Course of Algebra 1

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Contents

Chapter 1

Introduction to Mathematical logic

1.1 Statement

Definition 1.1.1

1. *A **Statement** (or assertion, proposition) is a sentence which is either true or false, but not both at the same time. Propositions are often denoted by capital letter $P, Q, R, \cdots$*

2. ***The truth value** is one of the two values. true (T) or False(F), that can be taken by a given statement. Sometimes the truth value T is denoted by 1, and F by 0.*

3. *A **truth table** is a table showing the truth value of a statement (generally a compound one) given the possible truth values of the simple statement of which it is composed.*

P
T
F

Example 1.1.1

1. *"2 is a prime number". T*

2. *"every prime number is odd". F*

3. *"The sum of two odd numbers is always odd" F*

4. *"The sum of two even numbers is always even" T*

5.

Definition 1.1.2 (Axiom) *An axiom, postulate, or assumption is a statement accepted as true as the basis for argument.*

Example 1.1.2 *Euclids axiom in geometry are :*

1. *A straight line may be drawn between any two points.*

2. *Any terminated straight line may be extended indefinitely.*

3. *A circle may be drawn with any given point as center and any given radius.*

4. *All right angles are equal.*

5. *For any given point not on a given line, there is exactly one line through the point that does not meet the given line.*

Example 1.1.3 (Peano Axioms) *If M is a subset of $\mathbb{N}$ containing 0 such that every successor of elemnet of M is an element of M then $M = \mathbb{N}$.*

Definition 1.1.3

Theorem *is a statement that has been proven true, or can be proven.*

Corollary *is a proposition that follows from one already proved.*

Lemma *is a proposition proved or accepted for immediate use in the proof of some other proposition.*

Definition *is a statement that explains the meaning of a word or phrase.*

Definition 1.1.4 (Conjecture) *A statement or an idea which is unproven, but is thought to be true, a guess.*

Remark 1.1.1 *A **conjecture** is an important step in problem solving; it is not just a tool for professional mathematicians. In everyday problem solving, it is very rare that a problem's solution is immediately apparent. Instead, the problem solving process involves analyzing the problem structure, examining cases, developing a conjecture about the solution, and then confirming that conjecture through proof.*

Example 1.1.4 (Bertrand's Conjecture) *"At least one Prime between n and $2n$" after 5 years, In 1850 Tschebyschev published a proof of Bertrand's Conjecture.*

Example 1.1.5 (Fermat's conjecture or Fermat's Last Theorem) *states that no three positive integers $x, y,$ and z satisfy the equation $x^n + y^n = z^n$ for any integer value of n greater than 2.*
After 358 years of effort by mathematicians, the first successful proof was released in 1994 by Andrew Wiles and formally published in 1995.

1.2 The logical connectors

Connectors (or operators) are used to construct or define a new proposition from one or more initial propositions.

1.2.1 Negation

let P be a proposition, we define a proposition " not P" denoted as $no(P)$ or $\neg P$ or simply as $\overline{P}$.
If P is false, then $\overline{P}$ is true. If P is true, then $\overline{P}$ is false.

P	Q
T	F
F	T

Example 1.2.1 1. P : "$2+5=8$" *(F)*, $\bar{P}$: "$2+5 \neq 8$" *(T)*.

2. *we have P: "2 is an even number" is true (T), then $\overline{P}$:" 2 is an odd number" is false (F).*

1.2.2 Conjunction "and"

Let P and Q be two propositions. We define a new proposition " P **and** Q" which is denoted $P \wedge Q$ which is true when the propositions P and Q are both true, and false in all other cases.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.2.2 1. *The proposition "*$2+5=7$ ***and*** *3 is an odd number " is true.*

2. *" 4 is a number less than 7* ***and*** *5 is even" is false.*

1.2.3 Disjunction "or"

Let P and Q be two propositions. We define a new proposition " P **or** Q" denoted $P \vee Q$ which is true when at least one of the two proposition P or Q are true, and false in all other cases. ($P \vee Q$ false when both of the propositions are false).

P	Q	P∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.2.3 1. *The proposition "*$2+5=7$ ***or*** *2 is an odd number " is true.*

2. *" 4 is a number less than 3* ***or*** *5 is even" is false.*

1.2.4 Implication. "$\Longrightarrow$"

Let P and Q be two propositions. The proposition $P \Longrightarrow Q$ reads " **P implies Q**" or " **if P then Q**" It is false when P is true and Q is false, and true otherwise. The truth table for proposition $P \Longrightarrow Q$ is as follows :

P	Q	P$\Longrightarrow$ Q
T	T	T
T	F	F
F	T	T
F	F	T

Remark 1.2.1 *$P \Longrightarrow Q$ can also read as*

1. *If P, then Q*
2. *it is necessary for Q to have P*
3. *It is sufficient for P to have Q*
4. *Q is required for P*

So, if you hear one of these expressions in everyday language, it is actually an implication.

Example 1.2.4 *for x real number,* $0 \leqslant x \leqslant 4 \Longrightarrow 0 \leqslant \sqrt{x} \leqslant 2$ *(take the square root)*

1.2.5 Equivalence. " $\Longleftrightarrow$ "

Let P and Q be two propositions. The new proposition $P \iff Q$ read " P is equivalent to Q". It can be read "P **if and only if** Q"
$P \iff Q$ is true when P and Q have the same truth value (both true or both false) False otherwise.
Equivalence is defined by :

$$(P \iff Q) \text{ is the proposition } [(P \Longrightarrow Q) \text{ and } (Q \Longrightarrow P)].$$

The truth table for proposition $P \iff Q$ is as follows :

P	Q	P$\Longleftrightarrow$ Q
T	T	T
T	F	F
F	T	F
F	F	T

When proving an equivalence, the double implication rule is often employed, First we establish one direction of implication then we prove the reciprocal implication.

Example 1.2.5 *1. For a real number x,* $0 \leqslant x^2 \leqslant 4 \iff -2 \leqslant x \leqslant 2$

2. For $x, y \in \mathbb{R} : xy = 0 \iff x = 0$ *or* $y = 0$.

Proposition 1 (Properties) *Let P, Q and R three propositions. We have the following equivalences:*

1. $(P \wedge Q) \iff (Q \wedge P)$ *(commutativity of and).*
2. $(P \vee Q) \iff (Q \vee P)$ *(commutativity of or).*
3. $\overline{(P \wedge Q)} \iff (\overline{P} \vee \overline{Q})$ *(morgan's laws).*
4. $\overline{(P \vee Q)} \iff (\overline{P} \wedge \overline{Q})$ *(morgan's laws)*
5. $\overline{\overline{P}} \iff P$.
6. $(P \wedge P) \iff P$.
7. $(P \vee P) \iff P$.
8. $[(P \wedge Q) \wedge R] \iff [P \wedge (Q \wedge R)]$ *(associativity of and).*
9. $[(P \vee Q) \vee R] \iff [P \vee (Q \vee R)]$ *(associativity of or).*
10. $[(P \wedge Q) \vee R] \iff [(P \vee R) \wedge (Q \vee R)]$ *(distributiveness of and with respect to or).*
11. $[(P \vee Q) \wedge R] \iff [(P \wedge R) \vee (Q \wedge R)]$ *distributiveness of or with respect to and).*
12. $(P \Longrightarrow Q) \iff (\overline{P} \vee Q)$.
13. $\overline{(P \Longrightarrow Q)} \iff (P \wedge \overline{Q})$.
14. $(P \Longrightarrow Q) \iff (\overline{Q} \Longrightarrow \overline{P})$ *(**contraposition**).*
15. $(P \iff Q) \iff (P \Longrightarrow Q \wedge Q \Longrightarrow P)$.

Proof 1.2.5.1 *We prove proposition* 12.

P	Q	$\overline{Q}$	$P \Longrightarrow Q$	$\overline{P \Longrightarrow Q}$	$P \wedge \overline{Q}$	$\overline{(P \Longrightarrow Q)} \Longleftrightarrow (P \wedge \overline{Q})$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

Definition 1.2.1 *A **tautology** is a statement that is always true (regardless of the truth value of its elementary statements).*

Example 1.2.6 *The statement*

$$(P \wedge Q) \Longrightarrow (P \vee Q)$$

is a tautology. To prove it, we construct the truth table.

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \Longrightarrow (P \vee Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

1.3 Quantifiers

Definition 1.3.1 *A **predicate** is a statement that contains variables and that may be true or false depending on the value of these variables.*

Example 1.3.1

1. *Let $P(x) : x^2 > 0$ is a predicate, one has $P(0) : 0 > 0$ is false and $P(-1) : (-1)^2 = 1 > 0$ is true.*

2. *Let $P(x, y) : (x + y)^2 = x^2 + y^2$ is a predicate , $P(0,1)$ and $P(1,0)$ are true but $P(1,1)$ is false.*

A predicate can also be made a proposition by adding a quantifier. There are two quantifiers :

1.3.1 The universal quantifier "$\forall$"

We write mathematically the expression

for all element of E , the proposition P(x) is true

as

$$\forall x \in E, P(x).$$

The symbole $\forall$ is read " for all" or "for every",or " for any", $\cdots$. It is a quantifier that indicates that the property is true for all objects satisfying the given condition.

Example 1.3.2 1. *The proposition " every real number is greater than or equal to 5" is written "* $\forall x\mathbb{R}, x \geqslant 5$ *".*

2. *The proposition* $\forall x \in \mathbb{N}, x+1>0$ *is true.*

3. *The proposition* $\forall x\mathbb{Z}, x \geqslant 0$ *ids false.*

1.3.2 Existential quantifier "$\exists$"

The expression

There existe x of E such that $P(x)$

is written mathematically

$$\exists x \in E, P(x)$$

to express that the proposition $P(x)$ is true for at least one x of E.
The symbole $\exists$ represente the existential quantifier, and it signifies the existence of at least one element that satisfies the predicate.

Remark 1.3.1

1. *We use sometimes the symbole* $\exists!$ *to express*

   There exists a unique element $\cdots$.

   The proposition "$\exists! \in E, P(x)$*" means that the assertion* $P(x)$ *is true for a unique value x of* E.

2. *The order of* ***Existential quantifier*** *and* ***universal quantifier*** *in a statement is important. For example these propositions* "$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x+y=0$" *and* "$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x+y=0$" *are not the same.*
   The first proposition is true, , the existence of y depends on x, if you pick any x, we can find y that make $x+y=0$ (for example $y=-x$).
   The second proposition is false, the existence of y doex not depend on x, there is no y that make $x+y=0$ true for every x.

The negation of quantifiers

Let E be a set, and $P(x)$ be a proposition

1. The **negation** of "$\forall x \in E, P(x)$" is : "$\exists x \in E, \overline{P(x)}$"
2. The **negation** of "$\exists x \in E, P(x)$" is : "$\forall x \in E, \overline{P(x)}$"

Exercise 1.3.1 1. *Translate the following proposition into a logical expression " Every real number except zero has a multiplicative inverse"*

2. *Write the following proposition with quantifiers :*
   " f is not increasing on $\mathbb{R}$ (where f is a function of $\mathbb{R}$ in $\mathbb{R}$)
3. *Show that the function cos is note zero.*
4.

Solution 1.3.1 1. $\forall x \in \mathbb{R}^*, \exists y \in \mathbb{R} : xy = 1.$

2. *By defintion*

$$f \text{ is increasing} \iff (\forall x, y \in \mathbb{R}, x \leqslant y \Longrightarrow f(x) \leqslant f(y)).$$

   By applying the negation of an implication

$$\overline{P \Longrightarrow Q} \iff P \wedge \overline{Q},$$

   one has

$$\overline{\forall x, y \in \mathbb{R}, x \leqslant y \Longrightarrow f(x) \leqslant f(y)} \iff [\exists (x,y) \in \mathbb{R}^2, (x \leqslant y) \wedge (f(x) > f(y))].$$

3. $\exists x = 0, cos(0) = 1 \neq 0,$ *then* $\cos(x) \neq 0.$

Exercise 1.3.2 *If $A = \{1,3,5,7,9,10\}$, determine the truth value of each of the following*

1. $\exists x \in A, x+4=7$.
2. $\forall x \in A, 4-x \in \mathbb{N}$.
3. $\exists x \in A$, *x is even.*

Solution 1.3.2

1. *Since x=3, satisfies $x+4=7$, then the proposition is true, its truth value is "T".*
2. $\exists x \in A, x=5$, *do not satisfy* $4-5=-1 \notin \mathbb{N}$, *then the proposition is false, so its truth value is "F".*
3. *Since $x=10 \in A$ satisfies the given proposition , then the proposition is true, its truth value is "T".*

Exercise 1.3.3 *Write the following sentence using quantifiers.*

- *The square of any real number is positive.*
- *Every integer has an opposite (there is another integer such that the sum equal to zero)*
- *there is at least one integer which is opposite to all integers.*
- *For all real numbers , the square of the sum of two numbers is equal to the sum of their squares.*

Solution 1.3.3

- $\forall x \in \mathbb{R}, x^2 \geqslant 0.$ *(T)*
- $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x+y=0$ *(T)*
- $\exists x \in \mathbb{Z}, \forall y \mathbb{Z}, x+y=0.$ *(F)*
- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (x+y)^2 = x^2+y^2.$*(F)*

1.4 Reasoning methods

1.4.1 Direct Proof

To show that ($P \Longrightarrow Q$) is true , we assume that P is true and we prove that Q is also true.

Example 1.4.1 *Prove that*

$$\forall a, b \in \mathbb{R}, (a^2 = b^2) \Longrightarrow (|a| = |b|).$$

Solution 1.4.1 *Let a, b be two real numbers, we assume that $a^2 = b^2$, since a^2, b^2 are positive real numbers then we can consider their square roots , so we have*

$$\begin{aligned} a^2 = b^2 &\Longrightarrow \sqrt{a^2} = \sqrt{b^2} \\ &\Longrightarrow |a| = |b|. \end{aligned}$$

1.4.2 Proof by cases

Suppose that we want to prove ($P \Longrightarrow Q$) and this is difficult for somme reason. We notice however that P can be broken, namely there exist $P_0, P_1, \cdots, P_n$ and we show that $P_i \Longrightarrow Q$ is true for all $i \in \{1, \cdots, n\}$, then we conclude that $P \Longrightarrow Q$ is true.

Example 1.4.2 *Let n be an integer, prove by cases*

- *2 divides $n(n+1)$*
- *3 divides $n(n+1)(n+2)$*

Solution 1.4.2

- *Let n be an integer. Then n is either even or odd.*
  - *If n is even, it exists $k \in \mathbb{N}$ such that $n = 2k$.*
    Therefore, $n(n+1) = 2k(2k+1) = 2k'$, with $k' = k(2k+1) \in \mathbb{N}$, so 2 divides $n(n+1)$.
  - *If n is odd, it exists $k \in \mathbb{N}$ such that $n = 2k+1$.*
    Therefore, $n(n+1) = (2k+1)(2k+2) = 2k'$, with $k' = (2k+1)(k+1) \in \mathbb{N}$, so 2 divides $n(n+1)$.

  ***Conclusion**: The proposition is true for all $n \in \mathbb{N}$.*

- ***hint** : Any natural number n can be written as $3k$, or $3k+1$, or $3k+2$, for some natural number k.(3 cases to study).*

1.4.3 Proof by the contrapositive

It is based on the following property

$$(P \Longrightarrow Q) \iff (\overline{Q} \Longrightarrow \overline{P}).$$

If the direct method does not help to prove that $(P \Longrightarrow Q)$, then we prove that its contrapositive hold.

Example 1.4.3
Let $x, y \in \mathbb{Z}$, show that if $y \neq 0$ then $x + y\sqrt{2} \notin \mathbb{Q}$.

Solution 1.4.3 *We can write the proposition by using implication*

$$(y \neq 0) \Longrightarrow (x + y\sqrt{2} \notin \mathbb{Q}).$$

By using the contrapositive property, it remaind the same to show if the proposition

$$(x + y\sqrt{2} \in \mathbb{Q}) \Longrightarrow (y = 0).$$

is true .

if $x + y\sqrt{2} \in \mathbb{Q}$ then there exist $(p, q) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$, such that $x + y\sqrt{2} = \frac{p}{q}$.
So $y\sqrt{2} = \frac{p}{q} - x$ is a rational number , if $y \neq 0$ then $\sqrt{2} = \frac{\frac{p}{q} - x}{y}$ will be a rational number, contradiction since $\sqrt{2} \notin \mathbb{Q}$, hence $y = 0$.

1.4.4 Proof by contradiction (or absurd)

In order to prove that a proposition P is true, we assume that it is false, in other word we assume that $\overline{P}$ is true and from this assumption, we are able to deduce a statement that contradicts some assumption we made in the proof or some known fact.
To prove that $(P \Longrightarrow Q)$ is true, we assume that P is true and Q is false, such assumption will lead to a contradiction.

Example 1.4.4 *Using reasoning by contradiction, show that for every integer n, we have*

$$[n^2 \textit{ is even}] \Longrightarrow [n \textit{ is even}].$$

Solution 1.4.4 *We suppose that ($[n^2$ is even$] \Longrightarrow [n$ is even$]$) is false , that means (n^2 is even and n is odd.*
since n is odd, then n can be written as $n = 2k + 1$ for some natural number k.
So $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ is odd, contradiction, since we assumed that n^2 is even.
Hence the assumption (n^2 is even $\Longrightarrow$ n is even) is true, for all natural number n.

1.4.5 Proof by counter example

To prove that the proposition $(\forall x \in E, P(x))$ is not true, it is enough to find an element x_0 in E such that $P(x_0)$ is false.

Example 1.4.5 *Is the following proposition true ?why?*

$$\forall x \in \mathbb{R}, x^2 > 0.$$

1.4.6 Proof by induction

Let $P(n)$ be a family of statement indexing by the naturel number. Suppose

(i) $P(0)$ is true. (**initial step**)

(ii) for any $n \in \mathbb{N}$ if $P(n)$ is true then $P(n+1)$ is true. (**Induction step**)

then $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.4.6 *Prove by induction that :*

$$\forall n \in \mathbb{N}, \sum_{k=0}^{n} k = \frac{1}{2} n(n+1)$$

Solution 1.4.5
- *P(0) is true since for n=0, in the left hand side (**LHS**), we obtain* $\sum_{k=0}^{k=0} k = 0$ *and in the Right hand side (**RHS**) , we have* $\frac{1}{2}0(0+1) = 0$.
- *Suppose that* $P(n)$ *is true , show that* $P(n+1)$ *is true.*

$$\begin{aligned}\sum_{k=0}^{k=n+1} k &= \sum_{k=0}^{k=n} k + (n+1)\\ &= \frac{1}{2} n(n+1) + (n+1) \text{ (using inductive hypothesis)}\\ &= \frac{1}{2}(n+1)(n+2).\end{aligned}$$

so $P(n+1)$ *is also true, by induction* $P(n)$ *is true for all* $n \in \mathbb{N}$.

In some cases, we can use this second form of induction called "**strong induction**"

Theorem 1.4.1 (Strong induction)
Let $P(n)$ *be a family of statement indexed by n, suppose :*

(I) *$P(0)$ is true.(0 represent the first element of the set)*

(II) *For any $n \in \mathbb{N}$, if $P(0), P(1), \ldots, P(n)$ are true, then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.4.7 *Every natural number greater than 1 can be expressed as a product of one or more primes.*

Solution 1.4.6 *Let $P(n)$ be the statement that n can be expressed as a product of primes.*

- *$P(2)$ is true since 2 is itself prime.*
- *Let $n > 2$ and suppose $P(n)$ holds for all $m < n$.*
    - *If n is prime , then $P(n)$ is true.*
    - *If n is not prime, then $n = rs$ for some $r, s, \in \mathbb{N}$ with $r, s < n$. By induction hypothesis r and s can be expressed as product of primes and hense $n = rs$, so $P(n)$ is true.*

By strong induction, $P(n)$ holds for all natural number greater than 1.