Corrigé de la série de TD n°1

Exercice 1

Etudier la limite de f en x_0 :

* $\lim_{x_0 = 0} (x^2+x+1) = 1$

* $\lim_{x \to 1} \left(\frac{3x-1}{7x-4} \right) = \frac{2}{3}$

* $\lim_{x \to -1} \left(\frac{x^2-1}{x+1} \right) = \frac{0}{0}$ (C.I)

$\frac{x^2-1}{x+1} = \frac{(x-1)(x+1)}{x+1} = x-1$

$\lim_{x \to -1} \left(\frac{x^2-1}{x+1} \right) = \lim_{x \to -1} (x-1) = -2$

* $\lim_{x \to 1} \left(\frac{x^2+2x-3}{3x^2-2x-1} \right) = \frac{0}{0}$ (C.I)

$\frac{x^2+2x-3}{3x^2-2x-1} = \frac{(x-1)(x+3)}{3(x-1)(x+\frac{1}{3})} = \frac{x+3}{3x+1}$

$\lim_{x \to 1} \left(\frac{x^2+2x-3}{3x^2-2x-1} \right) = \lim_{x \to 1} \left(\frac{x+3}{3x+1} \right) = \frac{4}{4} = 1.$

* $\lim_{x \to 3} \frac{\sqrt{x+1}-2}{x-3} = \frac{0}{0}$ (C.I)

$\frac{\sqrt{x+1}-2}{x-3} = \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)} = \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$

$= \frac{x-3}{(x-3)(\sqrt{x+1}+2)} = \frac{1}{\sqrt{x+1}+2}$

$$\lim_{x \to 3} \left(\frac{\sqrt{x+1} - 2}{x - 3} \right) = \lim_{x \to 3} \left(\frac{1}{\sqrt{x+1} + 2} \right) = \frac{1}{4}.$$

* $$\lim_{x \to 3} \left(\frac{\sqrt{x+1} - 2}{\sqrt{x+6} - 3} \right) = \frac{0}{0} \; (C.I)$$

$$\frac{\sqrt{x+1} - 2}{\sqrt{x+6} - 3} = \frac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)(\sqrt{x+6} + 3)}{(\sqrt{x+6} - 3)(\sqrt{x+6} + 3)(\sqrt{x+1} + 2)}$$

$$= \frac{(x-3)(\sqrt{x+6} + 3)}{(x-3)(\sqrt{x+1} + 2)} = \frac{\sqrt{x+6} + 3}{\sqrt{x+1} + 2}$$

$$\lim_{x \to 3} \left(\frac{\sqrt{x+1} - 2}{\sqrt{x+6} - 3} \right) = \lim_{x \to 3} \left(\frac{\sqrt{x+6} + 3}{\sqrt{x+1} + 2} \right) = \frac{6}{4} = \frac{3}{2}.$$

* $$\lim_{x \to 2} \left(\frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3} \right) = \frac{0}{0} \; (C.I)$$

$$\frac{x - \sqrt{x+2}}{\sqrt{4x+1} - 3} = \frac{(x - \sqrt{x+2})(x + \sqrt{x+2})(\sqrt{4x+1} + 3)}{(\sqrt{4x+1} - 3)(\sqrt{4x+1} + 3)(x + \sqrt{x+2})}$$

$$= \frac{(x^2 - x - 2)(\sqrt{4x+1} + 3)}{(4x - 8)(x + \sqrt{x+2})} = \frac{\cancel{(x-2)}(x+1)(\sqrt{4x+1} + 3)}{4\cancel{(x-2)}(x + \sqrt{x+2})}$$

$$= \frac{(x+1)(\sqrt{4x+1} + 3)}{4(x + \sqrt{x+2})}$$

$$\lim_{x \to 2} \left(\frac{x - \sqrt{x+2}}{\sqrt{4x-1} - 3} \right) = \lim_{x \to 2} \left(\frac{(x+1)(\sqrt{4x+1} + 3)}{4(x + \sqrt{x+2})} \right) = \frac{9}{8}.$$

* $$\lim_{x \to 4} \left(\frac{\sqrt{x-1} - \sqrt{3}}{x^2 - 16} \right) = \frac{0}{0} \; (C.I)$$

$$\frac{\sqrt{x-1}-\sqrt{3}}{x^2-16} = \frac{(\sqrt{x-1}-\sqrt{3})(\sqrt{x-1}+\sqrt{3})}{(x^2-16)(\sqrt{x-1}+\sqrt{3})}$$

$$= \frac{(x-4)}{(x-4)(x+4)(\sqrt{x-1}+\sqrt{3})} = \frac{1}{(x+4)(\sqrt{x-1}+\sqrt{3})}$$

$$\lim_{x\to 4}\left(\frac{\sqrt{x-1}-\sqrt{3}}{x^2-16}\right) = \lim_{x\to 4}\left(\frac{1}{(x+4)(\sqrt{x-1}+\sqrt{3})}\right) = \frac{1}{16\sqrt{3}},$$

Exercice 2

Trouver les limites suivantes,

* $$\lim_{x\to\pm\infty}\left(\frac{x^2-3x+8}{-2x^2+5x-7}\right) = \lim_{x\to\pm\infty}\left(\frac{x^2}{-2x^2}\right) = -\frac{1}{2},$$

* $$\lim_{x\to\pm\infty}\left(\frac{5x+3}{x^2+11x-17}\right) = \lim_{x\to\pm\infty}\left(\frac{5x}{x^2}\right) = \lim_{x\to\pm\infty}\left(\frac{5}{x}\right) = 0^{\pm}$$

* $$\lim_{x\to\pm\infty}\left(\frac{5x^2+4x-1}{x+2}\right) = \lim_{x\to\pm\infty}\left(\frac{5x^2}{x}\right) = \lim_{x\to\pm\infty}(5x) = \pm\infty$$

Exercice 3

Soit la fonction,

$$f(x) = x + \frac{\sqrt{x^2}}{x}$$

Etudions la limite de f au voisinage de 0,

$$f(x) = x + \frac{|x|}{x}$$

or, $|x| = \begin{cases} -x & \text{si } x<0 \\ +x & \text{si } x>0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x - \frac{x}{x}\right) = \lim_{x\to 0^-}(x-1) = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x + \frac{x}{x}\right) = \lim_{x\to 0^+}(x+1) = +1$$

On remarque que: $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^-} f(x)$

Conclusion: La fonction

Exercice 04

Trouvons les limites suivantes:

- $\lim_{x\to+\infty}\left[\sqrt{x^2-x-1} - (x-1)\right] = (+\infty) - (+\infty)$ (C.I)

$$\sqrt{x^2-x-1} - (x-1) = \frac{\left[\sqrt{x^2-x-1} - (x-1)\right]\left[\sqrt{x^2-x-1} + (x-1)\right]}{\left[\sqrt{x^2-x-1} + (x-1)\right]}$$

$$= \frac{x^2-x-1-(x-1)^2}{\sqrt{x^2-x-1}+(x-1)} = \frac{x^2-x-1-x^2+2x-1}{\sqrt{x^2-x-1}+(x-1)}$$

$$= \frac{x-2}{\sqrt{x^2-x-1}+(x-1)} = \frac{x\left(1-\frac{2}{x}\right)}{\sqrt{x^2\left(1-\frac{1}{x}-\frac{1}{x^2}\right)}+x\left(1-\frac{1}{x}\right)}$$

$$= \frac{x\left(1-\frac{2}{x}\right)}{|x|\sqrt{1-\frac{1}{x}-\frac{1}{x^2}}+x\left(1-\frac{1}{x}\right)}$$

$$\lim_{x\to+\infty}\left[\sqrt{x^2-x-1} - (x-1)\right] = \lim_{x\to+\infty}\left[\frac{x\left(1-\frac{2}{x}\right)}{x\sqrt{1-\frac{1}{x}-\frac{1}{x^2}}+x\left(1-\frac{1}{x}\right)}\right]$$

$$=$$

$$= \lim_{x \to +\infty} \left[\frac{1 - \frac{2}{x}}{\left[\sqrt{1 - \frac{1}{x} - \frac{1}{x^2}} + \left(1 - \frac{1}{x}\right)\right]} \right] = \frac{1}{2}$$

$$\cdot \lim_{x \to -\infty} \left[\frac{x - \sqrt{x^2+x+1}}{2x - \sqrt{4x^2+x}} \right] = \frac{-\infty}{-\infty} \; (C.I)$$

$$\frac{x - \sqrt{x^2+x+1}}{2x - \sqrt{4x^2+x}} = \frac{x - |x|\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}}{2x - |x|\sqrt{4 + \frac{1}{x}}}$$

$$= \frac{x + x\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}}{2x + x\sqrt{4 + \frac{1}{x}}} \quad \left(|x| = -x \; (x \to -\infty)\right)$$

$$= \frac{x\left(1 + \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}\right)}{x\left(2 + \sqrt{4 + \frac{1}{x}}\right)} = \frac{1 + \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}}{2 + \sqrt{4 + \frac{1}{x}}}.$$

$$\lim_{x \to -\infty} \left(\frac{x - \sqrt{x^2+x+1}}{2x - \sqrt{4x^2+x}} \right) = \lim_{x \to -\infty} \left[\frac{1 + \sqrt{1 + \cancel{\frac{1}{x}} + \cancel{\frac{1}{x^2}}}}{2 + \sqrt{4 + \cancel{\frac{1}{x}}}} \right]$$

$$= \frac{2}{4} = \frac{1}{2}.$$

Exercice 5

La continuité, suivant les valeurs du paramètre a, de la fonction suivante

$$\begin{cases} f(x) = 0 & \text{pour } x \leq 0 \\ f(x) = x & \text{pour } 0 \leq x \leq 1 \\ f(x) = 3 - ax^2 & \text{pour } x \geq 1 \end{cases}$$

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- Continuité en $x_0 = 0$:

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{0 \to 0^-} (0) = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (x) = 0$

Donc : $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = 0$

La fonction est continue en $x_0 = 0$

- Continuité en $x_0 = 1$.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (1) = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - ax^2) = 3 - a$.

$f(1) = 1$

La fonction f est continue en $x_0 = 1$ si et seulement si.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = f(1)$

Soit : $3 - a = 1 \Rightarrow a = 2$.

Donc :

Si $a = 2 \Rightarrow f$ est continue en $x_0 = 1$

si $a \neq 2 \Rightarrow f$ n'est pas continue en $x_0 = 1$

Conclusion

si $a = 2 \Rightarrow f$ est continue sur $D_f = \mathbb{R}$

si $a \neq 2 \Rightarrow f$ " " sur $\mathbb{R} - \{1\}$.

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Exercice 06

Soit la fonction :

$$f(x) = \frac{x^3 - 2x^2 + x - 2}{x^2 - 3x + 2} \qquad D_f =]1, +\infty[.$$

La fonction f est prolongeable par continuité en $x_0 = 2$ si seulement et si :

$x_0 = 2 \in D_f$ et $\lim_{x \to 2} f(x)$ existe.

$x_0 = 2 \in]1, +\infty[.$

$$\lim_{x \to 2} f(x) = \frac{0}{0} \ (C.I)$$

$$\frac{x^3 - 2x^2 + x - 2}{x^2 - 3x + 2} = \frac{(x - 2)(x^2 + 1)}{(x - 2)(x - 1)}$$