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| **A. Mira-Bejaia university** | **Faculty of Exact Sciences** | **Department of Mathematics** |
| **First year of mathematics** | **Algebra 1** | **2024-2025** |
| **Tutorial Series N°3** | **Functions** | |

**Exercise 1** Let $f:\mathbb{R}\longrightarrow\mathbb{R}$, $g:\mathbb{R}\longrightarrow\mathbb{R}$ be the functions defined by $f(x)=4x-3$, $g(x)=4x^2-7x+3$. Find $(f+g)(x)$, $(f+g)(2)$, , $(fg)(x)$, $(fg)(0)$ and $(f\circ g)(x)$, $(g\circ f)(x)$.

$(f+g)(x)=f(x)+g(x)=(4x-3)+(4x^2-7x+3)=4x^2-3x,$

$(f+g)(2)=10$

$(fg)(x)=f(x)g(x)=(4x-3)(4x^2-7x+3)=16x^3-28x^2+12x-12x^2+21x-9=16x^3-40x^2+33x-9,$

$(fg)(0)=-9.$

$(f\circ g)(x)=f(g(x))=4g(x)-3=4(4x^2-7x+3)-3=16x^2-28x+12-3=16x^2-28x+9.$

$(g\circ f)(x)=g(f(x))=4(4x-3)^2-7(4x-3)+3=4(16x^2-24x+9)-28x+21+3$

$=64x^2-96x+36-28x+24=64x^2-124x+60.$

**Exercise 2**

❶ Let $f:E\longrightarrow F$ be a constant function $f(x)=y_0$, for all $x\in E$. Determine $f^{-1}(B)$ for $B\subset F$?

$$\begin{aligned} f^{-1}(B) &= \{x\in E:\ f(x)\in B\}\\ &= \{x\in E:\ y_0\in B\}\\ &= \begin{cases} E & \text{if } y_0\in B.\\ \varnothing & \text{if } y_0\notin B.\end{cases}\end{aligned}$$

❷ Let $f:\mathbb{R}\longrightarrow\mathbb{R}$, and $g:\mathbb{R}\longrightarrow\mathbb{R}$ be the functions given by $f(x)=x^2$ and $g(x)=(x-1)^2$.

(a) Determine $f(\{-1,\ 1\})$, $f([-1,1])$, $g([0,2])$.

$f(\{-1,\ 1\})=\{f(-1),\ f(1)\}=\{1\}.$

$f([-1,1)=\{f(x)\ x\in[-1,1]\}$

$=f([-1,1])=f([-1,0]\cup[0,1])=f([-1,0])\cup f([0,1])$

$=[f(0),f(-1)]\cup[f(0),f(1)]$ (because $f$ is increasing in $[0,1]$ and decreasing in $[-1,0]$)

$=[0,1]\cup[0,1]=[0,1].$

$g([0,2])=g([0,1]\cup[1,2])=g([0,1])\cup g([1,2])$

$=[g(1),g(0)]\cup[g(1),g(2)]$ (because $g$ is increasing in $[1,2]$ and decreasing in $[0,1]$)

$=[0,1]\cup[0,1]=[0,1].$

(b) Determine $f^{-1}(\{1\})$, $f^{-1}([0,1])$, $f^{-1}([-1,1])$, $g^{-1}(\{-1\})$, $g^{-1}([-1,1])$.

$f^{-1}(\{1\})=\{x\in\mathbb{R}:\ f(x)\in\{1\}\}=\{x\in\mathbb{R}:x^2=1\}=\{-1,\ 1\}.$

$f^{-1}([0,1])=\{x\in\mathbb{R}:x^2\in[0,1]\}=[-1,1].$

$f^{-1}([-1,1])=\{x\in\mathbb{R}:x^2\in[-1,1]\}=\{x\in\mathbb{R}:x^2\in[0,1]\}=[-1,1].$

$g^{-1}(\{-1\})=\{x\in\mathbb{R}:\ g(x)=-1\}=\{x\in\mathbb{R}:\ (x-1)^2=-1\}=\varnothing.$

$g^{-1}([-1,1])=\{x\in\mathbb{R}:(x-1)^2\in[-1,1]\}=\{x\in\mathbb{R}:(x-1)^2\in[0,1]\}=[0,2]$

❸ Let $f : ]0,+\infty[ \longrightarrow ]0,+\infty[$ be the function given by $f(x) = \frac{1}{x}$. Determine $f^{-1}(]0,1[)$, and $f^{-1}([1,+\infty[)$?

$$f^{-1}(]0,1[) = \{x \in ]0,+\infty[ : f(x) \in ]0,1[\} = \{x \in ]0,+\infty[ : \frac{1}{x} \in ]0,1[\} = ]1,+\infty[$$
$$f^{-1}([1,+\infty[) = \{x \in ]0,+\infty[ : f(x) \in [1,\infty[\} = \{x \in ]0,+\infty[ : \frac{1}{x} \in [1,\infty[\} = ]0,1]$$

**Exercise 3** Let $f : E \longrightarrow F$ be a function. Let $A$ and $B$ be two subsets of $E$, and let $C$ and $D$ be two subsets of $F$. Prove that

❶ $f(A \cup B) = f(A) \cup f(B)$.

(a) First, show the inclusion ($\subset$)
Show that $f(A \cup B) \subset f(A) \cup f(B)$. Let $x \in f(A \cup B$, check if $x \in f(A) \cup f(B)$

$$\begin{aligned} x \in f(A \cup B) &\Longrightarrow \exists t \in A \cup B : x = f(t) \\ &\Longrightarrow (\exists t \in A : x = f(t)) \text{ or, } (\exists t \in B : x = f(t)) \\ &\Longrightarrow x \in f(A) \text{ or } x \in f(B) \\ &\Longrightarrow x \in f(A) \cup f(B). \end{aligned}$$

(b) Second show the inclusion ($\supset$).
Show that $f(A) \cup f(B) \subset f(A \cup B)$. Let $x \in f(A) \cup f(B)$, check if $x \in f(A \cup B)$

$$\begin{aligned} x \in f(A) \cup f(B) &\Longrightarrow x \in f(A) \text{ or } x \in f(B) \\ &\Longrightarrow (\exists t_1 \in A : x = f(t_1)) \text{ or} (\exists t_2 \in B : x = f(t_2)) \\ &\Longrightarrow \exists t \in A \cup B : x = f(t) \\ &\Longrightarrow x \in f(A \cup B). \end{aligned}$$

We conclude that $f(A \cup B) \subset f(A) \cup f(B)$.

❷ $f^{-1}(C_F(C)) = C_E(f^{-1}(C))$

(a) First, show the first inclusion ($\subset$)
Show that $f^{-1}(C_F(C)) \subset C_E(f^{-1}(C))$. Let $x \in f^{-1}(C_F(C))$, check if $x \in C_E(f^{-1}(C))$

$$\begin{aligned} x \in f^{-1}(C_F(C)) &\Longrightarrow f(x) \in C_F(C) \\ &\Longrightarrow f(x) \in F \text{ and } f(x) \notin C \\ &\Longrightarrow x \in E \text{ and } x \notin f^{-1}(C) \\ &\Longrightarrow x \in C_E(f^{-1}(C)). \end{aligned}$$

(b) Second show the inclusion ($\supset$).
Show that $C_E(f^{-1}(C)) \subset f^{-1}(C_F(C))$. Let $x \in C_E(f^{-1}(C))$, check if $x \in f^{-1}(C_F(C))$

$$\begin{aligned} x \in C_E(f^{-1}(C)) &\Longrightarrow x \in E \text{ and } x \notin f^{-1}(C) \\ &\Longrightarrow f(x) \in F \text{ and } f(x) \notin C \\ &\Longrightarrow f(x) \in C_F(C) \\ &\Longrightarrow x \in f^{-1}(C_F(C)). \end{aligned}$$

We conclude that $f^{-1}(C_F(C)) = C_E(f^{-1}(C))$.

❸ If $C \subset D$, then $f^{-1}(C) \subset f^{-1}(D)$.
We suppose that $C \subset D$ and show that $f^{-1}(C) \subset f^{-1}(D)$ Let $x \in f^{-1}(C)$, check if $x \in f^{-1}(D)$

$$\begin{aligned} x \in f^{-1}(C) &\Longrightarrow f(x) \in C \\ &\Longrightarrow f(x) \in D \text{ (because } C \subset D) \\ &\Longrightarrow x \in f^{-1}(D). \end{aligned}$$

So, if $C \subset D$, then $f^{-1}(C) \subset f^{-1}(D)$.

**Exercise 4** Let $f : E \longrightarrow F$ be a function. Show that

❶ $\forall A \in \mathscr{P}(E),\ A \subset f^{-1}(f(A))$. Give an example of a function $f$ and a subset $A \subset E$, such that $f^{-1}(f(A)) \nsubseteq A$

Show that $A \subset f^{-1}(f(A))$

Let $x \in A$ and show that $x \in f^{-1}(f(A))$

$$\begin{aligned} x \in A &\Longrightarrow f(x) \in f(A) \\ &\Longrightarrow x \in f^{-1}(f(A)). \end{aligned}$$

An example to show that $f^{-1}(f(A)) \nsubseteq A$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $f(x) = x^2$, and $A$ be a subset of $\mathbb{R}$, $A = \{-1\}$.Then we have $f(A) = \{1\}$, $f^{-1}(f(A)) = \{-1,\ 1\}$, so, $f^{-1}(f(A)) \nsubseteq A$

❷ $\forall B \in \mathscr{P}(F), f(f^{-1})(B) \subset B$. Give an example of a function $f$ and a subset $B \subset F$, such that $B \nsubseteq f(f^{-1}(B))$.

Show that $f(f^{-1}(B)) \subset B$

Let $x \in f(f^{-1}(B))$ and show that $x \in B$

$$\begin{aligned} x \in f(f^{-1}(B)) &\Longrightarrow \exists t \in f^{-1}(B)\ :\ x = f(t) \in B \text{ (By definition of inverse image of} B) \\ &\Longrightarrow x \in B. \end{aligned}$$

An example to show that $B \nsubseteq f(f^{-1}(B))$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $f(x) = x^2$, and $B$ be a subset of $\mathbb{R}$, $B = \{-1, 1\}$.Then we have $f^{-1}(B) = \{-1,\ 1\}$, $f(f^{-1}(B)) = \{1\}$, so, $B \nsubseteq f(f^{-1}(B))$

**Exercise 5** Let $f$ be a function $E$ defined by

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \frac{2x}{1+x^2}. \end{aligned}$$

❶ Determine $f^{-1}\left(\left\{\frac{1}{2}\right\}\right)$, $f^{-1}\left(\left\{2\right\}\right)$. Is $f$ injective? Surjective?

$$\begin{aligned} f^{-1}\left(\left\{\frac{1}{2}\right\}\right) &= \{x \in \mathbb{R}\ :\ f(x) = \frac{1}{2}\} \\ f(x) = \frac{1}{2} &\iff \frac{2x}{1+x^2} = \frac{1}{2} \iff 4x = 1 + x^2 \iff x^2 - 4x + 1 = 0. \end{aligned}$$

Compute $\Delta$

$\Delta = (-4)^2 - 4(1) = 12 > 0$, so the equation $x^2 - 4x + 1 = 0$ has two solutions. $x_1 = \frac{4-\sqrt{12}}{2} = 2 - \sqrt{3}$. and $x_2 = \frac{4+\sqrt{12}}{2} = 2 + \sqrt{3}$. Thus $f^{-1}\left(\left\{\frac{1}{2}\right\}\right) = \{2 - \sqrt{3},\ 2 + \sqrt{3}\}$.

$$\begin{aligned} f^{-1}\left(\left\{2\right\}\right) &= \{x \in \mathbb{R}\ :\ f(x) = 2\} \\ f(x) = 2 &\iff \frac{2x}{1+x^2} = 2 \iff 2x = 2 + 2x^2 \iff 2x^2 - 2x + 2 = 0. \end{aligned}$$

Compute $\Delta$

$\Delta = (-2)^2 - 4(2)(2) = -12 < 0$, so the equation $x^2 - 4x + 1 = 0$ does not have a solutions. Thus $f^{-1}\left(\left\{2\right\}\right) = \emptyset$, therefore, we deduce that $f$ is not injective and $f$ is not surjective.

❷ For which $y \in \mathbb{R}$ the equation $f(x) = y$ has solutions in $\mathbb{R}$? Show that $f(\mathbb{R}) = [-1,\ 1]$.

$$f(x) = y \iff \frac{2x}{1+x^2} = y \iff 2x = y(1+x^2) \iff yx^2 - 2x + y = 0.$$

if $y = 0$, then $x = 0$ If $y \neq 0$ then the equation is the second degree equation, we have to calculate $\Delta$. Compute $\Delta$

$\Delta = (-2)^2 - 4(y)(y) = 4 - 4y^2 = 4(1 - y^2)$

(a) If $y \in [-1, 0[\cup]0, 1]$ then $\delta \geqslant 0$, so the equation $f(x) = y$ has solutions.

(b) If $y \in ]-\infty, -1[\cup]1, +\infty[$ then $\delta < 0$, so the equation $f(x) = y$ does not have solutions.

In conclusion, the equation $f(x) = y$ has solution when $y \in [-1, 1]$. thus $f(\mathbb{R}) = [-1, 1]$.

❹ Show that the function $g$ defined by

$$g : [-1, 1] \longrightarrow [-1, 1]$$
$$x \mapsto g(x) = f(x).$$

is bijective and find its inverse $g^{-1}$.

(a) First show that $g$ is injective or one to one.

$$\forall x, y \in [-1, 1],\ g(x) = g(y) \Longrightarrow x = y$$

$$\begin{aligned} g(x) = g(y) &\Longrightarrow \frac{2x}{1+y^2} = \frac{2y}{1+x^2} \\ &\Longrightarrow 2x(1+y^2) = 2y(1+x^2) \\ &\Longrightarrow 2x + 2xy^2 - 2y - 2yx^2 = 0 \\ &\Longrightarrow 2x - 2y + 2xy^2 - 2yx^2 = 0 \\ &\Longrightarrow 2(x-y) + 2xy(y-x) = 0 \\ &\Longrightarrow 2(x-y)(1-xy) = 0 \\ &\Longrightarrow x - y = 0. \text{or } 1 - xy = 0. \\ &\Longrightarrow x = y \text{ or} (x = y = 1 \text{or} x = y = -1) \\ &\text{(because the only solution which belong to } [-1, 1], \text{is } x = y = 1 \text{ or } x = y = -1). \\ &\Longrightarrow x = y \end{aligned}$$

So $g$ is injective.

(b) Second , show that $g$ is surjective ( onto)

$$\forall y \in [-1, 1], \exists x \in [-1, 1] \,:\, y = g(x)$$

$$g(x) = y \iff \frac{2x}{1+x^2} = y \iff 2x = y(1+x^2) \iff yx^2 - 2x + y = 0.$$

if $y = 0$, then $x = 0$

If $y \neq 0$ then the equation is the second degree equation, we have to calculate $\Delta$.

$\Delta = (-2)^2 - 4(y)(y) = 4 - 4y^2 = 4(1-y^2)$ since $y \in [-1, 0[\cup]0\,1]$ then $\Delta \geqslant 0$, so the equation $f(x) = y$ has solutions

$$x_1 = \frac{2 - \sqrt{4(1-y^2)}}{2y} = \frac{1 - \sqrt{1-y^2}}{y} \in [-1, 0[\cup]0, 1],$$
$$x_2 = \frac{2 + \sqrt{4(1-y^2)}}{2y} = \frac{1 + \sqrt{1-y^2}}{y} \notin [-1, 0[\cup]0, 1].$$

Therefore $g$ is surjective.

we deduce that $g$ is bijective, so the inverse function of $g$ exist

$g(x) = y \iff x = g^{-1}(y) = \frac{1-\sqrt{1-y^2}}{y}$ if $y \neq 0$.

$$g^{-1} : [-1, 1] \longrightarrow [-1, 1]$$
$$x \mapsto g^{-1}(x) = \begin{cases} \frac{1-\sqrt{1-x^2}}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

**Exercise 1** (**Homework**) Let $f(x) = \frac{1}{x^2}$, $x \neq 0$, $x \in \mathbb{R}$

❶ Determine the direct image, $f(E)$ where $E = \left\{x \in \mathbb{R} : 1 \leqslant x \leqslant 2\right\}$.

$f(E) = f([1, 2]) = [f(2), f(1)] = [\frac{1}{4}, 1]$ ( because $f$ is decreasing function in $]0, +\infty[$ and increasing in $]-\infty, 0[$.)

❷ Determine the inverse image $f^{-1}(G)$, where $G = \left\{x \in \mathbb{R} : 1 \leqslant x \leqslant 4\right\}$.

$$\begin{aligned} f^{-1}(G) &= \{x \in \mathbb{R} \; : \; f(x) \in G\} \\ &= \{x \in \mathbb{R} \; : \; f(x) \in [1, 4]\} \\ &= \{x \in \mathbb{R} \; : \; 1 \leqslant \frac{1}{x^2} \leqslant 4\} \\ &= \{x \in \mathbb{R} \; : \; \frac{1}{4} \leqslant x^2 \leqslant 1\} \\ &= \{x \in \mathbb{R} \; : \; \frac{1}{2} \leqslant x \leqslant 1 \text{ or } -1 \leqslant x \leqslant \frac{-1}{2}\} \\ &= [-1, \frac{-1}{2}] \cup [\frac{1}{2}, 1]. \end{aligned}$$

**Exercise 2**(**Homework**) Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be a functions. Let $H \subset G$. Show that

$$(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H)).$$

1. First, show the inclusion ($\subset$)
   Show that $(g \circ f)^{-1}(H) \subset f^{-1}(g^{-1}(H))$. Let $x \in (g \circ f)^{-1}(H)$, check if $x \in f^{-1}(g^{-1}(H))$

$$\begin{aligned} x \in (g \circ f)^{-1}(H) &\Longrightarrow (g \circ f)(x) \in H \\ &\Longrightarrow g(f(x)) \in H \\ &\Longrightarrow f(x) \in g^{-1}(H) \\ &\Longrightarrow x \in f^{-1}(g^{-1}(H)). \end{aligned}$$

2. Second show the inclusion ($\supset$).
   Show that $f^{-1}(g^{-1}(H)) \subset (g \circ f)^{-1}(H)$. Let $x \in f^{-1}(g^{-1}(H))$, check if $x \in (g \circ f)^{-1}(H)$

$$\begin{aligned} x \in f^{-1}(g^{-1}(H)) &\Longrightarrow f(x) \in g^{-1}(H) \\ &\Longrightarrow g(f(x)) \in H \\ &\Longrightarrow (g \circ f)^{-1}(x) \in H \\ &\Longrightarrow x \in (g \circ f)^{-1}(H). \end{aligned}$$

We conclude that $(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$.