrow x = a$$

That is, there is no element x *in* A, *other than* a, *such that* a *is related to* x. *(or, there is no element in the set* A *greater than* a, *with respect to hàs the relation* $\mathfrak{R}$*).*

❷ *We say that* $b \in A$ *is a minimal element of* A *if* $\forall x \in A, : x\mathfrak{R}\, b \Longrightarrow x = b$.

That is, there exists no element x *in* A, *other than* b, *such that* x *is related to* b. *(or else, there is no element in the set* A *lower than* $, b$, *with respect to the relation* $\mathfrak{R}$*).*

❸ *We say that an element of* E *is extremal if it is or maximal or minimal.*

Exercise 3.3.2 (Exercise $N°5$ ***from the series*** $N°3$***)***

Solution 3.3.1 *to enter*