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First year of mathematics	**Algebra 1**	**2024-2025**
Tutorial Series N°1	**Mathematical logic**	

Exercise 1

Which of the following expressions are propositions? In the case of a proposition , say whether it is true or false :

❶ $\sqrt{3}$ is an irrational number. This is a proposition which is true.

❷ The integer n divides 12. this expression is not a proposition, because we can not decide if it is true or false, its truth value depend on the value of n

❸ $\forall n \in \mathbb{N}$, $n+1=5$.This is a proposition which is false, for $n=1$, we have $n+1 \neq 5$

❹ $\exists n \in \mathbb{N}$, $n+1=5$. This is a proposition which is true, there exist $n=4$, for which $n+1=5$

❺ 25 is a multiple of 5 and 2 divides 7. This is a proposition, which is false , because here we have a conjunction between two propositions, one of them is false (2 doesn't divide 7)

❻ 25 is a multiple of 5 or 2 divides 7. This is a proposition, which is true, because here we have a disjunction between two propositions, one of them is true (25 is a multiple of 5)

Exercise 2

Let P, Q be propositions. Give the truth table of these propositions.

❶ $(P \Longrightarrow Q) \wedge (\overline{P} \Longrightarrow Q)$.

P	Q	$\overline{P}$	$P \Longrightarrow Q$	$(\overline{P} \Longrightarrow Q)$	$(P \Longrightarrow Q) \wedge (\overline{P} \Longrightarrow Q)$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

❷ $(\overline{P \Longrightarrow Q}) \Longleftrightarrow (P \wedge \overline{Q})$.

P	Q	$\overline{Q}$	$P \Longrightarrow Q$	$(\overline{P \Longrightarrow Q}$	$(P \wedge \overline{Q}$	$(\overline{P \Longrightarrow Q}) \Longleftrightarrow (P \wedge \overline{Q})$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

❸ $(P \Longrightarrow Q) \Longleftrightarrow (\overline{Q} \Longrightarrow \overline{P})$.

P	Q	$\overline{P}$	$\overline{P}$	$P \Longrightarrow Q$	$(\overline{Q} \Longrightarrow \overline{P})$	$(P \Longrightarrow Q) \Longleftrightarrow (\overline{Q} \Longrightarrow \overline{Q})$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Exercise 3

Let f and g be two functions of $\mathbb{R}$ in $\mathbb{R}$, write in terms of quantifiers the following expressions :

❶ f never equals zero. $\forall x \in \mathbb{R}, f(x) \neq 0$

❷ f is even.$\forall x \in \mathbb{R}, f(-x) = f(x)$

❸ f is bounded.$\exists M \in \mathbb{R} : |f(x)| \leqslant M$

❹ f is strictly increasing function. $\forall x, y \in \mathbb{R} : x < y \Longrightarrow f(x) < f(y)$

❺ f less than g. $\forall x \in \mathbb{R}, f(x) \leqslant g(x)$.

Exercise 4

Show which of the following propositions are true and which are false, then give their negation :

❶ $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0$. This proposition is false, for a given x , we can take $y = -x$, then we get $x + y = 0$

❷ $(\exists x \in \mathbb{R}, x + 1 = 0) \wedge (\exists x \in \mathbb{R}, x + 2 = 0)$. This proposition is True because for the first proposition we take x=-1, and for the second we take $x = -2$

❸ $\exists x \in \mathbb{R}, (x + 1 = 0 \wedge x + 2 = 0)$. This proposition is false, we can not find a value of x , for which both equation are satisfied

❹ $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0$.This proposition is true, for each x we can take $y = -x + 1$, then we have $x + y > 0$

Exercise 5

❶ Using the proof by contradiction prove that $\sqrt{2}$ is not a rational number. We assume that $\sqrt{2}$ is rational which means that there exist two integers p and $q \neq 0$, which are coprimes or no commun factor. such that $\sqrt{2} = \frac{p}{q}$.

$$\begin{aligned}\sqrt{2} = \frac{p}{q} &\Longrightarrow 2 = \frac{p^2}{q^2} \\ &\Longrightarrow 2q^2 = p^2.\end{aligned}$$

We deduce that 2 divides p^2, so it divides p. Then p is even, so it is written by $p = 2k$. We replace the value of p in the equation $2q^2 = p^2$ we obtain

$$\begin{aligned}2q^2 = p^2 &\Longrightarrow 2q^2 = (2k)^2 \\ &\Longrightarrow 2q^2 = 4k^2 \\ &\Longrightarrow q^2 = 2k^2\end{aligned}$$

we deduce then 2 divides q^2 so it divides q, which means that q is also even. Contradiction because we assumed already that p and q are coprimes

❷ Prove by induction : $\forall n \in \mathbb{N}^*, 1 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.
Let $P(n)$ be 1+ $2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

— For $n = 1$, we check if $P(1)$ is true. In the left side of equality we have 1 and the right side we get $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$, so $P(1)$ is true.

— We assume that $P(n)$ is true and prove that $P(n+1)$ is also true which means we show that $1 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}$.

$$\begin{aligned}1 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= (n+1)^2[\frac{n^2}{4} + (n+1)] \text{ (by hypothesis induction)} \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4}.\end{aligned}$$

Thus $P(n+1)$ is true. By induction we conclude that $\forall n \in \mathbb{N}^*, 1 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

❸ By contrapositive, prove that $[(n^2-1)$ is not divisible by $8] \Longrightarrow (n$ is even$)$. By contrapositive, il is the same to show that (n is not even) $\Longrightarrow$ $[(n^2-1)$ is divisible by $8]$
If n is odd then it is written by $n = 2k+1$, for some integer k.
so we have

$$\begin{aligned} n^2-1 &= (2k+1)^2-1 = 4k^2+4k+1-1 = 4k^2+4k \\ &= 4k(k+1) = 4.2k^{'} = 8k^{'} \text{(because} k.(K+1) \text{ is always even)} \end{aligned}$$

So (n^2-1) is divisible by 8 .

❹ Let a be an integer. Prove by cases : 2 divides $a(a+1)$. By cases

case 1 If a is even then a is written by $a = 2k$ for some integer k. We have $a(a+1) = 2k(2k+1)$ it is obvious even integer.

case 2 If a is odd then a is written by $a = 2k+1$ for some integer k. So, we have $a(a+1) = (2k+1)((2k+1)+1) = (2k+1)(2k+2) = 2(2k+1)(2k+1)$ which also an even integer.

By case , we deduce that $a(a+1)$ is an even integer, for integer number a.