$* E_{\lambda_1} = Ker(A-\lambda_1 I_3) \Leftrightarrow \begin{cases} -x+y=0 \Rightarrow y=x \\ -x+z=0 \Rightarrow z=x \\ x-z=0 \end{cases}$

$E_{\lambda_1} = vect\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ 0,5 , $\dim E_{\lambda_1} = 1 = \alpha_1$

$* E_{\lambda_2} = Ker(A-\lambda_2 I_3) \Leftrightarrow \begin{cases} -ix+y=0 \\ -x+(1-i)y+z=0 \\ x-iz=0 \end{cases}$

$y = ix \ , \ z = -ix$ 0,5

$E_{\lambda_2} = vect\left\{ v_2 = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix} \right\}$, $\dim E_{\lambda_2} = 1 = \alpha_2$

$* E_{\lambda_3} = Ker(A-\lambda_3 I_3) \Leftrightarrow \begin{cases} ix+y=0 \\ -x+(1+i)y+z=0 \\ x+iz=0 \end{cases}$

$y = -ix \ , \ z = ix$ 0,5

$E_{\lambda_3} = vect\left\{ v_3 = \begin{pmatrix} 1 \\ -i \\ i \end{pmatrix} \right\}$, $\dim E_{\lambda_3} = 1 = \alpha_3$

A est diagonalisable $\Rightarrow \exists$ P inversible et D diagonale tq $A = PDP^{-1}$

$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & i & -i \\ 1 & -i & i \end{pmatrix}$ 0,25 , $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix}$ 0,25

$X' = AX \Leftrightarrow X' = PDP^{-1}X$

On pose $Y = P^{-1}X$ 0,25 $\Rightarrow X' = PDY$

$\Rightarrow Y' = P^{-1}X' = P^{-1}AX = P^{-1}PDP^{-1}X$

d'où $Y' = DY$ 0,25

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2y_1 \\ (1+i)y_2 \\ (1-i)y_3 \end{pmatrix} \Rightarrow \begin{cases} y_1 = \alpha e^{2x} \\ y_2 = \beta e^{(1+i)x} \\ y_3 = \gamma e^{(1-i)x} \end{cases}$$ 0,75

où α, β, γ trois cstes réelles.

On a posé $Y = P^{-1}X \Rightarrow X = PY$ 0,5

D'où $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & i & -i \\ 1 & -i & i \end{pmatrix} \begin{pmatrix} \alpha e^{2x} \\ \beta e^{(1+i)x} \\ \gamma e^{(1-i)x} \end{pmatrix}$

$x = \alpha e^{2x} + \beta e^{(1+i)x} + \gamma e^{(1-i)x}$

$y = \alpha e^{2x} + \beta i e^{(1+i)x} - \gamma i e^{(1-i)x}$

$z = \alpha e^{2x} - \beta i e^{(1+i)x} + \gamma i e^{(1-i)x}$

Exercice n° 03 :

$\begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$X_{n+1} = AX_n + b \Rightarrow X_n = AX_{n-1} + b$

On trouve $X_n = A^n X_0 + (A^{n-1} + A^{n-2} + \dots + I)b$

On remarque que:

$A = \frac{3}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$ 1

$= 3/2\, U + 1/2\, V$

$UV = VU = 0$. 1

On peut donc utiliser la formule de binôme matricielle.

On trouve $A^n = \frac{3^n}{2} U + \frac{1^n}{2} V$

$A^n = \frac{3^n}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} \frac{3^n+1}{2} & \frac{3^n-1}{2} \\ \frac{3^n-1}{2} & \frac{3^n+1}{2} \end{pmatrix}$ 1 $\forall n \in \mathbb{N}^*$.

$A^{n-1} + A^{n-2} + \dots + A + I = H$

$H = \begin{pmatrix} \underbrace{\frac{3^{n-1}+1}{2} + \frac{3^{n-2}+1}{2} + \dots + 2 + 1} & \frac{3^{n-1}-1}{2} + \frac{3^{n-2}-1}{2} + \dots + 1 + 0 \\ \frac{3^{n-1}-1}{2} + \frac{3^{n-2}-1}{2} + \dots + 1 + 0 & \frac{3^{n-1}+1}{2} + \frac{3^{n-2}+1}{2} + \dots + 2 + 1 \end{pmatrix}$

$= \frac{1}{2}\left[(3^{n-1}+1) + (3^{n-2}+1) + \dots + (3^{n-(n-1)}+1) + (3^{n-\dots}\right.$

$= \frac{1}{2}(3^{n-1} + 3^{n-2} + \dots + 3^{n-n} + n) = \frac{1}{2}\left[3(1-\frac{1}{\dots}\right.$