Serie 2

Exo1 $f(x) = \dfrac{x^2+2x-3}{3x^2-2x-1}$

1) $D_f = \{x \in \mathbb{R},\ 3x^2-2x-1 \neq 0\}$.

$\Delta = 4-4(-1)(3) = 16 > 0$, $x_1 = \dfrac{2+4}{6} = 1$, $x_2 = \dfrac{2-4}{6} = -\dfrac{1}{3}$.

$D_f =]-\infty, -\frac{1}{3}[\cup]-\frac{1}{3}, 1[\cup]1, +\infty[= \mathbb{R} - \{-\frac{1}{3}, 1\}$.

2) prolongement par continuite au pt (1).

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+3)}{3(x-1)(x+\frac{1}{3})} = \lim_{x\to 1} \frac{(x+3)}{3(x+\frac{1}{3})} = \frac{4}{4} = 1$$

donc f est prolongeable par continuite,

soit $g(x) = \begin{cases} f(x) & \text{si } x \neq 1 \\ 1 & \text{si } x = 1 \end{cases}$

g(x) est le prolongement par continuite de f au pt 1.

prolongement par continuité au pt $(-\frac{1}{3})$.

$$\lim_{x\to -\frac{1}{3}} f(x) = \lim_{x\to -\frac{1}{3}} \frac{(\cancel{x-1})(x+3)}{3(\cancel{x-1})(x+\frac{1}{3})} = \lim_{x\to -\frac{1}{3}} \frac{(x+3)}{3(x+\frac{1}{3})} = +\infty$$

donc f n'est pas prolongeable par continuite au pt $-\frac{1}{3}$.

Exo3. $f'(x) = (\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$.

$f'(x) = \ln(1+x^2) = \dfrac{2x}{1+x^2}$

$f'(x) = (x+e^x)' = 1+e^x$

$f'(x) = \left(\dfrac{5x-3}{x-1}\right)' = \dfrac{5(x-1)-(5x-3)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$.

$f'(x) = \left(\dfrac{x^3-2x-1}{x^3}\right)' = \dfrac{(3x^2-2)x^3-3x^2(x^3-2x-1)}{x^6} = \dfrac{4}{x^3}+\dfrac{3}{x^4}$

$$f'(x) = \left(\left(1+\sqrt{1+x^2}\right)^{\frac{1}{2}}\right)' = \frac{1}{2}\left(1+\sqrt{1+x^2}\right)^{\frac{1}{2}-1} \frac{\cancel{2}x}{\cancel{2}\sqrt{1+x^2}} \quad (2)$$

$$= \frac{1}{2\sqrt{1+\sqrt{1+x^2}}} \cdot \frac{x}{\sqrt{1+x^2}}$$

$$f'(x) = \left(\frac{\sqrt{x}}{1+x}\right)' = \frac{\frac{1}{2\sqrt{x}}(1+x) - \sqrt{x}}{(1+x)^2} = \frac{(1+x)}{2\sqrt{x}(1+x)^2} - \frac{\sqrt{x}}{(1+x)^2}$$

$$f'(x) = \left(\left(\frac{1+x}{1-x}\right)^{\frac{1}{3}}\right)' = \frac{1}{3}\left(\frac{1+x}{1-x}\right)^{\frac{1}{3}-1}\left(\frac{(1-x)-(1+x)}{(1-x)^2}\right)$$

$$f'(x) = \frac{1}{3}\left(\frac{1+x}{1-x}\right)^{-\frac{2}{3}} \frac{-2x}{(1-x)^2} = -\frac{2}{3}\,\frac{x}{\left(\frac{1+x}{1-x}\right)^{\frac{2}{3}}(1-x)^2}$$

Exo4, l'equation de la tangente à la courbe $f(x) = x^3 - x^2 - x$ au pt $x_0 = 2$.

$$y = f'(2)(x-2) + f(2) = 7(x-2) + 2 = 7x - 12,$$

$$f'(2) = 3(2)^2 - 2(2) - 1 = 7$$

$$f(2) = 2^3 - 2^2 - 2 = 8 - 4 - 2 = 2$$

Exo5 $f(x) = x + \frac{1}{x}$ f admet un minimum.

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

$$f''(x) = \frac{2x}{x^4} = \frac{2}{x^3} > 0 \quad \forall x \in]0, +\infty[$$

donc f admet un minimum au pt $x = 1 \in]0, +\infty[$

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Exo 6. $f : \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto x^3 + \lambda x$.

$f'(x) = 3x^2 + \lambda = 0 \Rightarrow ?$

$\Delta = -4 \times 3\lambda = -12\lambda > 0$ si et seulement si $\lambda < 0$

si $\lambda < 0$ $\quad f''(x) = 6x > 0$ si $x > 0$

$f''(x) < 0$ si $x < 0$

donc si $x \in]0, +\infty[$ f admet un minimum au pt $x = \dfrac{\sqrt{-12\lambda}}{6}$.

si $x \in]-\infty, 0[$ f admet un maximum au pt $x = \dfrac{-\sqrt{-12\lambda}}{6}$.

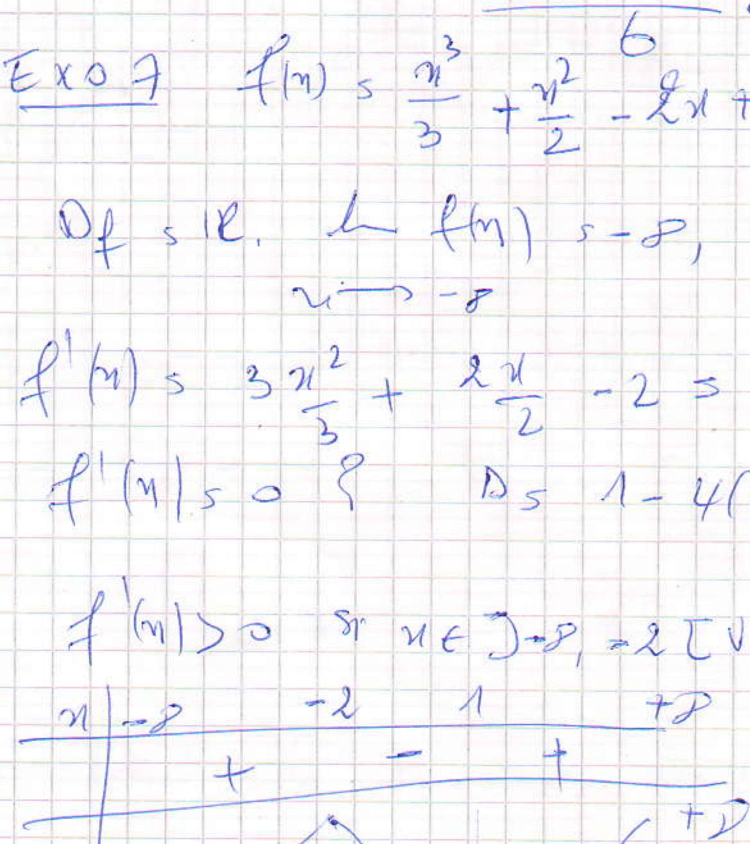
Exo 7 $\quad f(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + 2$

$D_f = \mathbb{R}$. $\lim\limits_{x \to -\infty} f(x) = -\infty$, $\lim\limits_{x \to +\infty} f(x) = +\infty$

$f'(x) = \dfrac{3x^2}{3} + \dfrac{2x}{2} - 2 = x^2 + x - 2$.

$f'(x) = 0$? $\quad \Delta = 1 - 4(-2) = 9$, $x_1 = \dfrac{-1+3}{2} = 1$

$f'(x) > 0$ si $x \in]-\infty, -2[\cup]1, +\infty[$ $\quad x_2 = \dfrac{-1-3}{2} = -2$



$f(1) = \frac{1}{3} + \frac{1}{2} - 2 + 2 = 5/6$

$f(-2) = \dfrac{-8}{3} + \dfrac{4}{2} - 2(-2) + 2$

$f(-2) = \dfrac{16}{3}$

$f(1) = \dfrac{1}{3} + \dfrac{1}{2} - 2 + 2 = \dfrac{5}{6}$

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l'intersection avec les axes :

l'axe $y'y$: $x = 0 \Rightarrow y = 2$ donc $(0, 2)$

l'axe $x'x$: $f(-2) = 14/3$; $f(-3) = \frac{21}{6}$; $f(-4) = \frac{-10}{3}$

f continue sur $[-4, -3]$

$f(-4) \cdot f(-3) < 0$

D'après le TVI, $\exists c \in]-4, -3[$ tq $f(c) = 0$

2) $f''(x) = 2x + 1 > 0$ si $x > -\frac{1}{2}$

Donc dans $]-\frac{1}{2}, +\infty[$ f admet un minimum

$f''(x) < 0$ si $x \in]-\infty, -\frac{1}{2}[$.

dans $]-\infty, -1/2[$ f admet ~~un~~ maximum.

Exo 8 : $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x}{(1+x)^4 - 1} = \frac{0}{0}$ cas I

on applique la règle de l'hopital.

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{4(1+x)^3} = \frac{1}{4}$.

$$\lim_{x\to 0} \frac{L_n(1+x)}{\sqrt{x}} = \frac{0}{0}$$ (5)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\frac{1}{1+x}}{\frac{1}{2\sqrt{x}}} = \frac{1}{\frac{1}{0}} = \frac{1}{\infty} = 0$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \frac{0}{0} = \lim_{x\to 0} \frac{+\sin x}{6x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1 - \cos x}{tg\, x} = \frac{0}{0} = \lim_{x\to 0} \frac{+\sin x}{\frac{1}{\cos^2 x}} = \frac{0}{1} = 0$$

<u>Ex 2</u> $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longrightarrow \frac{2x}{1+x^2}$$

1) f injective ? $\forall x_1, x_2 \in \mathbb{R}.\ f(x_1) = f(x_2) \overset{?}{\Rightarrow} x_1 = x_2$

$$\frac{2x_1}{1+x_1^2} = \frac{2x_2}{1+x_2^2} \Rightarrow 2x_1(1+x_2^2) = 2x_2(1+x_1^2)$$

$$\Rightarrow 2x_1 + 2x_1 x_2^2 = 2x_2 + 2x_2 x_1^2$$

$$\Rightarrow \cancel{2}(x_1 - x_2) = \cancel{2}x_1 x_2 (x_1 - x_2)$$

$$\Rightarrow x_1 x_2 = 1 \quad \text{si } x_1 \neq x_2$$

$$\Rightarrow x_2 = \frac{1}{x_1} \quad \text{si } x_1 \neq x_2,\ x_1 \neq 0$$

donc f n'est pas injective

f surjective si $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}.\ y = f(x)$ (6)

$y = f(x) \Leftrightarrow \frac{2x}{1+x^2} = y \Leftrightarrow y + yx^2 - 2x = 0$

$\Rightarrow yx^2 - 2x + y = 0$

$\Delta = (-2)^2 - 4(y)(y) = 4 - 4y^2 \overset{?}{>} 0$

$\Delta = 0 \Leftrightarrow \cancel{4}(1-y^2) = 0 \Rightarrow y = \pm 1.$

$\Delta > 0 \Rightarrow y \in [-1, 1]$

si $\Delta > 0$ ou bien si $y \in [-1, 1]$

$yx^2 - 2x + y = 0$ admet 2 racines.

$x_1 = \frac{+2 + 2\sqrt{1-y^2}}{2y} = \frac{1+\sqrt{1-y^2}}{y}$

$x_2 = \frac{2 - 2\sqrt{1-y^2}}{2y} = \frac{1-\sqrt{1-y^2}}{y}$

si $\Delta < 0 \Leftrightarrow y \in]-\infty, -1[\cup]1, +\infty[$

$yx^2 - 2x + y = 0$ n'admet pas de racine

($\nexists$ il n'existe pas) $\nexists x \in \mathbb{R}.$ tq $y = f(x)$

donc si $y \in [-1, 1]$ f surjective

si $y \in]-\infty, -1[\cup]1, +\infty[$. f n'est pas surject

$\Rightarrow f$ n'est pas surjective sur $\mathbb{R}$

2) $f(\mathbb{R}) = \{y \in \mathbb{R}, \exists x \in \mathbb{R}, y = f(x)\}$

$y = f(x) \Rightarrow y = \frac{2x}{1+x^2} \Rightarrow yx^2 - 2x + y = 0$

si $y \in [-1, 1]$ on trouve $x_1 = \frac{1+\sqrt{1-y^2}}{y}$, $x_2 = \frac{1-\sqrt{1-y^2}}{y}$

si $y \in]-\infty, -1[\cup]1, +\infty[$. $\nexists x \in \mathbb{R}$, $y = f(x)$ (7)

donc $f(\mathbb{R}) = [-1, 1]$.

3) $g : [-1, 1] \longmapsto [-1, 1]$

$g(x) = f(x)$.

· g bijective $\forall y \in [-1, 1]$, $\exists x!$ (x unique) / $y = f(x)$

$y = f(x) \Leftrightarrow y = \dfrac{2x}{1+x^2} \Rightarrow y x^2 - 2x + y = 0$

$\Rightarrow x_1 = \dfrac{1+\sqrt{1-y^2}}{y}$, $x_2 = \dfrac{1-\sqrt{1-y^2}}{y}$. $y \neq 0$

$x_1 \in [-1, 1] \Leftrightarrow -1 \leq x_1 \leq 1 \Leftrightarrow -1 \leq \dfrac{1+\sqrt{1-y^2}}{y} \leq 1$

$\Leftrightarrow -y \leq 1 + \sqrt{1-y^2} \leq y$

$\Leftrightarrow -1-y \leq \sqrt{1-y^2} \leq y-1$

$\Leftrightarrow (-1-y)^2 \leq 1-y^2 \leq (y-1)^2$

$\Leftrightarrow \cancel{1} + 2y + y^2 \leq \cancel{1} - y^2 \leq y^2 - 2y + \cancel{1}$

$\Rightarrow \begin{cases} 2y + 2y^2 \leq 0 \\ \wedge \\ 2y^2 - 2y \geq 0 \end{cases} \Rightarrow \begin{cases} 2y(1+y) \leq 0 & -(1) \\ \wedge \\ 2y(y-1) \geq 0 & -(2) \end{cases}$

$(1) \Rightarrow y \in [-1, 0]$

$(2) \Rightarrow y \in]-\infty, 0] \cup [1, +\infty[$

$(1) \wedge (2) \Rightarrow y \in [-1, 0]$

donc $g : [-1, 1] \longmapsto [-1, 0]$

$x \longmapsto \dfrac{2x}{1+x^2}$.

g bijective et g^{-1} (la fonction réciproque existe

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$g^{-1} : [-1, 0[\longmapsto [-1, 1]$

$x \longmapsto g^{-1}(x) = \dfrac{1 \pm \sqrt{1-x^2}}{x}$,

$x_2 \in [-1, 1] \Leftrightarrow -1 \leq x_2 \leq 1$

$\Rightarrow -1 \leq \dfrac{1-\sqrt{1-y^2}}{y} \leq 1 \Leftrightarrow -y \leq 1-\sqrt{1-y^2} \leq y$

$\Rightarrow \cancel{-1}-y \leq -\sqrt{1-y^2} \leq -1+y \quad , y \neq 0$

$\Rightarrow 1-y \leq \sqrt{1-y^2} \leq y+1$

$\Rightarrow (1-y)^2 \leq 1-y^2 \leq (y+1)^2$

$\Rightarrow \cancel{1}-2y+y^2 \leq \cancel{1}-y^2 \leq y^2+2y+\cancel{1}$

$\Rightarrow \begin{cases} 2y^2+2y \geq 0 \\ \wedge \\ 2y^2-2y \leq 0 \end{cases} \Rightarrow \begin{cases} 2y(y+1) \geq 0 & -(1) \\ \wedge \\ 2y(y-1) \leq 0 & (2) \end{cases}$

(1) $\Rightarrow y \in]-\infty, -1] \cup [0, +\infty[$

$\wedge$

(2) $y \in [0, 1]$

(1) $\wedge$ (2) $\Rightarrow y \in]0, 1]$

donc $g : [-1, 1] \longmapsto]0, 1]$

$x \longmapsto \dfrac{2x}{1+x^2}$

$g^{-1} :]0, 1[\longmapsto [-1, 1]$

$x \longmapsto g^{-1}(x) = \dfrac{1 - \sqrt{1-x^2}}{x}$