Exercise 1

Let E be a set E defined by $E = \{1,2,3,4,5,6,7,8,9,10\}$

❶ Let A, B be two subsets of E, such that : $A = \{3,4,6,7,1,9\}, B = \{5,6,8,4\}, C = \{5,8,4\}$. Determine

$$A \cap B,\ A \cup B,\ A \setminus B,\ B \setminus A,\ A\Delta B,\ C_E(A), C_E(B).$$

$$A \cap B = \{x \in E : x \in A \text{ and } x \in B\} = \{4,6\}, \quad A \cup B = \{x \in E : x \in A \text{ or } x \in B\} = \{1,3,4,5,6,7,8,9\}$$
$$A \setminus B = \{x \in E : x \in A \text{ and } x \notin B\} = \{3,7,9\}, \quad B \setminus A = \{x \in E : x \in B \text{ and } x \notin A\} = \{5,8\}.$$
$$A\Delta B = A \setminus B \cup B \setminus A = \{3,5,7,8,9\},\ C_E(A) = \{x \in E : x \notin A\} = \{2,5,8,10\}$$
$$C_E(B) = \{x \in E : x \notin B\} = \{1,2,3,5,7,9,10\}$$

❷ Suppose that $F = \{0,\ 1\},\ G = \{1,\ 2\}$. Find

$$F \times G, G \times F,\ F \times F,\ (F \cap G) \times F, \mathscr{P}(F),\ \mathscr{P}(G),\ \mathscr{P}(F \cap G),\ \mathscr{P}(F) \cap \mathscr{P}(G).$$

$$F \times G = \{(x,y) : x \in F \text{ and } y \in G\} = \{(0,1), (1,1), (0,2), (1,2)\},$$
$$G \times F = \{(x,y) : x \in G \text{ and } y \in F\} = \{(1,0), (1,1), (2,0), (2,1)\},$$
$$F \times F = \{(x,y) : x \in F \text{ and } y \in F\} = \{(0,0), (0,1), (1,0), (1,1)\}$$
$$(F \cap G) \times F = \{(x,y) : x \in F \cap G \text{ and } y \in F\} = \{(1,0), (1,1)\}$$
$$\mathscr{P}(F) = \{\emptyset, \{0\}, \{1\}, F\}, \quad \mathscr{P}(G) = \{\emptyset, \{1\}, \{2\}, G\}, \quad \mathscr{P}(F \cap G) = \mathscr{P}(\{1\}) = \{\emptyset, \{1\}\}$$
$$\mathscr{P}(F) \cap \mathscr{P}(G) = \{\emptyset, \{1\}\}.$$

❸ Suppose $A_1 = \{a,b,d,e,g,f\}, A_2 = \{a,b,c,d\},\ A_3 = \{b,d,a\}$ and $A_4 = \{a,b,h\}$. Find A and B such that :

$$A = \bigcup_{i=1}^{4} A_i, \quad B = \bigcap_{i=1}^{4} A_i.$$

Let E be a set of alphabet.

$$A = \bigcup_{i=1}^{4} A_i = \{x \in E : x \in A_1 \text{ or } x \in A_2 \text{ or } x \in A_3 \text{ or } x \in A_4\} = \{a,\ b,\ c,\ d,\ e,\ f,\ g,\ h\}$$
$$B = \bigcap_{i=1}^{4} A_i = \{x \in E : x \in A_1 \text{ and } x \in A_2 \text{ and } x \in A_3 \text{ and } x \in A_4\} = \{a,\ b\}$$

Exercise 2

❶ Let $E = \left\{a, b, c\right\}$. Determine whether the following statements are true

$a \in E$ True, $a \subset E$ (False because a is not a set), $\emptyset \in E$ (False), $\left\{\emptyset\right\} \subset E$ (False because, $E \cup \emptyset = E$ (True), $\emptyset \subset E$ (True).

❷ Let $A = \left\{1, 2, \{1\}, \{1,2\}\right\}$. Are the following statements true or false?

(a) $\{1\} \in A$(True) (b) $\left\{\{1\}\right\} \in A$ (False) (c) $2 \in A$. (True) (d) $\{2\} \in A$ (False)

(e) $\{2\} \subset A$ (True) (f) $\{1\} \subset A$ (True) (g) $\left\{\{1\}\right\} \subset A$ (True) (h) $2 \subset A$ (False).

Exercise 3

❶ List all the subsets of the following sets

$$\left\{1, 2, \emptyset\right\}, \quad \left\{\emptyset\right\}, \quad \left\{\{\mathbb{R}\}\right\}.$$

$$\mathscr{P}(\{1, 2, \emptyset\}) = \{\emptyset, \{1\}, \{2\}, \{\emptyset\}, \{1, 2\}, \{1, \emptyset\}, \{2, \emptyset\}, \{1, 2\ \emptyset\}\}$$
$$\mathscr{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$
$$\mathscr{P}(\{\{\mathbb{R}\}\}) = \{\emptyset, \{\mathbb{R}\}\}$$

❷ Find the cardinality of each of the following sets

$$A = \left\{1, 2, \{1\}, \{1,2\}\right\}, B = \left\{1, \{2\}\right\}, C = \left\{1, 2, 2, 2\right\}, D = \left\{\emptyset, \ \{\emptyset\}\right\} \text{ and } E = \left\{5n : n \in \mathbb{N}\right\}.$$

$|A| = |\left\{1, 2, \{1\}, \{1,2\}\right\}| = 4, |B| = |\left\{1, \{2\}\right\}| = 2, |C| = |\left\{1, 2, 2, 2\right\}| = 2, |D| = |\left\{\emptyset, \ \{\emptyset\}\right\}| = 2$ and $|E| = |\left\{5n : n \in \mathbb{N}\right\}| = \infty$.

Exercise 4

Let A, B, C be subsets of a set E. Show that

❶ $A \cap B = \emptyset \iff A \subset C_E(B)$.

(a) First, show the first implication ($\Longrightarrow$)
We assume that $A \cap B = \emptyset$ and show that $A \subset C_E(B)$ Let $x \in A$, check if $x \in C_E(B)$

$$x \in A \Longrightarrow x \notin B (\text{because } A \cap B = \emptyset)$$
$$\Longrightarrow x \in C_E(B).$$

(b) Second show the second implication ($\Longleftarrow$)
We assume that $A \subset C_E(B)$ and show that $A \cap B = \emptyset$ By contradiction , we suppose that $A \cap B \neq \emptyset$ which means that there exist $x \in A \cap B$, so $x \in A$ and $x \in B$, since $A \subset C_E(B)$, then $x \in C_E(B)$ and $x \in B$ absurd.

We conclude that $A \cap B = \emptyset \iff A \subset C_E(B)$

❷ $A \subset B \iff C_E(B) \subset C_E(A)$.

(a) First, show the first implication ($\Longrightarrow$)
We assume that $A \subset B$ and show that $C_E(B) \subset C_E(A)$. Let $x \in C_E(B)$, check if $x \in C_E(A)$

$$\begin{aligned} x \in C_E(B) &\Longrightarrow x \in E \text{ and } x \notin B \\ &\Longrightarrow x \in E \text{ and } x \notin A. \text{ (because } A \subset B) \\ &\Longrightarrow x \in C_E(A) \end{aligned}$$

(b) Second show the implication ($\Longleftarrow$)
We assume that $C_E(B) \subset C_E(A)$ and show that $A \subset B$. Let $x \in A$, so $x \notin C_E(A)$, since $C_E(B) \subset C_E(A)$, then $x \notin C_E(B)$, so $x \in B$.

We conclude that $A \subset B \iff C_E(B) \subset C_E(A)$.

❸ $A \cup B = A \cap C \iff B \subset A \subset C$.

(a) First, show the first implication ($\Longrightarrow$)
We assume that $A \cup B = A \cap C$ and show that $B \subset A \subset C$.

i. Show $B \subset A$
Let $x \in B$, check if $x \in A$

$$\begin{aligned} x \in B &\Longrightarrow x \in A \cup B \\ &\Longrightarrow x \in A \cap C. \text{ (because } A \cup B = A \cap C) \\ &\Longrightarrow x \in A. \end{aligned}$$

Thus $B \subset A$

ii. Show $A \subset C$
Let $x \in A$, check if $x \in C$

$$\begin{aligned} x \in A &\Longrightarrow x \in A \cup B \\ &\Longrightarrow x \in A \cap C. \text{ (because } A \cup B = A \cap C) \\ &\Longrightarrow x \in C. \end{aligned}$$

Therefore $A \subset C$

(b) Second show the implication ($\Longleftarrow$)
We assume that $B \subset A \subset C$ and show that $A \cup B = A \cap C$. it's easy to see that $A \cup B = A$(because $B \subset A$) and $A \cap C = A$ (because $A \subset C$), so we have equality.

We conclude that $A \subset B \iff C_E(B) \subset C_E(A)$.

❹ $C_E(A \cap B) = C_E(A) \cup C_E(B)$.

(a) First, show the first inclusion ($\subset$)
Show that $C_E(A \cap B) \subset C_E(A) \cup C_E(B)$. Let $x \in C_E(A \cap B)$, check if $x \in C_E(A) \cup C_E(B)$

$$\begin{aligned} x \in C_E(A \cap B) &\Longrightarrow x \in E \text{ and } x \notin A \cap B \\ &\Longrightarrow x \in E \text{ and} (x \notin A. \text{ or } x \notin B) \\ &\Longrightarrow (x \in E \text{ and } x \notin A) \text{ or } (x \in E \text{ and } x \notin B) \\ &\Longrightarrow x \in C_E(A) \text{ or } x \in C_E(B) \\ &\Longrightarrow x \in C_E(A) \cup C_E(B) \end{aligned}$$

(b) Second show the inclusion ($\supset$).
Show that $C_E(A) \cup C_E(B) \subset C_E(A \cap B)$. Let $x \in C_E(A) \cup C_E(B)$, check if $x \in C_E(A \cap B)$

$$\begin{aligned} x \in C_E(A) \cup C_E(B) &\Longrightarrow x \in C_E(A) \text{ or } x \in C_E(B) \\ &\Longrightarrow (x \in E \text{ and } x \notin A) \text{ or } (x \in E \text{ and } x \notin B) \\ &\Longrightarrow x \in E \text{ and} (x \notin A. \text{ or } x \notin B) \\ &\Longrightarrow x \in E \text{ and } x \notin A \cap B \\ &\Longrightarrow x \in C_E(A \cap B). \end{aligned}$$

We conclude that $C_E(A\cap B) = C_E(A)\cup C_E(B)$.

❺ $C_E(A\cup B) = C_E(A)\cap C_E(B)$ (homework).

(a) First, show the first inclusion ($\subset$)
Show that $C_E(A\cup B)\subset C_E(A)\cap C_E(B)$. Let $x\in C_E(A\cup B)$, check if $x\in C_E(A)\cap C_E(B)$

$$\begin{aligned} x\in C_E(A\cup B) &\Longrightarrow x\in E \text{ and } x\notin A\cup B \\ &\Longrightarrow x\in E \text{ and} (x\notin A. \text{ and } x\notin B) \\ &\Longrightarrow (x\in E \text{ and } x\notin A) \text{ and } (x\in E \text{ and } x\notin B) \\ &\Longrightarrow x\in C_E(A) \text{ and } x\in C_E(B) \\ &\Longrightarrow x\in C_E(A)\cap C_E(B) \end{aligned}$$

(b) Second show the inclusion ($\supset$).
Show that $C_E(A)\cap C_E(B)\subset C_E(A\cup B)$. Let $x\in C_E(A)\cap C_E(B)$, check if $x\in C_E(A\cup B)$

$$\begin{aligned} x\in C_E(A)\cap C_E(B) &\Longrightarrow x\in C_E(A) \text{ and } x\in C_E(B) \\ &\Longrightarrow (x\in E \text{ and } x\notin A) \text{ and } (x\in E \text{ and } x\notin B) \\ &\Longrightarrow x\in E \text{ and} (x\notin A. \text{ and } x\notin B) \\ &\Longrightarrow x\in E \text{ and } x\notin A\cup B \\ &\Longrightarrow x\in C_E(A\cup B). \end{aligned}$$

We conclude that $C_E(A\cup B) = C_E(A)\cap C_E(B)$.

❻ $(A\times C)\cup(B\times C) = (A\cup B)\times C$.

(a) First, show the first inclusion ($\subset$)
Show that $(A\times C)\cup(B\times C)\subset(A\cup B)\times C$. Let $(x,y)\in(A\times C)\cup(B\times C)$, check if $(x,y)\in(A\cup B)\times C$

$$\begin{aligned} (x,y)\in(A\times C)\cup(B\times C) &\Longrightarrow (x,y)\in A\times C \text{ or } (x,y)\in B\times C \\ &\Longrightarrow (x\in A \text{ and } y\in C) \text{ or } (x\in B \text{ and } y\in C) \\ &\Longrightarrow (x\in A \text{ or } x\in B) \text{ and } y\in C \\ &\Longrightarrow x\in A\cup B \text{ and } y\in C \\ &\Longrightarrow (x,y)\in(A\cup B)\times C. \end{aligned}$$

(b) Second show the inclusion ($\supset$).
Show that $(A\cup B)\times C\subset(A\times C)\cup(B\times C)$. Let $(x,y)\in(A\cup B)\times C$, check if $(x,y)\in(A\times C)\cup(B\times C)$

$$\begin{aligned} (x,y)\in(A\cup B)\times C &\Longrightarrow x\in A\cup B \text{ and } y\in C \\ &\Longrightarrow (x\in A \text{ or } x\in B) \text{ and } y\in C \\ &\Longrightarrow (x\in A \text{ and } y\in C) \text{ or } \ (x,y)\in B\times C \text{ and } y\in C) \\ &\Longrightarrow (x,y)\in A\times C \text{ or } (x,y)\in A\times C \\ &\Longrightarrow (x,y)\in A\times C\cup A\times C. \end{aligned}$$

We conclude that $(A\times C)\cup(B\times C) = (A\cup B)\times C$.