A. Mira-Bejaia university	**Faculty of Exact Sciences**	**Department of Mathematics**
First year of mathematics	**Algebra 1**	**2024-2025**
Tutorial Series N° 4	**Binary relations**	

Exercise 1 Determine whether the relations defined below are reflexive, symmetric, antisymmetric, transitive.

1. Let $\mathscr{R}$ be the relation given by $a\mathscr{R}b \iff a=|b|$, for all $a,b\in\mathbb{Z}$.
   - This relation is not reflexive because we don't have for all integer a, $a\mathscr{R}a$, for example for $a=-1$, $-1\not\mathscr{R}-1$. $(-1\neq|-1|)$.
   - This relation is not symmetric because we don't have for all integer a, b if $a\mathscr{R}b$, then $b\mathscr{R}a$ for example for $a=-1$, $b=1$, $1\mathscr{R}-1$. $(1=|-1|)$, but $-1\not\mathscr{R}1$. $(-1\neq|1|)$
   - This relation is antisymmetric because we have for all integer a, b if $a\mathscr{R}b$ and $b\mathscr{R}a$ then $a=b$ if $a\mathscr{R}b$ and $b\mathscr{R}a$ then $a=|b|$ and $b=|a|$ so a and b positive thus $a=b$.
   - This relation is transitive because we have for all integer $a, b\,c$ if $a\mathscr{R}b$ and $b\mathscr{R}c$ then $a\mathscr{R}c$ if $a\mathscr{R}b$ and $b\mathscr{R}c$ then $a=|b|$ and $b=|c|$ so $a=|c|$ thus $a\mathscr{R}c$.
2. Let S be the relation given by $aSb \iff a+b$ is even, for all $a,b\in\mathbb{Z}$.
   - This relation is reflexive because we have for all integer a, aSa, which means $a+a=2a$ is always even.
   - This relation is symmetric because we have for all integer a, b if aSb, then bSa, because $a+b=b+a$
   - This relation is not antisymmetric because we don't have for all integer a, b if aSb and bSa then $a=b$ For example we have $1S2$ and $2S1$ but $1\neq2$.
   - This relation is transitive because we have for all integer $a, b\,c$ if aSb and bSc then aSc if aSb and bSc then $a+b$is even and $b+c$ is even so $a+c$ is even thus aSc.

Exercise 2 Let $\mathscr{R}$ be the relation on the set of ordered pairs of real numbers, $\mathbb{R}\times\mathbb{R}$, such that

$$(x,y)\,\mathscr{R}\,(u,v) \iff x^2+y^2=u^2+v^2.$$

❶ Show that $\mathscr{R}$ is an equivalence relation.

(a) Show that $\mathscr{R}$ is reflexive.$\forall(x,y)\in\mathbb{R}\times\mathbb{R}$, $(x,y)\mathscr{R}(x,y)$
We have $x^2+y^2=x^2+y^2$ which means that $(x,y)\mathscr{R}(x,y)$, so $\mathscr{R}$ is reflexive.

(b) Show that $\mathscr{R}$ is symmetric.$\forall(x,y),(u,v)\in\mathbb{R}\times\mathbb{R}$, if $(x,y)\mathscr{R}(u,v)$ then $(u,v)\mathscr{R}(x,y)$
We have

$$\begin{aligned}(x,y)\mathscr{R}(u,v) &\Longrightarrow x^2+y^2=u^2+v^2\\ &\Longrightarrow u^2+v^2=x^2+y^2\\ &\Longrightarrow (u,v)\mathscr{R}(x,y)\end{aligned}$$

so $\mathscr{R}$ is symmetric.

(c) Show that $\mathscr{R}$ is transitive.$\forall(x,y),(u,v),(a,b)\in\mathbb{R}\times\mathbb{R}$, if $(x,y)\mathscr{R}(u,v)$ and $(u,v)\mathscr{R}(a,b)$ then $(x,y)\mathscr{R}(a,b)$
We have

$$\begin{aligned}(x,y)\mathscr{R}(u,v) \text{ and } (u,v)\mathscr{R}(a,b) &\Longrightarrow x^2+y^2=u^2+v^2 \text{ and } u^2+v^2=a^2+b^2\\ &\Longrightarrow x^2+y^2=a^2+b^2\\ &\Longrightarrow (x,y)\mathscr{R}(a,b)\end{aligned}$$

so $\mathscr{R}$ is Transitive.

❷ Find the equivalence class of $(0,0)$ and $(0,1)$, deduce the equivalence class of $(1,0)$.Interpret these equivalence classes geometrically.

$$\begin{aligned}
\dot{\widehat{(0,0)}} &= \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : (x,y)\mathscr{R}(0,0)\right\}\\
&= \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : x^2+y^2=0^2+0^2\right\}\\
&= \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : x^2+y^2=0\right\}\\
&= \left\{(0,0)\right\}\\
\dot{\widehat{(0,1)}} &= \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : (x,y)\mathscr{R}(0,1)\right\}\\
&= \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : x^2+y^2=0^2+1^2\right\}\\
&= \left\{(x,y)\in\mathbb{R}\times\mathbb{R} : x^2+y^2=1\right\}
\end{aligned}$$

$\dot{\widehat{(0,1)}}=\dot{\widehat{(1,0)}}$,
geometrically $\dot{\widehat{(0,1)}}$ is a circle with origin $(0,0)$ and radius 1.

❸ Find the quotient set $(\mathbb{R}^2)/\mathscr{R}$.

$$(\mathbb{R}^2)/\mathscr{R}=\left\{\dot{\widehat{(x,y)}} : (x,y)\in\mathbb{R}\times\mathbb{R}\right\} \text{ is the the union of singleton } \{(0,0)\}$$
and all circle with origin $(0,0)$ and radius r, with $r>0$

$$(\mathbb{R}^2)/\mathscr{R}=\bigcup_{r\geqslant 0}\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=r^2\right\}$$

Exercise 3 We define the following relation S on $\mathbb{N}^*$

$$\boxed{\forall n,m,\in\mathbb{N}^*,\ nSm \iff m \text{ divides } n}$$

❶ Verify that $6S2$ and $5S1$. we have $6=3.2$ which means that 2 divides 6 so $6S2$.
The same method to prove that $5S1$, we have $5=5.1$ so 1 divides 5, thus $5S1$.

❷ Show that the relation S is a partial order relation on $\mathbb{N}^*$.

(a) Show that S is reflexive.$\forall n\in\mathbb{N}^*,\ nSn$
We have $n=1.n$ which means that nSn, so S is reflexive.

(b) Show that S is antisymmetric $\forall n,m\in\mathbb{N}^*$, if nSm and nSm then $n=m$
We have

$$\begin{aligned}
nSm \text{ and } mSn &\Longrightarrow (\exists k\in\mathbb{N}^* : n=k.m) \text{ and } (\exists k'\in\mathbb{N}^* : m=k'.n)\\
&\Longrightarrow \exists k,k'\in\mathbb{N}^* : n=k.m \text{ and}, m=k'.k.m)\\
&\Longrightarrow \exists k,k'\in\mathbb{N}^* : n=k.m \text{ and}, k'.k=1\\
&\Longrightarrow \exists k,k'\in\mathbb{N}^* : n=k.m \text{ and}, k'=k=1\\
&\Longrightarrow n=m
\end{aligned}$$

so S is antisymmetric.

(c) Show that S is transitive.$\forall n,m,t\in\mathbb{N}^*$, if nSm and mSt then nSt
We have

$$\begin{aligned}
nSm \text{ and } mSt &\Longrightarrow (\exists k\in\mathbb{N}^* : n=k.m) \text{ and } (\exists k'\in\mathbb{N}^* : m=k'.t)\\
&\Longrightarrow \exists k,k'\in\mathbb{N}^* : n=k.k'.t\\
&\Longrightarrow \exists k''\in\mathbb{N}^* : n=k''.t\\
&\Longrightarrow nSt
\end{aligned}$$

so S is transitive.

❸ Let $A = \{2, 3, 4\}$ be a subset of $\mathbb{N}^*$

(a) What are the bounds of A (upper bound and lower bound) ?

i. Find the upper bound of the set A Let $M \in \mathbb{N}^*$,

$$\begin{aligned} M \text{ is an upper bound of } A &\iff \forall n \in A, nSM \\ &\iff 2SM \text{ and } 3SM \text{ and } 4SM. \\ &\iff M \text{ is a commun divisor of } 2, 3 \text{ and } 4. \\ &\iff M = 1. \end{aligned}$$

So the only upper bound of A is $M = 1$
The least upper bound of A (or supremum) denoted sup(A) is 1. (sup(A)=1), since $sup(A) \notin A$ then the greatest element of A does not exist. (maximum of A does not exist.)

ii. Find the lower bound of the set A Let $m \in \mathbb{N}^*$,

$$\begin{aligned} m \text{ is a lower bound of } A &\iff \forall n \in A, mSm \\ &\iff mS2 \text{ and } mS3 \text{ and } mS4. \\ &\iff M \text{ is a commun multiple of } 2, 3 \text{ and } 4. \\ &\iff m \text{ is a multiple of } 12. \end{aligned}$$

So the set of lower bound of A is the set of multiple of 12
The greatest lower bound of A (or infimum) denoted inf(A) is 12. (Inf(A)=12), since $Inf(A) \notin A$ then the least element of A does not exist. (minimum of A does not exist.)

(b) What are the minimal elements? maximal elements?

i. Let $a \in A$, a is said to be maximal if

$$\forall b \in A, \text{if } aSb \text{ then } a = b.$$

We check all the elements of A We have $2 \not S 3$ and $2 \not S 4$, so 2 is maximal.
$3 \not S 2$ and $3 \not S 4$, so 3 is maximal.
$4 \not S 3$ but $4S2$, so 4 is not maximal.

ii. Let $a \in A$, a is said to be minimal if

$$\forall b \in A, \text{ if } bSa \text{ then } a = b.$$

We check all the elements of A We have $3 \not S 2$ but $4S2$, so 2 is not minimal.
$2 \not S 3$ and $4 \not S 3$, so 3 is minimal.
$2 \not S 4$ and $3 \not S 4$, so 4 is minimal.

Homework

Exercise 1 Determine whether the relation $\mathscr{R}$ on the set of all integers is reflexive, symmetric, antisymmetric and transitive. Where $x \mathscr{R} y$ if and only if

❶ $x \neq y$

— This relation is not reflexive because we don't have for all integer a, $a\mathscr{R}a$, for example for $a = -1$, $-1 \not\mathscr{R} -1$. $(-1 \neq -1)$.

— This relation is symmetric because for all integer a, b if $a\mathscr{R}b$, then $b\mathscr{R}a$

— This relation is not antisymmetric because we have $a\mathscr{R}b$ and $b\mathscr{R}a$ but $a \neq b$

— This relation is not transitive because if $a\mathscr{R}b$ and $b\mathscr{R}c$ does not imply that $a\mathscr{R}c$ for example $a\mathscr{R}b$ and $b\mathscr{R}a$ but $a\not\mathscr{R}a$.

❷ x y ⩾ 1

— This relation is not reflexive because we don't have for all integer a, $a\mathscr{R}a$, for example for $a = 0$, $0\not\mathscr{R}0$.

— This relation is symmetric because for all integer a, b if $a\mathscr{R}b$, then $b\mathscr{R}a$

— This relation is not antisymmetric because we have $a\mathscr{R}b$ and $b\mathscr{R}a$ but $a \neq b$

— This relation is transitive because if $a\mathscr{R}b$ and $b\mathscr{R}c$ then $a\mathscr{R}c$

❸ $x = y + 1$ or $x = y - 1$

❹ $x \equiv y(mod 7)$

❺ $x + y = 7$

Exercise 2 Let $\mathscr{R}$ be the relation on the set of ordered pairs of positive integers such that

$$\boxed{\forall (a, b), (c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+, \quad (a, b)\mathscr{R}(c, d) \iff ad = bc.}$$

Show that $\mathscr{R}$ is an equivalence relation.

Exercise 3 Let $\mathscr{R}$ be a relation on the set $\mathbb{N}^*$

$$\boxed{\forall a, b \in \mathbb{N}^*, \ a\mathscr{R}b \iff \exists k \in \mathbb{N} : \tfrac{b}{a} = 2^k}$$

1. Show that $\mathscr{R}$ is an order relation on $\mathbb{N}^*$.
2. Decide whether $\mathscr{R}$ is a totally order relation on $\mathbb{N}^*$. Why?

Remark 1. *For the first exercise of homewok, choose only two relations.*