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First year of mathematics	**Algebra 1**	**2024-2025**
Tutorial Series N° 5	**Algebraic structures**	

Exercise 1 We define a binary operation $\star$ on the set $\mathbb{R}$ by : $x \star y = x + y + 1$.

❶ Show that $(\mathbb{R}, \star)$ is an abelian group.

❷ Let $H = \left\{2k+1, \text{ such that } k \in \mathbb{Z}\right\}$ be a subset of $\mathbb{R}$. Show that $(H, \star)$ is a subgroup of $(\mathbb{R}, \star)$.

❸ Let $\lambda \in \mathbb{R}$ and f be a function from the group $(\mathbb{R}, +)$ to the group $(\mathbb{R}, \star)$ defined by :

$$\forall x \in \mathbb{R},\ f(x) = x + \lambda.$$

Determine λ for which f be a group homomorphism.

Exercise 2 Recall that $(\mathbb{R}, +, .)$ be a field, notice that 0 is an identity element and 1 an unit element of $\mathbb{R}$.

(I) We define on $\mathbb{R}$ two other binary operations by

$$\forall a, b \in \mathbb{R},\ a \oplus b = a + b + 1 \quad \text{and} \quad a \otimes b = ab + a + b$$

❶ Show that $(\mathbb{R}, \oplus, \otimes)$ is a ring with unity. Is it a field?

❷ Show that the function f defined by

$$f : (\mathbb{R}, \oplus, \otimes) \longrightarrow (\mathbb{R}, +, .)$$
$$a \longmapsto f(a) = a + 1$$

is a ring isomorphism.

(II) Let E be the subset of $\mathbb{R}$ defined by

$$E = \left\{x + y\sqrt{3} :\ x, y \in \mathbb{Q}\right\}$$

❶ Show that E is closed under " + " and "."

❷ Show that every element of $E \setminus \{0\}$ has an inverse (under the multiplication) in $E \setminus \{0\}$

❸ Show that E is a subfield of the field $(\mathbb{R}, +, .)$.

Homework

Exercise 1

1. Determine which of the following sets are group under addition :
   (a) the set of all rationals (including 0) in lowest term whose denominators are odd.
   (b) the set of all rationals (including 0) in lowest term whose denominators are even.
   (c) the set of rationals whose absolute value <1.
   (d) the set of rationals whose absolute value 1 together with 0.
2. Let $G := \left\{a + b\sqrt{2}\ a, b \in \mathbb{Q}\right\}$.
   (a) Show that G is a group under addition.
   (b) Show that non-zero elements of G forms a group under multiplication.
3. Let $G = \left\{z \in \mathbb{C} \text{ such that } z = 1 \text{ for some } n \in \mathbb{N}\right\}$
   (a) Show that G forms a group with respect to multiplication.
   (b) Show that G does not form a group with respect to addition.